Research Statement

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1. Overview

My research interests are partial differential equations, fluid dynamics and dynamical systems. Particularly, my works are focused on the following topics:

(i) Dynamics of viscous, incompressible fluids related to the Navier–Stokes equations.
(ii) Generalized Forchheimer (non-Darcy) flows of compressible fluids in porous media.
(iii) Asymptotic analysis for dynamical systems.

The first two subjects are quite different from problems to techniques. They both deal with fundamental and challenging nonlinear partial differential equations in fluid dynamics. Their studies require both insights and innovative techniques. The last subject consists of analysis for both ordinary differential equations and abstract dynamical systems, which can be applied to partial differential equations.

1.1. **Dynamics of viscous, incompressible fluids.** Fluid dynamics is important in classical mechanics with real life applications. My long-term interest is the long-time dynamics of the fluid flows with the ultimate goal being to understand the fluid turbulence. Fluid turbulence, of course, is one of the most, if not the most, challenging problems in classical physics. Despite its long history, the phenomena are still not well-understood, especially from the mathematical point of view. However, in some cases things are much known. One of the situation is the decaying turbulence. Understanding decaying turbulence can help shed insights into the other types of turbulence. This is part of my contribution to the subject. My current research in this direction contains two subcategories that correspond to two descriptions of fluids: the Eulerian and the Lagrangian.

1.1.1. *The Navier–Stokes equations.* The NSE is a system of nonlinear partial differential equations that describe the dynamics of the incompressible, viscous fluid flows. They are formulated using the Eulerian description of fluids. They are one of the most important and challenging systems of nonlinear partial differential equations. Although their research is vast, the fundamental questions about the three-dimensional NSE are still open.

My strength in this direction is the asymptotic expansions for the solutions of the three-dimensional NSE in different contexts. I have published extensively on the subject and are continuing to expand the theory. Based on the pioneering works by Foias and Saut [29–33], my collaborators and I developed this theory further in [24–28, 48, 52]. We have detailed asymptotic analysis of solutions of the NSE: asymptotic expansions of solutions, normalization map and normal form theory. We develop analytic tools for studying dynamical systems in infinite dimensional spaces, particularly infinite integrable systems, the normal form theory and Poincaré-Dulac normal forms. We apply them to mathematical theory of decaying turbulence based on the NSE with potential body forces. It includes asymptotic analysis of statistical solutions, ensemble averages of physical quantities and relations between them, and connections to Kolmogorov's theory on turbulence. The current development is the asymptotic theory for the NSE with non-potential, decaying forces [6, 7, 49]. It is very promising in deriving a rigorous theory of decaying turbulence, at least in certain interesting, but still rather general, scenarios. Our methods and techniques invented for the NSE can be easily adapted to other dissipative dynamical systems.

1.1.2. *Analysis of the Lagrangian trajectories.* The Lagrangian formulation of fluids results in more complicated equations than the NSE. Therefore, the analysis of their solutions is very limited, and often restricted to short time properties. In a recent paper [39], I surprisingly obtained the asymptotic expansions, as time tends to infinity, for the Lagrangian trajectories associated with the solutions of the three-dimensional NSE. These asymptotic expansions provide very fine details for the long-time behaviors of these trajectories. This result may open up more systematic study of long-time Lagrangian dynamics, at least for decaying turbulence, which was out of reach previously.

1.2. **Generalized Forchheimer flows of compressible fluids in porous media.** In the studies of porous media, Darcy's law is widely used to describe the fluid flows. This law is a linear equation between the pressure gradient and the fluid velocity. However, it is known that Darcy's law is not accurate in modeling fluid flows in many situations [3, 34, 54]. In fact, even in their original works [22,23], Darcy and Dupuit already noted the deviation from the linear relation. Forchheimer [34,35], see also [3, 55, 60], later proposed three nonlinear equations to capture such deviations. They are called two-term, three-term and power laws. Compared to Darcy's law, Forchheimer equations are much less studied, particularly from the mathematical point of view. Also, mathematical papers on Forchheimer or related Brinkman-Forchheimer equations [10, 18, 36, 56–58, 62] are mostly for incompressible fluids.

My research in this direction is rigorous analysis of the so-called generalized Forchheimer equations which we propose in [1, 40] to cover many possible variations of the Forchheimer equations. Their genuine nonlinearity makes it much more difficult to analyze than the well-known linear Darcy equation. In joint works [1, 11, 12, 14–16, 40–47], my collaborators and I extend significantly studies of the generalized Forchheimer equations to compressible fluids in porous media. We model and analyze single-phase compressible fluids, two-phase mixed fluids, fluids of mixed regimes (pre-Darcy, Darcy and post-Darcy), and rotating fluids. We develop techniques for dynamical systems of degenerate parabolic equations, particularly asymptotic stability, long-time continuous dependence and structural stability. Our research also brings forth new nonlinear models that are practical, but still rigorous, and may stimulate the development of new PDE techniques.

1.3. **Asymptotic analysis for dynamical systems.**

1.3.1. *Attractors of abstract dynamical systems.* The dynamical systems arising in scientific and engineering problems usually depend on many parameters. While their long-time dynamics are captured by the global attractors, the dependence of these attractors on the parameters are hard to obtain. My collaborators and I [50] study a family of general dynamical systems and their global attractors with parameters belonging to a metric space. We prove that these attractors are continuous, with respect to the Hausdorff distance, at values of the parameters that belong to a residual (large) set. The results are then extended to the non-autonomous systems, and then applied to the celebrated Lorenz and the two-dimensional Navier–Stokes systems [51]. Although the dynamics are sensitive to the change of the parameters, our results show certain degree of robustness of the attractors. With this knowledge, we hope that more reliable numerical schemes can be designed to capture more precisely the long-time dynamics of fluid flows.

1.3.2. *Asymptotic expansions for nonlinear systems of ordinary differential equations.* This subject grows out of my research in the NSE. However, when considered in the context of more general nonlinear ordinary differential equations it becomes an independent topic itself and raises many interesting problems. My collaborators and I have obtained some key results for different types of equations and forcing functions [7,9,38]. They, in turn, provide insights into the dynamics of partial differential equations. They have many potential applications in other branches of mathematics including partial differential equations, geometry and mathematical biology.

We will discuss each of the above topics in details in sections 2–6 below. Section 7 contains some ideas about my future research.

2. Long-time dynamics for the Navier–Stokes equations

We consider the initial value problem for the viscous, incompressible Navier–Stokes equations (NSE) in the three-dimensional space:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla p + \mathbf{f}, \\ \text{div } \mathbf{u} = 0, \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}^0(\mathbf{x}), \end{cases} \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the spatial variable, $\nu > 0$ is the kinematic viscosity, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the unknown velocity field, $p = p(\mathbf{x}, t)$ is the unknown pressure, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is the body force and $\mathbf{u}^0(\mathbf{x})$ is the known initial velocity field. We study the periodic case when both the solution $\mathbf{u}$ and force $\mathbf{f}$ are periodic in x_1, x_2, x_3 of the same period $L > 0$, that is, they are L-periodic functions. By a remarkable Galilean transformation, we can assume the following zero spatial average condition

$$\int_\Omega \mathbf{u}(\mathbf{x}, t)\mathrm{d}\mathbf{x} = 0, \text{ where } \Omega = (-L/2, L/2)^3.$$

The force $\mathbf{f}$ is assumed to be divergence-free and has zero spatial average as well. By a change of scale we assume, without loss of generality, $L = 2\pi$ and $\nu = 1$.

Let H, respectively V, be the closure in $L^2(\Omega)^3$, respectively $H^1(\Omega)^3$, of $\mathcal{V}$ – the set of all vector-valued L-periodic trigonometric polynomials which are divergence-free and have zero spatial average.

The Stokes operator A is defined by $Au = -\Delta u$ for $u \in \mathcal{D}(A)$, where $\mathcal{D}(A)$, called the domain of A, is the closure of $\mathcal{V}$ in $H^2(\Omega)^3$. We recall that the spectrum $\sigma(A)$ of the Stokes operator A is a subset of $\mathbb{N} = \{1, 2, 3, \dots\}$ and the additive semi-group generated by $\sigma(A)$ is $\mathbb{N}$.

We define the bi-linear map associated with the nonlinear term in the NSE by

$$B(u, v) = P_L(u \cdot \nabla v) \quad \text{for all} \quad u, v \in \mathcal{D}(A),$$

where P_L denotes the orthogonal projection in $L^2(\Omega)^3$ onto H. The functional form of (2.1) is

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) + B(u(t), u(t)) = f(t), \quad t > 0. \tag{2.2}$$

When $f = 0$, it is proved in [32, 33] that for any Leray–Hopf weak solution $u(t)$ of (2.2) has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt}, \tag{2.3}$$

where $q_j(t)$, $j \geq 1$, is a polynomial in t with values in $\mathcal{V}$. This means that for any $N \in \mathbb{N}$ the correction term $\tilde{u}_{N+1}(t) = u(t) - \sum_{j=1}^N q_j(t) e^{-jt}$ satisfies

$$\|\tilde{u}_{N+1}(t)\|_{L^2(\Omega)} = O\big(e^{-(N+\varepsilon)t}\big) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad \varepsilon = \varepsilon_N > 0. \tag{2.4}$$

In fact, for each $m \in \mathbb{N}$ relation (2.4) holds for the Sobolev norm $\|\tilde{u}_{N+1}(t)\|_{H^m(\Omega)}$ and $\varepsilon = \varepsilon_{N,m} > 0$.

