Infinite series asymptotic expansions for dissipative differential equations with non-smooth nonlinearity

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Outline

1. Introduction
2. Main result
3. Sketch of Proof
4. Extended results and examples

1. Introduction

The Navier–Stokes equations

Viscous incrompressible fluids:

$$\begin{cases} u_t - \nu \Delta u + (u \cdot \nabla) u = -\nabla p + f, \\ \mathrm{div}\, u = 0. \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, and the unknowns are the velocity $u(x, t)$ and pressure $p(x, t)$.
Initial condition $u(x, 0) = u_0(x)$, where u_0 is a given initial vector field.

- Questions: Existence, uniqueness, dynamics.
- Here: long-time dynamics.

Foias–Saut asymptotic expansions

Functional form of NSE (after scaling to have $\nu = 1$):

$$u_t + Au + B(u,u) = f, \quad u(0) = u_0.$$

- If $f = const. \neq 0$, turbulence.
- If $f = 0$ or $f = f(t) \to 0$ as $t \to \infty$, turbulence for short time, then the flows settle (to zero) eventually.
- Consider $f = 0$. Foias–Saut (1987) proved that any Leray–Hopf weak solution $u(t)$ has an asymptotic expansion,

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t},$$

where $q_j(t)$'s are polynomials in t with values in functional spaces.
- In fact, there is a smallest n_0 such that $q_{n_0} \neq 0$ is independent of t, is an eigenfunction of the Stokes operator A.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative numbers. A function $f : [T, \infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X-valued polynomial, if one has, for any $N \geq 1$, that

$$\left\| f(t) - \sum_{n=1}^{N} f_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N) t}) \quad \text{as } t \to \infty,$$

for some $\varepsilon_N > 0$.

Exponential decaying rates

• Denote the spectrum of the Stokes operator A by $\{\Lambda_k : k \in \mathbb{N}\}$, where Λ_k's are positive, strictly increasing to infinity.
• Let $\mathcal{S}$ be the additive semigroup generated by Λ_k's, that is,

$$\mathcal{S} = \Big\{ \sum_{j=1}^{N} \Lambda_{k_j} : N, k_1, \ldots, k_N \in \mathbb{N} \Big\}.$$

• We arrange the set $\mathcal{S}$ as a sequence $(\mu_n)_{n=1}^{\infty}$ of positive, strictly increasing numbers. Clearly,

$$\lim_{n\to\infty} \mu_n = \infty,$$

$$\mu_n + \mu_k \in \mathcal{S} \quad \forall n, k \in \mathbb{N}.$$

Other NSE and PDE results

- H.–Martinez (2017, 2018) prove that the Foias–Saut expansion holds in Gevrey spaces with non-potential force

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\gamma_n t}.$$

- Cao–H. (2020)

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^{\infty} \chi_n t^{-\gamma_n}.$$

- H.–Titi (2020): Rotating fluids

$$u_t - \nu\Delta u + (u\cdot\nabla)u + Re_3 \times u = -\nabla p.$$

- Dissipative wave equations: Shi (2000)
- Navier–Stokes–Boussinesq system: Biswas–H.–Martinez (in preparation)

ODE results

A. With analytic nonlinear terms, no forcing.

$$y' + Ay = F(y).$$

- Normal forms: Poincaré, Dulac, Lyapunov (first method), Bruno.
- Power geometry: Bruno (1960s–present).
- Foias–Saut approach: Minea (1998).

B. Lagrangian trajectories. H. (2020):

$$y' = u(y, t).$$

C. With forcing.

$$y' + Ay = F(y) + f(t).$$

- Cao–H. (2020).

$$f(t) = \sum t^{-\mu}, (\ln t)^r, (\ln \ln t)^r (\ln \ln \ln t)^r, \ldots.$$

2. Main result

Consider ODE in $\mathbb{R}^d$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} + Ay = F(y), \quad t > 0,$$

where A is a $d \times d$ constant (real) matrix, and F is a vector field on $\mathbb{R}^d$.

Assumption

Matrix A is a diagonalizable with positive eigenvalues.

