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- Motivation
- Context
- → Theorem
- Techniques

Joint work with Oana Veliche, Northeastern U.

— x —

k - a field

$Q = k[x_1, \dots, x_e]$

$f_1, \dots, f_c \in Q \qquad I = (f_1, \dots, f_c)$

$V = \{ p \in k^e \mid f_1(p) = \dots = f_c(p) = 0 \}$

$Q/I = R$

$R_{(x_1, \dots, x_e)} = \{ f/s,\ f \in R,\ s \in R \setminus (x_1, \dots, x_e) \}$

no

regular = nonsingular $\qquad Q = \mathbb{C}[x,y,z]_{(x,y,z)}$

$\cap$

hypersurface $\qquad Q/(xy-z^2)$

$\cap$ $\qquad Q/(x^2, y^3, z^5)$

complete intersect.

$\cap$ $\qquad Q/(x^2, xz, xy-z^2, yz, y^2)$

Gorenstein

$\cap$

Cohen-Macaulay $\qquad \boxed{Q/(x^2, xy, xz, y^2, yz, z^2)}$

Golod

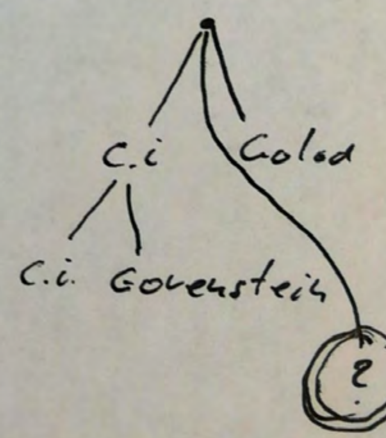
What does a generic local ring look like?!!

— x —

$Q = k[x_1, \dots, x_e]$

$e = 1$: hypersurface = Golod

$e = 2$: Eithe complete intersect or Golod



$e = 3$:

$$Q = k[x,y,z]_{(x,y,z)}$$

$$I \subseteq (x,y,z)^2$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & Q^a \arrow[rr] \arrow[dr] & & Q^b \arrow[rr] \arrow[dr] & & Q^c \arrow[rr] \arrow[dr] & & Q \arrow[r] & Q/I \arrow[r] & 0 \\
& & \Omega^2 \arrow[ur] & & \Omega^1 \arrow[ur] & & I \arrow[ur] \arrow[d] & & & \\
& & & & & & 0 & & &
\end{tikzcd}$$

<u>Exa</u> $I = (x^2, y^3, z^5)$

$$\partial_3 = \begin{bmatrix} z^5 \\ x^2 \\ y^3 \end{bmatrix} \qquad \partial_2 = \begin{bmatrix} -y^3 & 0 & z^5 \\ x^2 & -z^5 & 0 \\ 0 & y^3 & -x^2 \end{bmatrix} \qquad \partial_1 = \begin{bmatrix} x^2 & y^3 & z^5 \end{bmatrix}$$

$$F_\cdot \quad 0 \to \underset{3}{\overset{g}{Q}} \xrightarrow{\partial_3} \underset{2}{\overset{f_1, f_2, f_3}{Q^3}} \xrightarrow{\partial_2} \underset{1}{\overset{e_1, e_2, e_3}{Q^3}} \xrightarrow{\partial_1} \underset{0}{Q} \to Q/I \to 0$$

$F_\cdot$ has diff. graded comm. alg. structure.

$$\partial(ab) = \partial(a)b + (-1)^{|a|} a\,\partial(b)$$

$$ab = (-1)^{|a||b|} ba$$

$$a^2 = 0 \quad |a| \text{ odd}$$

– x –

$$\partial(e_1 \cdot e_2) = \partial(e_1)e_2 - e_1\partial(e_2)$$
$$= x^2 e_2 - e_1 y^3 = -y^3 e_1 + x^2 e_2$$
$$= \partial(f_1)$$

	e_1	e_2	e_3
e_1	0	f_1	$-f_3$
e_2	$-f_1$	0	f_2
e_3	f_3	$-f_2$	0

$$\partial(e_1 \cdot e_3) = x^2 e_3 - z^5 e_1$$

$$\partial(e_1 f_1) = x^2 f_1 - e_1(x^2 e_2 - y^3 e_1)$$
$$= x^2 f_1 - x^2 f_1 = 0$$

	f_1	f_2	f_3
e_1	0	g	0
e_2	0	0	g
e_3	g	0	0

- An alg. structure exists
- It is not unique, but the induced structure

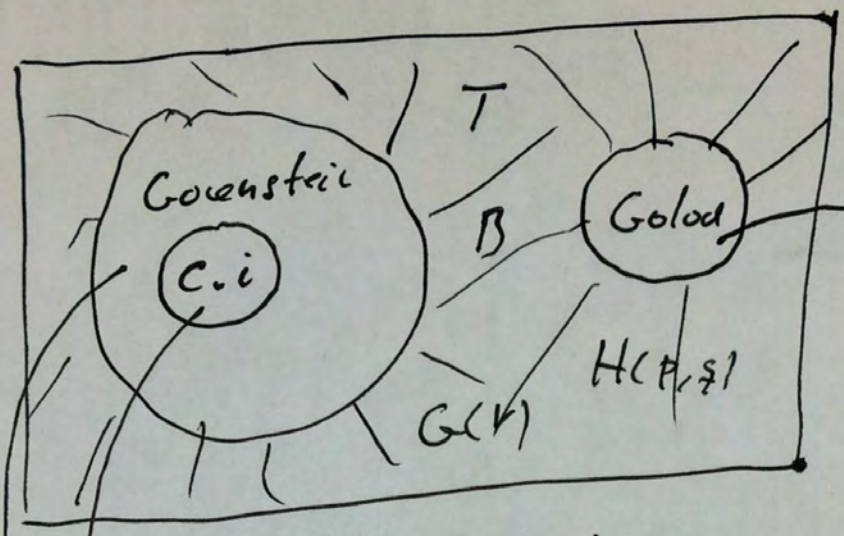
$$A_* = H_*(k \otimes F) = \mathrm{Tor}_*^{Q}(k, Q/I)$$