2.1. Asymptotic expansions with exponentially decaying functions.

The asymptotic expansions (2.3) can be generalized as follows.

Definition 2.1. *Let X be a vector space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.*

(a) *A function $g : \mathbb{R} \to X$ is an X-valued S-polynomial if it is a finite sum of the functions in the set*

$$\Big\{t^m \cos(\omega t) Z,\ t^m \sin(\omega t) Z : m \in \mathbb{N} \cup \{0\},\ \omega \in \mathbb{R},\ Z \in X\Big\}.$$

(b) *A function $g : \mathbb{R} \to X$ is an X-valued SS-polynomial if it is a finite sum of the functions in the set*

$$\Big\{t^m \cos\big(a\cos(\omega t) + b\sin(\omega t) + ct + d\big)Z,\ t^m \sin\big(a\cos(\omega t) + b\sin(\omega t) + ct + d\big)Z :$$
$$m \in \mathbb{N}\cup\{0\},\ a,b,c,d,\omega \in \mathbb{R},\ Z \in X\Big\}.$$

(c) *Denote by $\mathcal{F}_0(X)$, respectively, $\mathcal{F}_1(X)$ and $\mathcal{F}_2(X)$ the set of all X-valued polynomials, respectively, S-polynomials and SS-polynomials.*

Definition 2.2. *Let $(X, \|\cdot\|_X)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative real numbers. Let $\mathcal{F} = \mathcal{F}_0$, $\mathcal{F}_1$, or $\mathcal{F}_2$. A function $f : [T,\infty) \to X$, for some $T \in \mathbb{R}_+$, is said to have an asymptotic expansion*

$$f(t) \sim \sum_{n=1}^\infty f_n(t)e^{-\alpha_n t} \quad \text{in } X, \tag{2.5}$$

where each f_n belongs to $\mathcal{F}(X)$, if one has, for any $N \geq 1$, that

$$\Big\|f(t) - \sum_{n=1}^N f_n(t)e^{-\alpha_n t}\Big\|_X = \mathcal{O}(e^{-(\alpha_N+\varepsilon_N)t}), \text{ as } t \to \infty, \text{ for some } \varepsilon_N > 0.$$

For $\alpha,\sigma \in \mathbb{R}$ and $u = \sum \widehat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \in H$, define

$$A^\alpha u = \sum |\mathbf{k}|^{2\alpha}\widehat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad e^{\sigma A^{1/2}}u = \sum e^{\sigma|\mathbf{k}|}\widehat{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Denote $|\cdot| = \|\cdot\|_{L^2(\Omega)^3}$. For $\alpha,\sigma \geq 0$, the Gevrey spaces are defined by

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) \overset{\text{def}}{=} \{u \in H : |u|_{\alpha,\sigma} \overset{\text{def}}{=} |A^\alpha e^{\sigma A^{1/2}}u| < \infty\}.$$

In particular, when $\sigma = 0$ the domain of the fractional operator A^α is

$$\mathcal{D}(A^\alpha) = G_{\alpha,0} = \{u \in H : |A^\alpha u| = |u|_{\alpha,0} < \infty\}.$$

Thanks to the zero-average condition, the norm $|A^{m/2}u|$ is equivalent to $\|u\|_{H^m(\Omega)^3}$ on the space $\mathcal{D}(A^{m/2})$, for $m = 0,1,2,\ldots$

Note that $\mathcal{D}(A^0) = H$, $\mathcal{D}(A^{1/2}) = V$, and $\|u\| \overset{\text{def}}{=} |\nabla u|$ is equal to $|A^{1/2}u|$, for all $u \in V$. Also, the norms $|\cdot|_{\alpha,\sigma}$ are increasing in α, σ, hence, the spaces $G_{\alpha,\sigma}$ are decreasing in α, σ.

Denote for $\sigma \in \mathbb{R}$ the space $E^{\infty,\sigma} = \bigcap_{\alpha\geq 0} G_{\alpha,\sigma} = \bigcap_{m\in\mathbb{N}} G_{m,\sigma}$.

We will say that an asymptotic expansion (2.5) holds in $E^{\infty,\sigma}$ if it holds in $G_{\alpha,\sigma}$ for all $\alpha \geq 0$.

Let us also denote by $\mathcal{P}^{\alpha,\sigma}$ the space of $G_{\alpha,\sigma}$-valued polynomials in case $\alpha \in \mathbb{R}$, and the space of $E^{\infty,\sigma}$-valued polynomials in case $\alpha = \infty$.

Theorem 2.3 (Hoang–Martinez [49])**.** *Assume that there exist a number $\sigma_0 \geq 0$ and polynomials $f_n \in \mathcal{P}^{\infty,\sigma_0}$, for all $n \geq 1$, such that $f(t)$ has the asymptotic expansion*

$$f(t) \sim \sum_{n=1}^\infty f_n(t)e^{-nt} \quad \text{in } E^{\infty,\sigma_0}.$$

Let $u(t)$ be a Leray-Hopf weak solution of the NSE. Then there exist polynomials $q_n \in \mathcal{P}^{\infty,\sigma_0}$, for all $n \geq 1$, such that $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^\infty q_n(t)e^{-nt} \quad \text{in } E^{\infty,\sigma_0}. \tag{2.6}$$

Moreover, the polynomials $q_n(t)$'s satisfy the following ODEs in the space E^{∞,σ_0}

$$\frac{\mathrm{d}}{\mathrm{d}t}q_n(t) + (A - n\mathrm{Id})q_n(t) + \sum_{\substack{k,m\geq 1\\ k+m=n}} B(q_k(t), q_m(t)) = f_n(t), \quad t \in \mathbb{R}.$$

Thanks to the techniques and improvements in [48, 49], we can easily obtain other asymptotic expansions for the NSE in different contexts. One of them is the NSE for the rotating fluids, which are

$$\frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \Omega\mathbf{e}_3 \times \mathbf{u} = 0, \tag{2.7}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2.8}$$

where $\frac{1}{2}\Omega\mathbf{e}_3$ is the angular velocity of the rotation.

Let $\mathbf{L} = (L_1, L_2, L_3) \in (0,\infty)^3$. We study the solutions of (2.7) and (2.8) which are L_j-periodic in x_j, for $j = 1, 2, 3$. Denote $\mathbb{T}_\mathbf{L} = (\mathbb{R}/L_1\mathbb{Z}) \times (\mathbb{R}/L_3\mathbb{Z}) \times (\mathbb{R}/L_3\mathbb{Z})$. We can assume $\nu = 1$ and $\sigma(A) \subset [1,\infty)$. Arrange the additive semigroup generated by $\sigma(A)$ as a sequence $(\mu_n)_{n=1}^\infty$ which strictly increases to infinity.

Theorem 2.4 (Hoang–Titi [52])**.** *Let $u(t)$ be any Leray-Hopf weak solution of (2.7) and (2.8) with the zero spatial average over $\mathbb{T}_\mathbf{L}$. Then there exist unique $\mathcal{V}$-valued S-polynomials Q_n's, for all $n \in \mathbb{N}$, such that*

$$u(t) \sim \sum_{n=1}^\infty Q_n(t)e^{-\mu_n t} \quad \text{in } G_{\alpha,\sigma}, \text{ for all } \alpha, \sigma > 0. \tag{2.9}$$

Consider the case without the zero spatial average condition. Denote by $\mathbf{U}(t)$ the average of $\mathbf{u}(\mathbf{x}, t)$ over $\mathbb{T}_\mathbf{L}$. Integrating equation (2.7) over the domain $\mathbb{T}_\mathbf{L}$ and solving the resulting ODE give

$$\mathbf{U}(t) = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0\\ -\sin(\Omega t) & \cos(\Omega t) & 0\\ 0 & 0 & 1\end{pmatrix}\mathbf{U}(0).$$

Theorem 2.5 (Hoang–Titi [52])**.** *There exist $\mathcal{V}$-valued SS-polynomials $\mathcal{Q}_n(t)$'s, for all $n \in \mathbb{N}$, such that*

$$u(t) \sim \mathbf{U}(t) + \sum_{n=1}^\infty \mathcal{Q}_n(t)e^{-\mu_n t} \text{ in } G_{\alpha,\sigma}, \text{ for all } \alpha, \sigma > 0. \tag{2.10}$$

Note that the asymptotic expansion (2.10) with "double sinusoidal" functions is a new phenomenon, which was not observed before.

2.2. **Asymptotic expansions in general systems of decaying functions.** In our recent papers [6, 7], the force $f(t)$ is assumed to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^\infty \phi_n\psi_n(t).$$

We establish, for any Leray–Hopf weak solution $u(t)$ of the NSE, the asymptotic expansion

$$u(t) \sim \sum_{n=1}^\infty \xi_n\psi_n(t).$$

Above, the system $(\psi_n(t))_{n=1}^\infty$ can be very general and complicated. We streamline our presentation to the systems of the power, logarithmically and iterated logarithmically decaying functions only.

Let real numbers $\sigma \geq 0$, $\alpha \geq 1/2$, and $(\lambda_n)_{n=1}^\infty$ be a strictly increasing, divergent sequence of positive numbers such that the set $\{\lambda_n : n \in \mathbb{N}\}$ preserves the addition and unit increment.

