Texas Tech University. Pure Mathematics Colloquium.
Current Advances in Mathematics.

Topologically nontrivial counterexamples to Sard's theorem

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ABSTRACT. In 1979 Kaufman constructed a surprising counterexample to the classical Sard theorem: there is a surjective map $f : [0,1]^{n+1} \to [0,1]^n$ of class C^1 such that $\operatorname{rank} df \leq 1$ everywhere. A natural question is whether there is a topologically non-trivial version of this example: a mapping $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ such that $\operatorname{rank} df < n$ everywhere and f is not homotopic to a constant map. Clearly such a map has to be surjective and it cannot be of class C^2 (because of Sard's theorem). I will discuss the following result: If $n = 2, 3$ and $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ is not homotopic to a constant map, then there is an open set $\Omega \subset \mathbb{S}^{n+1}$ such that $\operatorname{rank} df = n$ on Ω, while for any $n \geq 4$, there is a map $f \in C^1(\mathbb{S}^{n+1}, \mathbb{S}^n)$ that is not homotopic to a constant map and such that $\operatorname{rank} df < n$ everywhere. The result in the case $n \geq 4$ answers a question of Larry Guth. I will also discuss an application of the result to a solution of a recent conjecture of Jacek Galeski. In particular I will show that there is a C^1 mappings in $\mathbb{R}^5$ with the derivative of rank at most 3 that cannot be uniformly approximated by C^2 mappings with the derivative of rank at most 3. The methods use analysis, algebraic topology and geometric measure theory. The talk will be accessible to graduate students. The presentation is based on my two joint papers. One with P. Goldstein and one with P. Goldstein and P. Pankka.