• The spectrum $\sigma(A)$ of matrix A consists of eigenvalues Λ_k's, for $1 \le k \le d$, which are positive and increasing in k.
• Then there exists an invertible matrix S such that

$$A = S^{-1} A_0 S, \text{ where } A_0 = \mathrm{diag}[\Lambda_1, \Lambda_2, \ldots, \Lambda_d].$$

• Denote the distinct eigenvalues of A by λ_j's that are strictly increasing in j, i.e.,

$$0 < \lambda_1 = \Lambda_1 < \lambda_2 < \ldots < \lambda_{d_*} = \Lambda_d \quad \text{with } 1 \le d_* \le d.$$

Positively homogeneous functions

Definition

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two (real) normed spaces.
A function $F : X \to Y$ is positively homogeneous of degree $\beta \geq 0$ if

$$F(tx) = t^\beta F(x) \text{ for any } x \in X \text{ and any } t > 0.$$

Define $\mathcal{H}_\beta(X, Y)$ to be the set of positively homogeneous functions of order β from X to Y, and denote $\mathcal{H}_\beta(X) = \mathcal{H}_\beta(X, X)$.
For a function $F \in \mathcal{H}_\beta(X, Y)$, define

$$\|F\|_{\mathcal{H}_\beta} = \sup_{\|x\|_X = 1} \|F(x)\|_Y = \sup_{x \neq 0} \frac{\|F(x)\|_Y}{\|x\|_X^\beta}.$$

The following are immediate properties.

1. If $F \in \mathcal{H}_\beta(X,Y)$ with $\beta > 0$, then taking $x = 0$ and $t = 2$ gives
$$F(0) = 0.$$
If, in addition, F is bounded on the unit sphere in X, then
$$\|F\|_{\mathcal{H}_\beta} \in [0,\infty) \text{ and } \|F(x)\|_Y \le \|F\|_{\mathcal{H}_\beta} \|x\|_X^\beta \quad \forall x \in X.$$
2. The zero function (from X to Y) belongs to $\mathcal{H}_\beta(X,Y)$ for all $\beta \ge 0$, and a constant function (from X to Y) belongs to $\mathcal{H}_0(X,Y)$.
3. Each $\mathcal{H}_\beta(X,Y)$, for $\beta \ge 0$, is a linear space.
4. If $F_1 \in \mathcal{H}_{\beta_1}(X,\mathbb{R})$ and $F_2 \in \mathcal{H}_{\beta_2}(X,Y)$, then $F_1 F_2 \in \mathcal{H}_{\beta_1+\beta_2}(X,Y)$.
5. If $F : X \to Y$ is a homogeneous polynomial of degree $m \in \mathbb{Z}_+$, then $F \in \mathcal{H}_m(X,Y)$.

Notation. $\mathbb{R}_* = \mathbb{R} \setminus \{0\}$, $\mathbb{R}_*^n = (\mathbb{R}_*)^n$ and $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{0\}$.

Assumption

The mapping $F : \mathbb{R}^d \to \mathbb{R}^d$ *has the the following properties.*

1. F *is locally Lipschitz on* $\mathbb{R}^d$ *and* $F(0) = 0$.
2. *Either (a) or (b) below is satisfied.*
   (a) There exist numbers β_k*'s, for* $k \in \mathbb{N}$*, which belong to* $(1, \infty)$ *and increase strictly to infinity, and functions* $F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0)$*, for* $k \in \mathbb{N}$*, such that it holds, for any* $N \in \mathbb{N}$*, that*

$$\left| F(x) - \sum_{k=1}^{N} F_k(x) \right| = \mathcal{O}(|x|^\beta) \text{ as } x \to 0, \text{ for some } \beta > \beta_N.$$

   (b) There exist $N_* \in \mathbb{N}$*, strictly increasing numbers* β_k*'s in* $(1, \infty)$*, and functions* $F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0)$*, for* $k = 1, 2, \ldots, N_*$*, such that*

$$\left| F(x) - \sum_{k=1}^{N_*} F_k(x) \right| = \mathcal{O}(|x|^\beta) \text{ as } x \to 0, \text{ for all } \beta > \beta_{N_*}.$$

Main Theorem

Theorem (Cao-H.-Kieu 2020)

Let $y(t)$ be a non-trivial, decaying solution. Then there exist polynomials $q_n\colon \mathbb{R} \to \mathbb{R}^d$ such that $y(t)$ has an asymptotic expansion

$$y(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^d,$$

where μ_n's are increasing strictly to infinity, and $q_n(t)$ satisfies, for any $n \geq 1$,

$$q_n' + (A - \mu_n I_d) q_n = \mathcal{J}_n \stackrel{\text{def}}{=\!=} \sum_{\substack{r\geq 1, m\geq 0, k_1,k_2,\ldots,k_m\geq 2,\\ \sum_{j=1}^m \tilde\mu_{k_j} + \alpha_r \lambda_* = \tilde\mu_n}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m}) \text{ in } \mathbb{R},$$

where $\mathcal{F}_{r,m}$ are m-linear mappings from $(\mathbb{R}^d)^m$ to $\mathbb{R}^d$.