We assume the force $f(t)$ has an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma}, \text{ where the sequence } (\phi_n)_{n=1}^{\infty} \subset G_{\alpha,\sigma}.$$

Theorem 2.6 (Cao–Hoang [6, 7])**.** *Any Leray-Hopf weak solution $u(t)$ of the NSE (2.2) has the asymptotic expansion*

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-\lambda_n} \quad \textit{in } G_{\alpha+1-\rho,\sigma} \textit{ for all } \rho \in (0,1),$$

where

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\Big(\phi_n + \chi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m)\Big) \quad \textit{for } n \ge 2, \tag{2.11}$$

with $\chi_n = \lambda_p \xi_p$ if there exists an integer $p \in [1, n-1]$ such that $\lambda_p + 1 = \lambda_n$, and $\chi_n = 0$ otherwise.

When the force $f(t)$ decays logarithmically, we have the following results. For $k, m \in \mathbb{N}$, let

$$L_k(t) = \ln(\ln(\cdots \ln(t))) \ (k\text{-times}) \quad \text{and} \quad \mathcal{L}_m(t) = (L_1(t), L_2(t), \cdots, L_m(t)).$$

Let $Q_0 : \mathbb{R}^m \to \mathbb{R}$ be a polynomial in m variables which has positive degree and positive leading coefficient (with the lexicographic order). Let Q_1 be a polynomial in one variable of positive degree with positive leading coefficient. Given a number $\beta > 0$, we define

$$\omega(t) = (Q_0 \circ \mathcal{L}_m \circ Q_1)(t^\beta)) \text{ with } t \in \mathbb{R}.$$

One can see that there exists $T_* > 0$ such that ω is a positive function defined on $[T_*, \infty)$ and $\omega(t) \to \infty$ as $t \to \infty$.

Let real numbers $\sigma \ge 0$, $\alpha \ge 1/2$, and $(\lambda_n)_{n=1}^{\infty}$ be a strictly increasing, divergent sequence of positive numbers such that the set $\{\lambda_n : n \in \mathbb{N}\}$ preserves the addition. Assume

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n \omega(t)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma}, \text{ for a sequence } (\phi_n)_{n=1}^{\infty} \text{ in } G_{\alpha,\sigma}.$$

Theorem 2.7 (Cao–Hoang [6])**.** *Let ξ_n's be defined by (2.11) with $\chi_n = 0$. Then any Leray-Hopf weak solution $u(t)$ of the NSE (2.2) has the same asymptotic expansion*

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n \omega(t)^{-\lambda_n} \quad \textit{in } G_{\alpha+1-\rho,\sigma} \textit{ for all } \rho \in (0,1),$$

As a consequence, we have the asymptotic expansions in terms of iterated logarithmic functions.

Corollary 2.8 (Cao–Hoang [6])**.** *Given $m \in \mathbb{N}$. If*

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n L_m(t)^{-\lambda_n} \quad \textit{in } G_{\alpha,\sigma},$$

then any Leray-Hopf weak solution $u(t)$ of the NSE (2.2) admits the same asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n L_m(t)^{-\lambda_n} \quad \textit{in } G_{\alpha+1-\rho,\sigma} \textit{ for all } \rho \in (0,1).$$

The main difference between the expansions in section 2.1 and this section is that the former depend on each solution, while the latter are independent of individual solutions. This is also a newly discovered feature for the NSE.

3. Asymptotic expansions for the Lagrangian trajectories

Theoretically speaking, there are two standard descriptions of fluid flows. One is the Lagrangian that is based on the trajectory $x(t)\in\mathbb{R}^3$ of each initial fluid particle (or material point) $x_0=x(0)$, where t is the time variable. The other is the Eulerian which yields the NSE (2.1) in section 2 for the velocity field $u(x,t)$ and pressure $p(x,t)$. The relation between the two descriptions is the following ordinary differential equations (ODE)

$$x'=u(x,t). \tag{3.1}$$

The solutions $x(t)$ of (3.1) are called the Lagrangian trajectories.

Because of the lack of understanding of the NSE and its solution $u(x,t)$, the analysis of the Lagrangian trajectories $x(t)$ is very limited. There have been results for the Lagrangian trajectories in small time intervals. See recent work [4, 5, 19–21, 37, 61] and references therein for short-time well-posedness, regularity, and analyticity, based on solutions of the Euler or Navier–Stokes related systems. Naturally, the long-term behavior of the Lagrangian trajectories is even lesser-known. Nonetheless, the problem is attacked in my paper [39] in the case of potential body forces. We consider the following two situations.

Dirichlet boundary condition (DBC). Let Ω be an bounded, open, connected set in $\mathbb{R}^3$ with C^∞ boundary. We consider (2.7) in $\Omega\times(0,\infty)$ with the boundary condition $u=0$ on $\partial\Omega\times(0,\infty)$.

Let $\mathcal{V}$ be the set of divergence-free vector fields in $C_c^\infty(\Omega)^3$. Denote $\Omega^*=\bar\Omega$.

Spatial periodicity condition (SPC). Fix a vector $\mathbf{L}=(L_1,L_2,L_3)\in(0,\infty)^3$. We consider (2.7) in $\mathbb{R}^3\times(0,\infty)$ with $u(\cdot,t)$ and $p(\cdot,t)$ being $\mathbf{L}$-periodic for $t>0$.

Let $\mathcal{V}$ be the set of $\mathbf{L}$-periodic trigonometric polynomial vector fields on $\mathbb{R}^3$ which are divergence-free and have zero average over Ω. Denote $\Omega^*=\mathbb{R}^3$.

Let H be the completion of $\mathcal{V}$ in $L^2(\Omega)^3$. For any $u_0\in H$, there exists a Leray–Hopf weak solution $u(x,t)$ of (2.7) on $[0,\infty)$ with initial condition $u(x,0)=u_0(x)$. By its eventual regularity, there is $T\ge 0$ such that $u\in C^\infty(\Omega^*\times[T,\infty))$ and satisfies the corresponding (DBC) or (SPC).

Let us fix such a Leray–Hopf weak solution $u(x,t)$ and a Lagrangian trajectory $x(t)\in C^1([T,\infty),\Omega)$ in the (DBC) case, or $x(t)\in C^1([T,\infty),\mathbb{R}^3)$ in the (SPC) case.

The starting point is the following convergence result.

Proposition 3.1 (Hoang [39])**.** *The limit $x_*\stackrel{\text{def}}{=}\lim_{t\to\infty}x(t)$ exists and belongs to Ω^*, and*

$$|x(t)-x_*|=\mathcal{O}(e^{-\mu_1 t}). \tag{3.2}$$

Consideration I: The (SPC) case, or $x_*\in\Omega$ in the (DBC) case.

Consideration II: The (DBC) case with $x_*\in\partial\Omega$.

It is proved in [32] that the solution $u(x,t)$ has an asymptotic expansion

$$u(\cdot,t)\sim\sum_{n=1}^\infty q_n(\cdot,t)e^{-\mu_n t} \text{ in } H^m(\Omega)^3,$$

for any $m\in\mathbb{N}$, where $q_j(\cdot,t)$'s are polynomials in t with values in a linear subspace $\mathcal{X}$ of $C^\infty(\Omega^*)^3$.

One can write each polynomial $q_n(x,t)$, for $n\ge 1$, explicitly as

$$q_n(x,t)=\sum_{k=0}^{d_n}t^k q_{n,k}(x), \text{ where } d_n\ge 0, \text{ and } q_{n,k}\in\mathcal{X}.$$

In fact, $q_1(x,t)$ is independent of t, hence we write $q_1(x,t)=q_1(x)\in\mathcal{X}$.

We focus on Consideration I first. The Taylor expansion about ξ_* for each $q_{n,k}(x)$ is

$$\sum_{m=0}^\infty\frac{1}{m!}D_x^m q_{n,k}(x_*)(x-x_*)^{(m)} \text{ as } x\to x_*,$$

where $D_x^m q_{n,k}$ denotes the m-th order derivative of $q_{n,k}$. Here, $D_x^m q_{n,k}$ is $q_{n,k}$ for $m = 0$, and is an m-linear mapping from $(\mathbb{R}^3)^m$ to $\mathbb{R}^3$, for $m \geq 1$. Define

$$\mathcal{Q}_{n,m}(x_*, t) = \sum_{k=0}^{d_n} \frac{t^k}{m!} D_x^m q_{n,k}(x_*) = \frac{1}{m!} D_x^m q_n(x_*, t).$$

Theorem 3.2 (Hoang [39]). *Under Consideration I, there exist polynomials $\zeta_n : \mathbb{R} \to \mathbb{R}^3$, for $n \geq 0$, such that solution $x(t)$ has an asymptotic expansion*

$$x(t) \sim x_* + \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3, \tag{3.3}$$

where each ζ_n, for $n \geq 1$, is the unique polynomial solution of the following differential equation

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)).$$

for all $t \in \mathbb{R}$. Explicitly, for $n \geq 1$ and $t \in \mathbb{R}$,

$$\zeta_n(t) = -\int_t^{\infty} e^{\mu_n(t-\tau)} \Big\{ q_n(x_*, \tau) + \sum_{m=1}^{s_n} \sum_{\substack{k, j_1, \ldots, j_m = 1, \\ \mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n}}^{n-1} \mathcal{Q}_{k,m}(x_*, \tau)(\zeta_{j_1}(\tau), \ldots, \zeta_{j_m}(\tau)) \Big\} \mathrm{d}\tau.$$

In particular, when $n = 1$, one has $\zeta_1(t) = -q_1(x_*)/\mu_1$ for $t \in \mathbb{R}$.