is unique $\Longrightarrow$ Classification

$Q = k[x,y,z]_{(x,y,z)}$

$I \subseteq (x,y,z)^2$

Q/I is C-M



A : $k \ltimes V$

Trivial

A : Exterior algebra

A : Poincaré Duality Algebra

$A_3 = \langle g \rangle$

$A_2 = \langle f_1, \dots, f_n \rangle$

$A_1 = \langle e_1, \dots, e_n \rangle$

$e_i f_i = g$ all other products 0

<u>Def^n</u> Q/I is of class $G(r)$ if $A = P \ltimes V$, ~~th~~ a Poincaré Duality Alg. P of total rank $2r+2$ trivially extended by V.

Notice : $G(0)$ means Golod

$Q = k[x,y,z]_{(x,y,z)}$

$(x,y,z)^{s+1} \subseteq I \subseteq (x,y,z)^2$ $\quad$ I homogeneous

s : socle degree

$\quad \operatorname{Soc} R = \operatorname{Ann}_R(x,y,z)$

<u>Ex</u> $\quad I = (x^2, xz, xy - z^2, yz, y^2)$

$R = Q/I$ is graded

$R_0 = 1$ $\qquad h_R(0) = 1$

$R_1 = x \; y \; z$ $\qquad h_R(1) = 3$

$R_2 = z^2$ $\qquad h_R(2) = 1$

<u>Def^n</u> $\quad h_{Q/I}(i) = \dim_k (Q/I)_i$

$\dim_k \operatorname{Soc} R = 1 \iff R$ Gorenstein

Thm A generic ring $R = Q/I$ with $\dim_k \mathrm{Soc}\, Q/I = 2$ is of class $G(r)$.

- If $\mathrm{Soc}\, Q/I$ is concentrated in deg s then $r = 0$ (i.e. Q/I Golod)
- If $\mathrm{Soc}\, Q/I$ sits in degs $s_1 << s$, then $0 < r$ grows as a function of $s - s_1$

———— X ————

Defn Q/I is compressed if $\sum_{i=0}^{s} h_R(i)$ is as large as poss.

Thm (not me). Generic rings are compressed.

$\dim_k \mathrm{Soc}\, Q/I = 2$

$$I = I_1 \cap I_2$$

Q/I_1 and Q/I_2 are Gorenstein

$$0 \to Q/I \to Q/I_1 \oplus Q/I_2 \to Q/I_1 + I_2 \to 0$$

$$h_{Q/I}(i) + h_{Q/I_1+I_2} = h_{Q/I_1}(i) + h_{Q/I_2}(i)$$

Hilbert coefficients	Format	Generic Class	Other compressed classes	s	s1
1 3 6 10 15 21 28 29 21 15 10 6 3 1	1 10 11 2	G(7)/10	**G(8)/11 G(9)/12 G(10)/13**	13	**7**
1 3 6 10 15 21 28 31 22 15 10 6 3 1	1 13 14 2	G(5)/13	**G(6)/14**	13	**8**
1 3 6 10 15 21 28 34 24 16 10 6 3 1	1 17 18 2	G(2)/17	-	13	**9**
1 3 6 10 15 21 28 36 27 18 11 6 3 1	1 18 19 2	H(0,0)/18	-	13	**10**
1 3 6 10 15 21 28 36 31 21 13 7 3 1	1 14 15 2	H(0,0)/14	**H(0,0)/15**	13	**11**
1 3 6 10 15 21 28 36 36 25 16 9 4 1	1 12 13 2	H(0,0)/12	**H(0,0)/13 H(0,0)/14**	13	**12**
1 3 6 10 15 21 28 36 42 30 20 12 6 2	1 19 20 2	H(0,0)/19	-	13	**13**
1 3 6 10 15 21 28 36 29 21 15 10 6 3 1	1 16 17 2	G(13)/16	-	14	**8**
1 3 6 10 15 21 28 36 31 22 15 10 6 3 1	1 14 15 2	G(9)/14	**G(8)/14 G(6)/14 G(10)/15 G(7)/15 G(9)/15**	14	**9**
1 3 6 10 15 21 28 36 34 24 16 10 6 3 1	1 11 12 2	G(2)/11	**G(3)/12 G(4)/13 H(0,1)/11 G(2)/12 G(3)/13**	14	**10**
1 3 6 10 15 21 28 36 38 27 18 11 6 3 1	1 14 15 2	H(0,0)/14	**H(0,1)/15 H(0,0)/15**	14	**11**
1 3 6 10 15 21 28 36 43 31 21 13 9 4 1	1 20 21 2	H(0,0)/20	-	14	**12**
1 3 6 10 15 21 28 36 45 36 25 16 9 4 1	1 19 20 2	H(0,0)/19	-	14	**13**
1 3 6 10 15 21 28 36 45 42 30 20 12 6 2	1 13 14 2	H(0,0)/13	**H(0,0)/14 H(0,0)/15**	14	**14**

$$0 \to I_1^1 + I_2^2 / I_1 \xrightarrow[\text{module map}]{} Q/I \xrightarrow[\text{ring map}]{} Q/I_2 \to 0$$