3. Sketch of Proof

Proof (I). First asymptotic approximation

Proposition (Cao-H.-Kieu 2020)

Let $y(t)$ *be a non-trivial, decaying solution. Then there exists a number* $C_1 > 0$ *such that*

$$|y(t)| \le C_1 e^{-\Lambda_1 t} \text{ for all } t \ge 0.$$

Moreover, for any $\varepsilon > 0$*, there exists a number* $C_2 = C_2(\varepsilon) > 0$ *such that*

$$|y(t)| \ge C_2 e^{-(\Lambda_d+\varepsilon)t} \text{ for all } t \ge 0.$$

Theorem (Cao-H.-Kieu 2020)

Let $y(t)$ *be a non-trivial, decaying solution. Then there exist an eigenvalue* λ_* *of* A *and a corresponding eigenvector* ξ_* *such that*

$$|y(t) - e^{-\lambda_* t}\xi_*| = \mathcal{O}(e^{-(\lambda_*+\delta)t}) \text{ for some } \delta > 0.$$

Recall Foias-Saut's approach: the use of the Dirichlet quotient.
Consider A symmetric, positive definite:

$$y' + Ay = F(y).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|y|^2 + \frac{Ay \cdot y}{|y|^2}|y|^2 = F(y) \cdot y.$$

To find asymtotic behavior of $|y(t)|^2$, we need to know that of

$$\chi(t) = \frac{Ay(t) \cdot y(t)}{|y(t)|^2} = \frac{|A^{1/2}y(t)|^2}{|y(t)|^2}.$$

Also, need the to find limit of $v(t) = y(t)/|y(t)|$ as $t \to \infty$, that will be ξ_* (hence, no more exponential part.)
Clever ways to write ODEs for $\chi(t)$ and $v(t)$.
We have a new proof.

Proof. There exist positive numbers $c_*, \varepsilon_*, \alpha$ such that

$$|F(x)| \le c_* |x|^{1+\alpha} \ \forall x \in \mathbb{R}^d \text{ with } |x| \le \varepsilon_*.$$

Define the set (of possible decaying rates)

$$S' = \left\{ \sum_{j=1}^{n} \lambda_j' + m\alpha\lambda_1 : \text{ for any numbers } n \in \mathbb{N}, \lambda_j' \in \sigma(A), 0 \le m \in \mathbb{Z} \right\}.$$

The set S' can be arranged as a strictly increasing sequence $\{\nu_n\}_{n=1}^{\infty}$.
Note that $\nu_1 = \lambda_1$ and $\nu_n \to \infty$ as $n \to \infty$.
Step 1. Recall that $|y(t)| \le Ce^{-\nu_1 t}$. Let $w_0(t) = e^{\nu_1 t} y(t)$. Then $w_0(t)$ satisfies

$$w_0'(t) + (A - \nu_1 I_d) w_0(t) = g_1(t) \stackrel{\text{def}}{=\!=} e^{\nu_1 t} F(t, y(t)).$$

Note

$$|g_1(t)| \le Ce^{\nu_1 t} |y(t)|^{1+\alpha} \le Ce^{\nu_1 t} e^{-\nu_1 (1+\alpha) t} = \mathcal{O}(e^{-\alpha\nu_1 t}).$$

Lemma (Approximation Lemma, Special Case)

Let $g : [T, \infty) \to \mathbb{R}^d$, $|g(t)| = \mathcal{O}(e^{-\alpha t})$ *for some* $\alpha > 0$. *Suppose* $\lambda > 0$ *and* $y \in C([T, \infty), \mathbb{R}^d)$ *is a solution of*

$$y'(t) = -(A - \lambda I_d)y(t) + g(t), \quad \text{for } t \in (T, \infty).$$

If $\lambda > \lambda_1$, *assume further that*

$$\lim_{t\to\infty}(e^{(\bar\lambda-\lambda)t}|y(t)|) = 0, \text{ where } \bar\lambda = \max\{\lambda_j : 1 \le j \le d_*, \lambda_j < \lambda\}.$$

Then there exists a a constant vector $\xi \in \mathbb{R}^d$ *such that*

$$(A - \lambda I_d)\xi = 0 \text{ and } |y(t) - \xi| = \mathcal{O}(e^{-\varepsilon t}).$$

Note: If $\xi \neq 0$, then λ is an eigenvalue of A, and ξ is a corresponding eigenvector.