Theorem 3.3 (Hoang [39]). *Under Consideration II, one has*

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu t}) \text{ for all } \mu > 0. \tag{3.4}$$

4. Generalized Forchheimer flows in porous media

Consider a porous medium with constant porosity $\phi \in (0, 1)$ and constant permeability $k > 0$. We study a fluid flow in porous media with velocity $v(x, t) \in \mathbb{R}^n$, pressure $p(x, t) \in \mathbb{R}$ and density $\rho(x, t) \in \mathbb{R}_+ = [0, \infty)$, where $x \in \mathbb{R}^n$ $(n \geq 2)$ and $t \in \mathbb{R}$ are the spatial and time variables.

Mathematical papers on fluid flows in porous media usually deal with the equation $u_t = \Delta(u^m)$. It results from the the Darcy Law

$$v = -\frac{k}{\mu} \nabla p, \tag{4.1}$$

where μ is the absolute viscosity, combined with the conservation of mass

$$\phi \rho_t + \nabla \cdot (\rho v) = 0, \tag{4.2}$$

and the constitutive law (for isentropic gas flows) $\rho = \bar{c} p^\gamma$.

The generalized Forchheimer equation that we propose [1, 40–42] to replace (4.1) is

$$g(|v|)v = -\nabla p, \tag{4.3}$$

where function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is given by

$$g(s) = a_0 + a_1 s^{\bar{\alpha}_1} + \cdots + a_N s^{\bar{\alpha}_N} = \sum_{i=0}^{N} a_i s^{\bar{\alpha}_i}, \quad \text{for } s \geq 0,$$

with the integer $N \geq 1$, the powers $\bar{\alpha}_0 = 0 < \bar{\alpha}_1 < \bar{\alpha}_2 < \ldots < \bar{\alpha}_N$ being real numbers, and coefficients $a_0, a_1, \ldots, a_N$ being positive constants. Note that the generalized Forchheimer equation (4.3) covers all cases of Forchheimer's two-term, three-term and power laws.

Solving for v implicitly from (4.3) we obtain

$$v = -K(|\nabla p|)\nabla p, \text{ where } K(\xi) = \frac{1}{g(s)} \text{ for } \xi \geq 0, \text{ with } sg(s) = \xi. \tag{4.4}$$

For (isothermal) slightly compressible fluids, the equation of state is

$$\frac{1}{\rho}\frac{d\rho}{dp} = \varpi, \quad \text{where the constant compressibility } \varpi > 0 \text{ is small.} \tag{4.5}$$

From (4.4), (4.2) and (4.5) we derive an equation for the pressure:

$$\phi\frac{\partial p}{\partial t} = \frac{1}{\varpi}\nabla\cdot(K(|\nabla p|)\nabla p) + K(|\nabla p|)|\nabla p|^2. \tag{4.6}$$

Since the constant $1/\varpi$ is very large, we can neglect the last term in (4.6) and study the reduced equation:

$$\phi\frac{\partial p}{\partial t} = \frac{1}{\varpi}\nabla\cdot(K(|\nabla p|)\nabla p). \tag{4.7}$$

(Note that this reduction is commonly used in engineering.)

This problem is studied in a series papers of ours for homegeneous porous media [1,40–44,47], for heterogeneous porous media when $a_i = a_i(x)$ [11,12], and for different mixed regimes [13].

Equation (4.12), however, is a simplified version and does not include the gravity. We fix these by using dimension analysis (Muskat [54]) to refine (4.3) as

$$g(\rho|v|)v = -\nabla p + \rho\vec{\mathbf{g}}, \tag{4.8}$$

where $\vec{\mathbf{g}}$ is the constant gravitational field. Multiplying (4.8) by ρ, and solving for ρv give

$$\rho v = -K(|\rho\nabla p - \rho^2\vec{\mathbf{g}}|)(\rho\nabla p - \rho^2\vec{\mathbf{g}}). \tag{4.9}$$

Combining the above equations, we derive a PDE

$$(u^\lambda)_t = \nabla\cdot(K(|\nabla u - cu^\ell\vec{\mathbf{g}}|)(\nabla u - cu^\ell\vec{\mathbf{g}})) \tag{4.10}$$

with constants $\lambda \in (0,1]$, $\ell = 2\lambda$, $c > 0$. Here, u plays the role of a pseudo-pressure and $u \sim p^{1+1/\gamma}$.

In [14,15], the fluid flows are considered in a bounded domain U, and are subject to the volumetric flux condition $v\cdot\vec{\nu} = \psi$ on $\Gamma = \partial U$, where $\vec{\nu}$ is the outward normal vector. This yields a nonlinear Robin boundary condition for u:

$$-K(|\nabla u - cu^\ell\vec{\mathbf{g}}|)(\nabla u - cu^\ell\vec{\mathbf{g}})\cdot\vec{\nu} = c^{1/2}\psi u^\lambda \text{ on } \Gamma.$$

In fact, a much more general problem is studied in [14,15], namely,

$$\begin{cases} \dfrac{\partial(u^\lambda)}{\partial t} = \nabla\cdot\big(K(|\nabla u + Z(u)|)(\nabla u + Z(u))\big) + f(x,t,u) & \text{on } U\times(0,\infty),\\ u(x,0) = u_0(x) & \text{on } U,\\ K(|\nabla u + Z(u)|)(\nabla u + Z(u))\cdot\vec{\nu} = B(x,t,u) & \text{on } \Gamma\times(0,\infty), \end{cases} \tag{4.11}$$

where the unknown $u(x,t)$ is non-negative on $\bar{U}\times(0,\infty)$, the initial data $u_0(x) \ge 0$ on U is given, $Z(u)$ is a function from $[0,\infty)$ to $\mathbb{R}^n$, $B(x,t,u)$ is a function from $\Gamma\times[0,\infty)\times[0,\infty)$ to $\mathbb{R}$, and $f(x,t,u)$ is a function from $U\times[0,\infty)\times[0,\infty)$ to $\mathbb{R}$.

Compared with (4.7), problem (4.11) is double nonlinear and is much more complicated. It is analyzed rigorously in our works [14,15], and requires the use and improvements of many techniques for degenerate/singular parabolic equations. Its study prompts us to investigate even more complex problems in [16,17]. We now study the dynamics of fluid flows in a porous medium rotated with a constant angular velocity $\Omega\vec{k}$, where $\Omega \ge 0$ is the constant angular speed, and $\vec{k}$ is a constant unit vector. The equation for the Darcy flows in rotating porous media written in a rotating frame is, see Vadasz [63],

$$\frac{\mu}{k}v + \frac{2\rho\Omega}{\phi}\vec{k}\times v + \rho\Omega^2\vec{k}\times(\vec{k}\times x) = -\nabla p + \rho\vec{\mathbf{g}},$$

where x is the position in the rotating frame, $\Omega^2\vec{k}\times(\vec{k}\times x)$ is centripetal acceleration, and $(2\rho\Omega/\phi)\vec{k}\times v$ represents the Coriolis effects.

For the generalized Forchheimer equation in rotating porous media, we similarly have

$$g(\rho|v|)v + \frac{2\rho\Omega}{\phi}\vec{k} \times v + \rho\Omega^2\vec{k} \times (\vec{k} \times x) = -\nabla p + \rho\vec{\mathbf{g}}. \tag{4.12}$$

Unfortunately, we cannot obtain (4.9) from (4.12), hence, cannot obtain a similar PDE to equation (4.10). We approach the problem differently this time.

Set $\mathcal{R}(\rho) = 2\rho\Omega/\phi$. Multiplying both sides of (4.12) by ρ gives

$$g(|\rho v|)\rho v + \mathcal{R}(\rho)\vec{k} \times (\rho v) = -\rho\nabla p + \rho^2\vec{\mathbf{g}} - \rho^2\Omega^2\vec{k} \times (\vec{k} \times x).$$

Then we can solve

$$\rho v = -F^{-1}_{\mathcal{R}(\rho)}(\rho\nabla p - \rho^2\vec{\mathbf{g}} + \rho^2\Omega^2\mathbf{J}^2 x), \tag{4.13}$$

where $\mathbf{J}$ is the 3×3 matrix for which $\mathbf{J}x = \vec{k} \times x$ for all $x \in \mathbb{R}^3$, and $F_z : \mathbb{R}^3 \to \mathbb{R}^3$, for any $z \in \mathbb{R}$, is defined by $F_z(v) = g(|v|)v + z\mathbf{J}v$ for $v \in \mathbb{R}^3$.

In the case of slightly compressible fluids, thanks to the new relation (4.13), we can derive a PDE for $u(x,t) \sim \rho(x,t)$:

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(X\left(u, \nabla u + u^2\mathcal{Z}(x,t)\right)\right), \tag{4.14}$$

where

$$X(z,y) = F^{-1}_{R_* z}(y) \text{ for } z \in \mathbb{R},\ y \in \mathbb{R}^3,\ \text{with } R_* \sim \Omega,$$

$$\mathcal{Z}(x,t) = -\mathcal{G}e_0(t) + \Omega^2\mathbf{J}^2 x \text{ with } \mathcal{G} = const.,\ |e_0(t)| = 1.$$

The obvious new feature and new difficulty in (4.14) is the first dependence of X on u. To analyze (4.14), we need to understand the structure of the function $X(z,y)$. Its essential properties are established in [16, 17] which show the explicit types of degeneracy in both z and y. Denote number $a = \bar{\alpha}_N/(1+\bar{\alpha}_N) \in (0,1)$ and $\chi_0 = g(1) = \sum_{i=0}^N a_i$.

Lemma 4.1 (Celik–Hoang–Kieu [17])**.** *For all $z \in \mathbb{R}_+, y \in \mathbb{R}^3$, one has*

$$\frac{c_1(\chi_0 + R_* z)^{-1}|y|}{(1+|y|)^a} \le |X(z,y)| \le \frac{c_2(\chi_0 + R_* z)^a|y|}{(1+|y|)^a},$$

$$(\chi_0 + R_* z)^{-(1-a)}|y|^{1-a} - 1 \le |X(z,y)| \le c_3|y|^{1-a},$$

$$\frac{c_4(\chi_0 + R_* z)^{-2}|y|^2}{(1+|y|)^a} \le X(z,y) \cdot y \le \frac{c_2(\chi_0 + R_* z)^a|y|^2}{(1+|y|)^a},$$

$$c_5(\chi_0 + R_* z)^{-2}(|y|^{2-a} - 1) \le X(z,y) \cdot y \le c_3|y|^{2-a}.$$

Regarding the partial derivatives of X we have the following.

Lemma 4.2 (Celik–Hoang–Kieu [17])**.** *For all $z \in \mathbb{R}_+$ and $y \in \mathbb{R}^3$, the matrix D_yX of partial derivatives in the variable y satisfies*

$$c_6(\chi_0 + R_* z)^{-1}(1+|y|)^{-a} \le |D_yX(z,y)| \le c_7(1 + \chi_0 + R_* z)^a(1+|y|)^{-a},$$

$$\xi^{\mathrm{T}} D_yX(z,y)\xi \ge c_8(\chi_0 + R_* z)^{-2}(1+|y|)^{-a}|\xi|^2 \text{ for all } \xi \in \mathbb{R}^3.$$

Based on this clear understanding of $X(z,y)$, we can establish a maximum principle for equation (4.14). This is far from obvious and, in fact, is a surprise.

For $T > 0$, denote $U_T = U \times (0,T]$, its closure $\overline{U_T}$ and its parabolic boundary $\partial_p U_T = \overline{U_T} \setminus U_T$.

Theorem 4.3 (Celik–Hoang–Kieu [17])**.** *Assume $u \in C(\overline{U_T}) \cap C^{2,1}_{x,t}(U_T)$, $u \ge 0$ on $\overline{U_T}$ and u satisfies (4.14) in U_T. Then one has*

$$\max_{\overline{U_T}} u = \max_{\partial_p U_T} u.$$

For the initial, boundary value problem, the initial data $u_0(x)$ and the Dirichlet boundary data $\psi(x,t)$ are given.

The above maximum principle allows us to estimate the solutions whenever the initial data is continuous on $\bar\Omega$, hence bounded. For unbounded initial data, we have to use more sophisticated technique, namely, the Moser iteration. However, the PDE (4.14) has extra dependence on u, in addition to $\nabla u + u^2\mathcal{Z}(x,t)$. This dependence turns out to yield new weights, which depend on the solution u itself, in the energy estimates. Therefore, more technical treatments are required. Indeed, we establish suitable weighted Poincaré–Sobolev inequalities to deal with these weights. We are then able to estimate the Lebesgue norms of the solutions, and, by the Moser iteration, their essential supremum. These short-time estimates are combined with the above maximum principle to give all time estimates. Moreover, our estimates provide explicit dependence on physical parameters including the angular speed of the rotation.

Let $\Psi(x,t)$ be an extension of the boundary data $\psi(x,t)$ from $\Gamma\times(0,T]$ to $\bar U\times[0,T]$. Define $\bar u_0(x) = u_0(x) - \Psi(x,0)$. Below, quantity $\mathcal{M}(t)>0$ involves some $L^\beta_{x,t}$-norms over $U\times[0,t]$ for $\Psi, \nabla\Psi, \Psi_t$. The powers ω_i, for $1\le i\le 5$, are positive numbers. Moreover, χ_* is positive number which is used to characterize the angular speed Ω relative to other physical parameters. It is of order of Ω when Ω is large.

Theorem 4.4 (Celik–Hoang–Kieu [17]). *Let $u\in C(\overline{U}\times(0,T_*))\cap C^{2,1}_{x,t}(U\times(0,T_*))\cap C([0,T_*),L^{\beta_1}(U))$ be a nonnegative solution of (4.14) for certain positive numbers β_1 and T_*. Then there exists $0<t_0<\min\{1,T_*\}$ such that the following estimates hold.*

If $t\in(0,t_0]$, then

$$\sup_{x\in U} u(x,t) \le \mathcal{S}(t) \overset{\text{def}}{=} C\chi_*^{\omega_1} t^{-\omega_2}\left(1+\|\bar u_0\|_{L^{\beta_1}}\right)^{\omega_3}\mathcal{M}(t)^{\omega_4}\mathscr{B}_*(t)^{\omega_5} + \sup_{x\in U}|\Psi(x,t)|,$$

where $C>0$, and $\mathscr{B}_*(t) = 1+\operatorname{ess\,sup}_{\tau\in(t/4,t)}\|\Psi(\cdot,\tau)\|_{L^{q_*}}$ with a fixed number $q_*>0$.

If $t\in(t_0,T_*)$ then

$$\sup_{x\in U} u(x,t) \le \max\Big\{\mathcal{S}(t_0), \sup_{(x,\tau)\in\Gamma\times[t_0,t]}\psi(x,\tau)\Big\}.$$

In our paper [16], $\mathcal{R}(\rho)$ is approximated by a constant $\mathcal{R} = 2\rho_*\Omega/\phi$, for some constant density ρ_*. This resulted in a simpler equation than (4.14) where $X = X(y)$ only. In that case, we obtain interior estimates for the L^s-norms of the gradients, for any $s>0$. These are achieved by using the Ladyzhenskaya–Uraltseva iteration technique. Define

$$M_* = \sup_{U\times[0,T]}|u|,\quad \mathcal{E}_* = \int_0^T\int_U(|\nabla\Psi|^2+|\Psi_t|^2+\Psi^2)dxdt,$$
$$\mathcal{N}_0 = \|\bar u_0\|^2_{L^2} + TM_*^2 + \mathcal{E}_*,\quad \mathcal{N}_* = \|\bar u_0\|^2_{L^2}+T+\mathcal{E}_*,\quad \mathcal{N}_2 = (\|\bar u_0\|^2_{L^2}+\|\nabla u_0\|^2_{L^2})+T+\mathcal{E}_*,$$
$$\mathcal{N}_s = (\|\bar u_0\|^2_{L^2}+\|\nabla u_0\|^2_{L^2}+\|\nabla u_0\|^s_{L^s})+T+\mathcal{E}_* \text{ for } s>2.$$

Theorem 4.5 (Celik–Hoang–Kieu [16]). *If $U'\Subset U$, then one has, for all $t\in[0,T]$, that*

$$\int_{U'}|\nabla u(x,t)|^s dx \le \int_U|\nabla u_0(x)|^s dx + C\begin{cases}\chi_*^{4(4+a)}(M_*+1)^6\mathcal{N}_0 & \text{if } s=2,\\ \chi_*^{(s+2)(4+a)}M_*^{s-2}(M_*+1)^{3s+2}\mathcal{N}_2 & \text{if } 2<s\le 4,\\ \chi_*^{(s+4)(4+a)}M_*^2(M_*+1)^{4(s+1)}\mathcal{N}_{s-2} & \text{if } s>4.\end{cases}$$

When $t>0$ stays away from 0, we obtain better estimates.

Theorem 4.6 (Celik–Hoang–Kieu [16]). *Let $U'\Subset U$ and $T_0\in(0,T)$. Then it holds, for $s\ge 2$ and $t\in[T_0,T]$, that*

$$\int_{U'}|\nabla u(x,t)|^s dx \le C\chi_*^{(4+a)(s+a+4)}(1+T_0^{-1})^{s+a+1}(M_*+1)^{4(s+a+2)}\mathcal{N}_*.$$

5. Attractors of abstract dynamical systems

Let Λ and X be complete metric spaces. We will suppose that $S_\lambda(\cdot)$ is a parameterized family of semigroups on X for $\lambda \in \Lambda$ that satisfies the following properties:

(G1) $S_\lambda(\cdot)$ has a global attractor $\mathscr{A}_\lambda$ for every $\lambda \in \Lambda$;
(G2) there is a bounded subset D of X such that $\mathscr{A}_\lambda \subseteq D$ for every $\lambda \in \Lambda$; and
(G3) for $t > 0$, $S_\lambda(t)x$ is continuous in λ, uniformly for x in bounded subsets of X.

Note that condition (G2) can be strengthened and (G3) weakened by replacing *bounded* by *compact*. These modified conditions will be referred to as conditions (G2′) and (G3′).

Theorem 5.1 (Babin–Pilyugin [2], Hoang–Olson–Robinson [50])**.** *Under assumptions* (G1–G3)*— or under the assumptions* (G1), (G2′) *and* (G3′)*—$\mathscr{A}_\lambda$ is continuous in λ at all λ_0 in a residual subset of Λ. In particular the set of continuity points of $\mathscr{A}_\lambda$ is dense in Λ.*

The proof developed in [50] of the above theorem is more direct than previous proofs (e.g. in [2]) and can be modified to establish analogous results for the pullback attractors and uniform attractors of non-autonomous systems [51]. We describe the main results of [51] here.