By Approximation Lemma (special case), there exists a vector $\xi_1 \in \mathbb{R}^d$ and a number $\varepsilon_1 > 0$ such that

$$A\xi_1 = \nu_1 \xi_1,$$
$$|w_0(t) - \xi_1| = \mathcal{O}(e^{-\varepsilon t}),\ \text{that is }, |y(t) - e^{-\nu_1 t}\xi_1| = \mathcal{O}(e^{-(\nu_1+\varepsilon_1)t}).$$

Step 2. (recursive argument) Set

$$M = \{n \in \mathbb{N} : |y(t)| = \mathcal{O}(e^{-(\nu_n+\delta)t}) \text{ for some } \delta > 0\}.$$

Suppose $n \in M$. Let $w_n(t) = e^{\nu_{n+1}t} y(t)$. Then

$$w_n'(t) + (A - \nu_{n+1} I_d) w_n(t) = g_{n+1}(t) \overset{\text{def}}{=\!=} e^{\nu_{n+1}t} F(t, y(t)).$$

Note that $\nu_n(1+\alpha) \ge \nu_n + \lambda_1\alpha \ge \nu_{n+1}$ (for, $\nu_n + \lambda_1\alpha \in S'$ and is $> \nu_n$.) Then, for large t,

$$|g_{n+1}(t)| \le Ce^{\nu_{n+1}t}|y(t)|^{1+\alpha} \le Ce^{\nu_{n+1}t} e^{-(\nu_n+\delta)(1+\alpha)t}| = \mathcal{O}(e^{-\delta(1+\alpha)t}).$$

Then there exists a vector $\xi_{n+1} \in \mathbb{R}^d$ and a number $\varepsilon > 0$ such that

$$A\xi_{n+1} = \nu_{n+1}\xi_{n+1},$$
$$|w_n(t) - \xi_{n+1}| = \mathcal{O}(e^{-\varepsilon t}),\ \text{that is }, |y(t) - e^{-\nu_{n+1}t}\xi_{n+1}| = \mathcal{O}(e^{-(\nu_{n+1}+\varepsilon)t}).$$

Step 3. If the vector ξ_1 in Step 1 is not zero, then the theorem is proved with $\lambda_* = \lambda_1$ and $\xi_* = \xi_1$.
Now, consider $\xi_1 = 0$. One has $1 \in M$. Since $|y(t)|$ is above a non-zero exponential function, and $\nu_n \to \infty$, the set M must be finite.
Let k be the maximum number of M, and $n_0 = k + 1$. By the result in Step 2 applied to $n = k$, there exist $\xi_{n_0} \in \mathbb{R}^d$ and $\varepsilon > 0$ such that

$$A\xi_{n_0} = \nu_{n_0}\xi_{n_0},$$
$$|y(t) - e^{-\nu_{n_0} t}\xi_{n_0}| = \mathcal{O}(e^{-(\nu_{n_0}+\varepsilon)t}).$$

If $\xi_{n_0} = 0$, then $n_0 \in M$, which is a contradiction.
Thus, $\xi_{n_0} \neq 0$, which implies $\lambda_* = \nu_{n_0}$ is an eigenvalue and $\xi_* = \xi_{n_0}$ is a corresponding eigenvector of A. □

Proof (II). Infinite series expansion

Consider

$$F(x) \sim \sum_{k=1}^{\infty} F_k(x), \quad F_k \in \mathcal{H}_{\beta_k}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d_0),$$

Definition

We define a set $\tilde{S} \subset [0, \infty)$ as follows. Let $\alpha_k = \beta_k - 1 > 0$ for $k \in \mathbb{N}$, and

$$\tilde{S} = \Big\{ \sum_{k=n_0}^{d_*} m_k(\lambda_k - \lambda_*) + \sum_{j=1}^{\infty} z_j \alpha_j \lambda_* : m_k, z_j \in \mathbb{Z}_+, \text{ with } z_j > 0 \text{ for only finitely many } j\text{'s} \Big\}.$$