For bounded sets A, C in a metric space (Y, d_Y), let $\rho_Y(A, C)$ denote the Hausdorff semi-distance

$$\rho_Y(A, C) = \sup_{a \in A} \inf_{c \in C} d_Y(a, c).$$

Definition 5.2. *A family of compact sets $\mathscr{A}(\cdot) = \{\mathscr{A}(t) \colon t \in \mathbb{R}\}$ in X is the* pullback attractor *for the process $S(\cdot, \cdot)$ if*

(A1) *$\mathscr{A}(\cdot)$ is invariant: $S(t,s)\mathscr{A}(s) = \mathscr{A}(t)$ for all $t \geq s$;*
(A2) *$\mathscr{A}(\cdot)$ is pullback attracting: for any bounded set B in X and $t \in \mathbb{R}$, $\rho_X\big(S(t,s)B, \mathscr{A}(t)\big) \to 0$ as $s \to -\infty$;*
(A3) *$\mathscr{A}(\cdot)$ is minimal, in the sense that if $C(\cdot)$ is any other family of compact sets that satisfies (A1) and (A2) then $\mathscr{A}(t) \subseteq C(t)$ for all $t \in \mathbb{R}$.*

The uniform attractor is obtained by taking the limit as $t \to \infty$.

Definition 5.3. *A set $\mathbb{A} \subseteq X$ is the* uniform attractor *if it is the minimal compact set such that*

$$\lim_{t \to \infty} \sup_{s \in \mathbb{R}} \rho_X\big(S(t+s, s)B, \mathbb{A}\big) = 0 \text{ for any bounded } B \subseteq X.$$

Let Λ be a complete metric space and $S_\lambda(\cdot, \cdot)$ a parameterized family of processes on X with $\lambda \in \Lambda$. Suppose that

(L1) $S_\lambda(\cdot, \cdot)$ has a pullback attractor $\mathscr{A}_\lambda(\cdot)$ for every $\lambda \in \Lambda$;
(L2) there is a bounded subset D of X such that $\mathscr{A}_\lambda(t) \subseteq D$ for every $\lambda \in \Lambda$ and every $t \in \mathbb{R}$;
(L3) for every $s \in \mathbb{R}$ and $t \geq s$, $S_\lambda(t,s)x$ is continuous in λ, uniformly for x in bounded subsets of X.

We denote by (L2′) and (L3′) the assumptions (L2) and (L3), respectively, with *bounded* replaced by *compact*.

Theorem 5.4 (Hoang–Olson–Robinson [51])**.** *Let $S_\lambda(\cdot, \cdot)$ be a family of processes on (X, d) each satisfying* (P1–P3) *and suppose that* (L1) *holds along with either*

(i) (L2′) *and* (L3′)*, or*
(ii) (L2), (L3)*, and for any $\lambda_0 \in \Lambda$ and $t \in \mathbb{R}$, there exists $\delta > 0$ such that $\overline{\bigcup_{B_\Lambda(\lambda_0, \delta)} \mathscr{A}_\lambda(t)}$ is compact.*

Then, there exists a residual set Λ_* in Λ such that for every $t \in \mathbb{R}$ the function $\lambda \mapsto \mathscr{A}_\lambda(t)$ is continuous at each $\lambda \in \Lambda_*$.

Theorem 5.5 (Hoang–Olson–Robinson [51])**.** *Suppose that there exists a compact set $K \subseteq X$ such that*

(a) *for every bounded* $B \subseteq X$ *and each* $\lambda \in \Lambda$ *there exists a* $t_{B,\lambda}$ *such that* $S_\lambda(t+s,s)B \subseteq K$ *for all* $t \geq t_{B,\lambda}$ *and* $s \in \mathbb{R}$*; and*
(b) *for any* $t > 0$ *the mapping* $S_\lambda(t+s,s)x$ *is continuous in* $\lambda \in \Lambda$ *uniformly for* $s \in \mathbb{R}$ *and* $x \in K$*.*

Then the uniform attractor $\mathbb{A}_\lambda$ *is continuous in* λ *at a residual subset of* Λ*.*

The above abstract results can be applied to ODEs (such as the Lorenz system) and PDEs. We present the latter case below.

Let Ω be a bounded, open and connected set in $\mathbb{R}^2$ with C^2 boundary. Consider the two-dimensional incompressible Navier–Stokes equations (2.1) in Ω with no-slip Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.

Denote by $CB(Y)$ the collection of all non-empty closed, bounded subsets of a metric space Y, which is itself a metric space with metric given by the symmetric Hausdorff distance.

Theorem 5.6 (Hoang–Olson–Robinson [51])**.** *There is a residual and dense subset* Λ_* *in* $L^\infty(\mathbb{R}, H)$ *such that the maps from* $L^\infty(\mathbb{R}, H) \to CB(H)$ *given by*

$$f \mapsto \mathscr{A}_f(t) \quad \text{for every} \quad t \in \mathbb{R} \qquad \text{and} \qquad f \mapsto \mathbb{A}_f$$

are continuous at every point $f \in \Lambda_*$*.*

Corollary 5.7 (Hoang–Olson–Robinson [51])**.** *There is a residual and dense set* Λ_* *in* H *such that the map from* $H \to CB(H)$ *given by* $f \mapsto \mathscr{A}_f$ *is continuous at every point* $f \in \Lambda_*$*.*

We now fix the forcing $f = f_0$ where $f_0 \in L^\infty(\mathbb{R}, H)$ and consider the family of attractors parameterized by viscosity ν.

Corollary 5.8 (Hoang–Olson–Robinson [51])**.** *There is a residual and dense set* Λ_* *in* $(0,\infty)$ *such that the maps from* $(0,\infty) \to CB(H)$ *given by*

$$\nu \mapsto \mathscr{A}_\nu(t) \quad \text{for every} \quad t \in \mathbb{R} \qquad \text{and} \qquad \nu \mapsto \mathbb{A}_\nu$$

are continuous at every point $\nu \in \Lambda_*$*.*

6. Asymptotic expansions for nonlinear ordinary differential equations

We present the obtained results for the non-autonomous systems with complicated forcing functions, and for the autonomous systems with the nonlinear terms lacking the smoothness.

6.1. Non-autonomous systems with coherently decaying forcing functions.

Consider the following system of nonlinear ODEs in $\mathbb{C}^n$:

$$y' = -Ay + G(y) + f(t), \tag{6.1}$$

where A is an $n \times n$ constant matrix of complex numbers, G is a vector field on $\mathbb{C}^n$, and f is a function from $(0,\infty)$ to $\mathbb{C}^n$.

Assumption 6.1. *All eigenvalues of the matrix* A *have positive real parts.*

Assumption 6.2. *Function* $G : \mathbb{C}^n \to \mathbb{C}^n$ *has the the following properties.*

(i) G *is locally Lipschitz.*
(ii) *There exist functions* $G_m : \mathbb{C}^n \to \mathbb{C}^n$*, for* $m \geq 2$*, each is a homogeneous polynomial of degree* m*, such that, for any* $N \geq 2$*, there exists* $\delta > 0$ *so that*

$$\left| G(x) - \sum_{m=2}^{N} G_m(x) \right| = \mathcal{O}(|x|^{N+\delta}) \ \text{as } x \to 0.$$

Define the iterated logarithmic functions as follows:

$$L_{-1}(t) = e^t, \quad L_0(t) = t \text{ for } t \in \mathbb{R}, \text{ and } L_{m+1}(t) = \ln(L_m(t)) \text{ for } m \in \mathbb{Z}_+.$$

For $k \in \mathbb{Z}_+$, define $\widehat{\mathcal{L}}_k = (L_{-1}, L_0, L_1, \ldots, L_k)$. Explicitly, $\widehat{\mathcal{L}}_k(t) = (e^t, t, \ln t, \ln\ln t, \ldots, L_k(t))$.

Define $z^\alpha = \prod_{j=-1}^k z_j^{\alpha_j}$ for

$$z = (z_{-1}, z_0, z_1, \ldots, z_k) \in (0,\infty)^{k+2} \text{ and } \alpha = (\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{C}^{k+2}, \tag{6.2}$$

For $\mu \in \mathbb{R}$, $m, k \in \mathbb{Z}$ with $k \ge m \ge -1$, denote by $\mathcal{E}(m,k,\mu)$ the set of vectors α in (6.2) such that $\mathbf{Re}(\alpha_j) = 0$ for $-1 \le j < m$ and $\mathbf{Re}(\alpha_m) = \mu$.

Definition 6.3. *Let $\mathbb{K}$ be $\mathbb{C}$ or $\mathbb{R}$, and X be a linear space over $\mathbb{K}$.*

(i) *For $k \ge -1$, define $\mathscr{P}(k,X)$ to be the set of functions of the form*

$$p(z) = \sum_{\alpha\in S} z^\alpha \xi_\alpha \text{ for } z \in (0,\infty)^{k+2}, \tag{6.3}$$

where S is some finite subset of $\mathbb{K}^{k+2}$, and each ξ_α belongs to X.

(ii) *Let $\mathbb{K} = \mathbb{C}$, $k \ge m \ge -1$ and $\mu \in \mathbb{R}$. Define $\mathscr{P}_m(k,\mu,X)$ to be set of functions of the form (6.3), where S is a finite subset of $\mathcal{E}(m,k,\mu)$.*

Define $\mathscr{F}_m(k,\mu,X) = \left\{ p \circ \widehat{\mathcal{L}}_k : p \in \mathscr{P}_m(k,\mu,X) \right\}$.

The types of asymptotic expansions we study are the following.

Definition 6.4. *Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $(X, \|\cdot\|_X)$ be a normed space over $\mathbb{K}$. Suppose g is a function from (T,∞) to X for some $T \in \mathbb{R}$, and $m_* \in \mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$.*

Let $(\gamma_k)_{k=1}^\infty$ be a divergent, strictly increasing sequence of positive numbers, and $(n_k)_{k=1}^\infty$ be a sequence in $\mathbb{N} \cap [m_*, \infty)$. We say

$$g(t) \sim \sum_{k=1}^\infty g_k(t), \text{ where } g_k \in \mathscr{F}_{m_*}(n_k, -\gamma_k, X) \text{ for } k \in \mathbb{N}, \tag{6.4}$$

if, for each $N \in \mathbb{N}$, there is some $\mu > \gamma_N$ such that $\left\| g(t) - \sum_{k=1}^N g_k(t) \right\|_X = \mathcal{O}(L_{m_*}(t)^{-\mu})$.

Because of the complex power functions, the asymptotic expansion (6.4) may contain, in addition to $\cos(\omega t)$ and $\sin(\omega t)$, the terms $\cos(\omega(L_m t))$ and $\sin(\omega L_m(t))$. Therefore, (6.4) is a new kind of asymptotic expansion, not seen before in [7, 32, 49, 52, 53, 59].

Definition 6.5. *Given an integer $k \ge -1$, let $p \in \mathscr{P}(k, \mathbb{C}^n)$ be given by (6.3) with z and α as in (6.2), and $\mathbb{K} = \mathbb{C}$, $X = \mathbb{C}^n$.*

Define, for $j = -1, 0, \ldots, k$, the function $\mathcal{M}_j p : (0,\infty)^{k+2} \to \mathbb{C}^n$ by

$$(\mathcal{M}_j p)(z) = \sum_{\alpha\in S} \alpha_j z^\alpha \xi_\alpha.$$

In the case $k \ge 0$, define the function $\mathcal{R}p : (0,\infty)^{k+2} \to \mathbb{C}^n$ by

$$(\mathcal{R}p)(z) = \sum_{j=0}^k z_0^{-1} z_1^{-1} \ldots z_j^{-1} (\mathcal{M}_j p)(z).$$

In the case $p \in \mathscr{P}_{-1}(k, 0, \mathbb{C}^n)$, define the function $\mathcal{Z}_A p : (0,\infty)^{k+2} \to \mathbb{C}^n$ by

$$(\mathcal{Z}_A p)(z) = \sum_{\alpha\in S} z^\alpha (A + \alpha_{-1} I_n)^{-1} \xi_\alpha.$$

In the following, we assume that there exists a number $T_0 \ge 0$ such that $y \in C^1((T_0,\infty))$ is a solution of (6.1) on (T_0,∞), and $y(t) \to 0$ as $t \to \infty$.

6.1.1. *Case of power-decay.*

Assumption 6.6. *The function $f(t)$ admits the asymptotic expansion, in the sense of Definition 6.4 with $X=\mathbb{C}^n$ and $m_*=0$,*

$$f(t) \sim \sum_{k=1}^{\infty} f_k(t), \text{ where } f_k \in \mathscr{F}_0(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}, \tag{6.5}$$

with $(\mu_k)_{k=1}^{\infty}$ being a divergent, strictly increasing sequence of positive numbers, and $(n_k)_{k=1}^{\infty}$ being an increasing sequence in $\mathbb{Z}_+$. Moreover, the set $\mathcal{S} \stackrel{\text{def}}{=} \{\mu_k : k \in \mathbb{N}\}$ preserves the addition and the unit increment.

Theorem 6.7 (Hoang [38]). *Under Assumption 6.6, there exist functions*

$$y_k \in \mathscr{F}_0(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}, \tag{6.6}$$

such that the solution $y(t)$ admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^{\infty} y_k(t) \text{ in the sense of Definition 6.4 with } m_*=0. \tag{6.7}$$

More specifically, assume, for all $k \in \mathbb{N}$,

$$f_k(t) = p_k(\widehat{\mathcal{L}}_{n_k}(t)) \text{ for some } p_k \in \mathscr{P}_0(n_k, -\mu_k, \mathbb{C}^n). \tag{6.8}$$

Then the functions y_k's in (6.7) can be constructed recursively as follows. For each $k \in \mathbb{N}$, $y_k(t) = q_k(\widehat{\mathcal{L}}_{n_k}(t))$, where

$$q_k = \mathcal{Z}_A\left(\sum_{m\ge 2}\ \sum_{\mu_{j_1}+\mu_{j_2}+\ldots\mu_{j_m}=\mu_k} \mathcal{G}_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k - \chi_k\right), \tag{6.9}$$

with

$$\chi_k = \begin{cases} \mathcal{R} q_\lambda & \text{if there exists } \lambda \le k-1 \text{ such that } \mu_\lambda + 1 = \mu_k, \\ 0 & \text{otherwise.} \end{cases}$$

6.1.2. *Case of logarithmic or iterated logarithmic decay.*

Assumption 6.8. *There exist a number $m_* \in \mathbb{N}$, a divergent, strictly increasing sequence $(\mu_k)_{k=1}^{\infty} \subset (0,\infty)$, and an increasing sequence $(n_k)_{k=1}^{\infty} \subset \mathbb{N} \cap [m_*, \infty)$ such that the function $f(t)$ admits the asymptotic expansion, in the sense of Definition 6.4 with $X=\mathbb{C}^n$,*

$$f(t) \sim \sum_{k=1}^{\infty} f_k(t), \text{ where } f_k \in \mathscr{F}_{m_*}(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}. \tag{6.10}$$

Moreover, the set $\mathcal{S} \stackrel{\text{def}}{=} \{\mu_k : k \in \mathbb{N}\}$ preserves the addition.

Theorem 6.9 (Hoang [38]). *Under Assumption 6.8, there exist functions*

$$y_k \in \mathscr{F}_{m_*}(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N}, \tag{6.11}$$

such that the solution $y(t)$ admits the asymptotic expansion $y(t) \sim \sum_{k=1}^{\infty} y_k(t)$ in the sense of Definition 6.4. More specifically, suppose $f_k(t) = p_k(\widehat{\mathcal{L}}_{n_k}(t))$ with $p_k \in \mathscr{P}_{m_*}(n_k, -\mu_k, \mathbb{C}^n)$ for all $k \in \mathbb{N}$. Then $y_k(t) = q_k(\widehat{\mathcal{L}}_{n_k}(t))$, where q_k is given by (6.9) with $\chi_k = 0$.

6.1.3. *Systems in the real linear spaces.* The results in sections 6.1.1 and 6.1.2 are for $\mathbb{C}^n$, but are still valid for systems of ODEs in $\mathbb{R}^n$ – when we have most applications.

Assumption 6.10. *Assume the following.*

(i) *The matrix A is an $n \times n$ matrix of real numbers that satisfies Assumption 6.1.*
(ii) *The function G is from $\mathbb{R}^n$ to $\mathbb{R}^n$, satisfies Assumption 6.2 with G_m's being functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.*
(iii) *The multi-linear mappings $\mathcal{G}_m$'s in ((ii)) are from $(\mathbb{R}^n)^m$ to $\mathbb{R}^n$.*
(iv) *The forcing function $f(t)$ and solution $y(t)$ are $\mathbb{R}^n$-valued.*

Instead of stating more technical results, we present some examples.

Definition 6.11. *Given integers $k \ge m \ge 0$. Define the class $\mathcal{P}^1_m(k,\mathbb{R}^n)$ to be the collection of functions which are the finite sums of the following functions*

$$z = (z_{-1}, z_0, \dots, z_k) \in (0,\infty)^{k+2} \mapsto z^\alpha \prod_{j=0}^{k} \sigma_j(\omega_j z_j)\xi,$$

where $\xi \in \mathbb{R}^n$, $\alpha \in \mathcal{E}(m,k,0) \cap \mathbb{R}^{k+2}$, ω_j's are real numbers, and, for each j, either $\sigma_j = \cos$ or $\sigma_j = \sin$.

Example 6.12 (Hoang [38])**.** If

$$f(t) = \frac{\cos(\alpha t)(\ln t)(\ln\ln t)^{-1/3}}{t^m}\xi \text{ for some } m \in \mathbb{N} \text{ and } \xi \in \mathbb{R}^n,$$

then any decaying solution $y(t)$ of (6.1) admits the asymptotic expansion

$$y(t) \sim \sum_{k=0}^{\infty} \frac{q_k(t)}{t^{m+k}},$$

where $q_k(t) = \widehat{q}_k(\widehat{\mathcal{L}}_2(t))$ with $\widehat{q}_k \in \mathcal{P}^1_0(2,\mathbb{R}^n)$. Roughly speaking, the functions $q_k(t)$'s are composed by

$$\cos(\omega L_j(t)),\ \sin(\omega L_j(t)),\ L_\ell(t)^\alpha, \tag{6.12}$$

for $j = 0,1,2$ and $\ell = 1,2$, with some real numbers ω's and α's.

Example 6.13 (Hoang [38])**.** If

$$f(t) = \frac{\cos(2t)\sin(3\ln\ln t)(\ln\ln\ln t)^2\sin(5\ln\ln\ln t)}{(\ln t)^{1/2}}\xi \text{ for some } \xi \in \mathbb{R}^n,$$

then any decaying solution $y(t)$ of (6.1) admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^{\infty} \frac{q_k(t)}{(\ln t)^{k/2}},$$

where, roughly speaking, $q_k(t)$'s are functions composed by the functions in (6.12) for $j = 0,1,2,3$ and $\ell = 2,3$.