The set $\tilde{S}$ has countably, infinitely many elements. Arrange $\tilde{S}$ as a sequence $(\tilde{\mu}_n)_{n=1}^{\infty}$ of non-negative and strictly increasing numbers. Set

$$\mu_n = \tilde{\mu}_n + \lambda_* \text{ for } n \in \mathbb{N}, \text{ and define } S = \{\mu_n : n \in \mathbb{N}\}.$$

Let $r \in \mathbb{N}$ and $s \in \mathbb{Z}_+$. Since F_r is a C^∞-function in a neighborhood of $\xi_* \neq 0$, we have the following Taylor's expansion, for any $h \in \mathbb{R}^d$,

$$F_r(\xi_* + h) = \sum_{m=0}^{s} \frac{1}{m!} D^m F_r(\xi_*) h^{(m)} + g_{r,s}(h),$$

where $D^m F_r(\xi_*)$ is the m-th order derivative of F_r at ξ_*, and

$$g_{r,s}(h) = \mathcal{O}(|h|^{s+1}) \text{ as } h \to 0.$$

For $m \geq 0$, denote

$$\mathcal{F}_{r,m} = \frac{1}{m!} D^m F_r(\xi_*).$$

When $m = 0$, $\mathcal{F}_{r,0} = F_r(\xi_*)$. When $m \geq 1$, $\mathcal{F}_{r,m}$ is an m-linear mapping from $(\mathbb{R}^d)^m$ to $\mathbb{R}^d$.

One has, for any $r, m \geq 1$, and $y_1, y_2, \ldots, y_m \in \mathbb{R}^d$, that

$$|\mathcal{F}_{r,m}(y_1, y_2, \ldots, y_m)| \leq \|\mathcal{F}_{r,m}\| \cdot |y_1| \cdot |y_2| \cdots |y_m|.$$

Proof of Main Theorem

First Step ($N = 1$). By the first asymptotic approximation.
Induction Step. Let $y_n(t) = q_n(t)e^{-\mu_n t}$, $u_n(t) = y(t) - \sum_{k=1}^n y_k(t)$. By induction hypotheses,

$$u_N(t) = \mathcal{O}(e^{-(\mu_N+\delta_N)t}).$$

Let $w_N(t) = e^{\mu_{N+1}t}u_N(t)$. Then

$$w_N' + (A - \mu_{N+1}I_d)w_N = e^{\mu_{N+1}t}F(y) - e^{\mu_{N+1}t}\sum_{k=1}^N (Ay_k + y_k').$$

$$F(x) = \sum_{r=1}^{r_*} F_r(x) + \mathcal{O}(|x|^{\beta_{r_*}+\varepsilon_{r_*}}) \text{ as } x \to 0.$$

$$e^{\mu_{N+1}t}F(y(t)) = E(t) + e^{\mu_{N+1}t}\mathcal{O}(e^{-\lambda_*(\beta_{r_*}+\varepsilon_{r_*})t}),$$

where

$$E(t) = e^{\mu_{N+1}t}\sum_{r=1}^{r_*} F_r(y(t)).$$

Denote

$$\tilde{y}_k(t) = y_k(t)e^{\lambda_* t} = q_k(t)e^{-\tilde{\mu}_k t} \text{ and } \tilde{u}_k(t) = u_k(t)e^{\lambda_* t}.$$

Then

$$\tilde{u}_N(t) = u_N(t)e^{\lambda_* t} = \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t}), \quad \tilde{u}_1(t) = u_1(t)e^{\lambda_* t} = \mathcal{O}(e^{-\delta_1 t}).$$

Write

$$F_r(y(t)) = F_r(y_1 + u_1) = F_r\big(e^{-\lambda_* t}(\xi_* + \tilde{u}_1)\big) = e^{-\beta_r \lambda_* t} F_r(\xi_* + \tilde{u}_1).$$