6.2. **Autonomous systems with non-smooth nonlinearity.** We study the following ODE systems in $\mathbb{R}^d$ of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Ay = F(y), \quad t > 0, \tag{6.13}$$

where A is a $d \times d$ constant (real) matrix, and F is a vector field on $\mathbb{R}^d$.

The goal is to obtain the asymptotic expansions for the solutions of (6.13) even when $F(x)$ may not be as good as the function $G(x)$ in (6.1).

Definition 6.14. *Suppose* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two (real) normed spaces.*

A function $F: X \to Y$ *is positively homogeneous of degree* $\beta \geq 0$ *if*

$$F(tx) = t^\beta F(x) \text{ for any } x \in X \text{ and any } t > 0.$$

Define $\mathcal{H}_\beta(X, Y)$ *to be the set of positively homogeneous functions of order* β *from* X *to* Y*, and denote* $\mathcal{H}_\beta(X) = \mathcal{H}_\beta(X, X)$.

Assumption 6.15. *The mapping* $F : \mathbb{R}^d \to \mathbb{R}^d$ *has the the following properties.*

(i) F *is locally Lipschitz on* $\mathbb{R}^d$ *and* $F(0) = 0$.

(ii) *There exist numbers* β_k*'s, for* $k \in \mathbb{N}$*, which belong to* $(1, \infty)$ *and increase strictly to infinity, and functions* $F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0)$*, for* $k \in \mathbb{N}$*, such that it holds, for any* $N \in \mathbb{N}$*, that*

$$\left| F(x) - \sum_{k=1}^N F_k(x) \right| = \mathcal{O}(|x|^\beta) \text{ as } x \to 0, \text{ for some } \beta > \beta_N.$$

We conveniently write Assumption 6.2(ii) as $F(x) \sim \sum_{k=1}^\infty F_k(x)$. Note that F_k may not have the Taylor expansion about the origin. Hence, the results in [8, 38, 53, 59] for the system (6.1) when $f = 0$ do not apply. We need a different approach. We start with the following first asymptotic approximation for solution $y(t)$.

Theorem 6.16 (Cao–Hoang–Kieu [9])**.** *Let* $y(t)$ *be a non-trivial, decaying solution of (6.13). Then there exist an eigenvalue* λ_* *of* A *and a corresponding eigenvector* ξ_* *such that*

$$|y(t) - e^{-\lambda_* t}\xi_*| = \mathcal{O}(e^{-(\lambda_*+\delta)t}) \text{ for some } \delta > 0. \tag{6.14}$$

Let eigenvalue $\lambda_* = \lambda_{n_0}$ and its corresponding eigenvector ξ_* be as in Theorem 6.16. For obtaining further expansion of $y(t)$, the idea is to rescale $y(t)$ by $e^{\lambda_* t}$ and shift the Taylor expansion of $F_k(x)$ to the new center $\xi_* \neq 0$.

Let $r \in \mathbb{N}$ and $s \in \mathbb{Z}_+$. Since F_r is a C^∞-function in a neighborhood of $\xi_* \neq 0$, we have the following Taylor's expansion, for any $h \in \mathbb{R}^d$,

$$F_r(\xi_* + h) = \sum_{m=0}^s \mathcal{F}_{r,m} h^{(m)} + g_{r,s}(h), \quad \mathcal{F}_{r,m} = \frac{1}{m!} D^m F_r(\xi_*), \tag{6.15}$$

where $D^m F_r(\xi_*)$ is the m-th order derivative of F_r at ξ_*, and $g_{r,s}(h) = \mathcal{O}(|h|^{s+1})$ as $h \to 0$.

When $m = 0$, $\mathcal{F}_{r,0} = F_r(\xi_*)$. When $m \geq 1$, $\mathcal{F}_{r,m}$ is an m-linear mapping from $(\mathbb{R}^d)^m$ to $\mathbb{R}^d$.

The exponential rates μ_n's in a possible asymptotic expansion of solution $y(t)$ are found in the following way. Let $\alpha_k = \beta_k - 1 > 0$ for $k \in \mathbb{N}$. Define the set $\tilde{S} \subset [0, \infty)$ by

$$\tilde{S} = \Big\{ \sum_{k=n_0}^{d_*} m_k(\lambda_k - \lambda_*) + \sum_{j=1}^\infty z_j \alpha_j \lambda_* : m_k, z_j \in \mathbb{Z}_+, \text{ with } z_j > 0 \text{ for only finitely many } j\text{'s} \Big\}.$$

The set $\tilde{S}$ has countably, infinitely many elements. Arrange $\tilde{S}$ as a sequence $(\tilde{\mu}_n)_{n=1}^\infty$ of non-negative and strictly increasing numbers. Set $\mu_n = \tilde{\mu}_n + \lambda_*$ for $n \in \mathbb{N}$.

Theorem 6.17 (Cao–Hoang–Kieu [9])**.** *Let* $y(t)$ *be a non-trivial, decaying solution of (6.13). There exist polynomials* q_n*:* $\mathbb{R} \to \mathbb{R}^d$ *such that* $y(t)$ *has an asymptotic expansion*

$$y(t) \sim \sum_{n=1}^\infty q_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^d, \tag{6.16}$$

where $q_n(t)$ *satisfies, for any* $n \geq 1$*,*

$$q_n' + (A - \mu_n I_d) q_n = \sum_{\substack{r \geq 1, m \geq 0, k_1, k_2, \ldots, k_m \geq 2, \\ \sum_{j=1}^m \tilde{\mu}_{k_j} + \alpha_r \lambda_* = \tilde{\mu}_n}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m}) \text{ in } \mathbb{R}.$$

The above general theorem can be extended and applied to many different situations. We provide some examples below.

For $p \in [1,\infty)$ and $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, the ℓ^p-norm of x is $\|x\|_p = \Big(\sum_{j=1}^d |x_j|^p\Big)^{1/p}$.

Example 6.18. (a) Given a constant $d\times d$ matrix M_0, even numbers $p_1, p_2 \ge 2$, and real numbers $\alpha, \beta > 0$, let

$$F(x) = \frac{\|x\|_{p_1}^\alpha M_0 x}{1 + \|x\|_{p_2}^\beta} \text{ for } x \in \mathbb{R}^d.$$

(b) For $1 \le k \le N_*$, let M_k be a constant $d\times d$ matrix, and $p_k \ge 2$ be an even number, and $\alpha_k > 0$. Let $F(x) = \sum_{k=1}^{N_*} \|x\|_{p_k}^{\alpha_k} M_k x$.

Example 6.19. Assume $\alpha > 0$ is not an even integer. Consider the following system in $\mathbb{R}^2$:

$$y_1' + y_1 = -|y_2|^\alpha y_1, \quad y_2' + y_1 + 2y_2 = -y_1^2 y_2.$$

Example 6.20. There are many other situations, especially in multi-dimensional spaces higher than $\mathbb{R}^2$. We present one example here. Let $d = 3$, and assume 3×3 matrix A has the following eigenvalues and bases of the corresponding eigenspaces

$$\lambda_1 = \lambda_2 = 1, \text{ basis } \{\xi_1 = (1,0,1), \xi_2 = (0,1,0)\}, \text{ and } \lambda_3 = 2, \text{ basis } \{\xi_3 = (1,1,-1)\}.$$

Let $F(x) = (x_1^2 + x_2^2)^{1/3} \cdot (x_2^6 + x_3^6)^{1/5} P(x)$, where P is a polynomial vector field on $\mathbb{R}^3$ of degree $m_0 \in \mathbb{N}$ with $P(0) = 0$.

For the previous three examples, any non-trivial, decaying solution $y(t)$ of (6.13) has an infinite series asymptotic expansion (6.16).

7. Future research

Each of the topics (i), (ii) and (iii) in the Overview section can be developed much more in depth and breadth. I list here only a few ideas.

A. Related to the Navier–Stokes equations and viscous, incompressible fluids.

- Continue expanding the theory of asymptotic expansions and investigate their convergence.
- Study the statistical solutions in the case of decaying forces and apply to the turbulence theory.
- Study asymptotic expansions when the forces have noise. Develop the stochastic theory for the asymptotic expansions.
- Study the asymptotic expansions for the Lagrangian trajectories. Is there an associated normalization map in some sense, or a normal form theory? Establish the expansions for the case of non-potential forces.

B. Related to the Forchheimer flows in porous media.

- *Forchheimer flows in heterogeneous porous media.* This subject is barely touched in mathematical research. It is due to the complex and sophisticated mathematics involved. We may need to utilize, modify or create new tools in analysis, theory of singular/degenerate partial differential equations.
- *Forchheimer flows with other effects.* Some examples are Klinkenberg's effect for gaseous flows and the dependence of the permeability on the pressure. The last one is known, for instance, for sea ice with the Darcy model, but the mathematical analysis is currently not available for the Forchheimer flows.
- *Multi-phase Forchheimer flows.* This is a challenging subject which my collaborators and I only managed to publish two papers [45, 46]. They deal with special steady states and the linearized problems. I plan to focus on the analysis of the non-stationary solutions.

- *Applications in agriculture.* I am interested in fluid dynamics in rhizosphere. My assessment is that our techniques are mature enough to analyze some nonlinear models without sacrificing the mathematical rigor.

C. Related to dynamical systems.

- For a family of parametric dynamical system, find the criteria for the existence of limiting dynamics. This may provide another view to the vanishing viscosity problem.
- Find applications, particularly in mathematical biology, for our results in [8, 9, 38].
- Develop the ODE theory to many more types of nonlinear systems and time-dependent forces. What are their counterparts for PDE?

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