By Taylor expansion about ξ_*:

$$F_r(y(t)) = e^{-\beta_r \lambda_* t}\left(F_r(\xi_*) + \sum_{m=1}^{s_*} \mathcal{F}_{r,m}\tilde{u}_1^{(m)}\right) + e^{-\beta_r \lambda_* t} g_{r,s_*}(\tilde{u}_1).$$

$$\mathcal{F}_{r,m}\tilde{u}_1^{(m)} = \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k + \tilde{u}_N\Big)^{(m)}$$

$$= \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k + \tilde{u}_N, \sum_{k=2}^N \tilde{y}_k + \tilde{u}_N, \ldots, \sum_{k=2}^N \tilde{y}_k + \tilde{u}_N\Big)$$

$$= \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k\Big)^{(m)} + \sum_{\text{finitely many}} \mathcal{F}_{r,m}(z_1, \ldots, z_N).$$

$$\sum_{m=0}^{s_*} \mathcal{F}_{r,m}\tilde{u}_1^{(m)} = F_r(\xi_*) + \sum_{m=1}^{s_*} \mathcal{F}_{r,m}\Big(\sum_{k=2}^N \tilde{y}_k\Big)^{(m)} + \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t})$$

$$= \sum_{m=0}^{s_*} \sum_{k_1,\ldots,k_m \ge 2}^N \mathcal{F}_{r,m}(\tilde{y}_{k_1}, \tilde{y}_{k_2}, \ldots, \tilde{y}_{k_m}) + \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t})$$

$$= \sum_{m=0}^{s_*} \sum_{k_1,\ldots,k_m \ge 2}^N e^{-t\sum_{j=1}^m \tilde{\mu}_{k_j}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m}) + \mathcal{O}(e^{-(\tilde{\mu}_N+\delta_N)t}).$$

Thus,

$$e^{-\beta_r \lambda_* t} \sum_{m=0}^{s_*} \mathcal{F}_{r,m} \tilde{u}_1^{(m)} = \sum_{m=0}^{s_*} \sum_{k_1,\ldots,k_m=2}^{N} e^{-t(\sum_{j=1}^m \tilde{\mu}_{k_j} + \beta_r \lambda_*)} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m}) + \mathcal{O}(e^{-(\tilde{\mu}_N + \beta_r \lambda_* + \delta_N)t}).$$

Exponent: $\mu = \tilde{\mu}_{k_1} + \ldots + \tilde{\mu}_{k_m} + \alpha_r \lambda_* \in \tilde{S}$, hence is some $\tilde{\mu}_p$. Then

$$\sum_{j=1}^m \tilde{\mu}_{k_j} + \beta_r \lambda_* = \mu + \lambda_* = \tilde{\mu}_p + \lambda_* = \mu_p \in S \quad \text{for some integer } p.$$

Recall $\mathcal{J}_n = \sum_{\substack{r\ge 1, m\ge 0, k_1, k_2, \ldots, k_m \ge 2, \\ \sum_{j=1}^m \tilde{\mu}_{k_j} + \alpha_r \lambda_* = \tilde{\mu}_n}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \ldots, q_{k_m})$.

More manipulations:

$$w_N' + (A - \mu_{N+1} I_d) w_N = -e^{\mu_{N+1} t} \sum_{k=1}^N e^{-\mu_k t} \chi_k + \mathcal{J}_{N+1} + \mathcal{O}(e^{-\delta_N' t}),$$

where

$$\chi_1 = q_1' + (A - \mu_1 I_d) q_1, \quad \chi_k = q_k' + (A - \mu_k I_d) q_k - \mathcal{J}_k \text{ for } 2 \le k \le N.$$

By the induction hypothesis, $\chi_k = 0$ for $2 \le k \le N$. Hence,

$$w_N' + (A - \mu_{N+1} I_d) w_N = \mathcal{J}_{N+1} + \mathcal{O}(e^{-\delta_N' t}),$$

where $\mathcal{J}_{N+1}(t)$ is a polynomial in t.

Lemma (Approximation Lemma)

Let $p(t)$ be an $\mathbb{R}^d$-valued polynomial and $g : [T, \infty) \to \mathbb{R}^d$, $|g(t)| = \mathcal{O}(e^{-\alpha t})$ for some $\alpha > 0$. Suppose $\lambda > 0$ and $y \in C([T, \infty), \mathbb{R}^d)$ is a solution of

$$y'(t) = -(A - \lambda I_d) y(t) + p(t) + g(t), \quad \text{for } t \in (T, \infty).$$

If $\lambda > \lambda_1$, assume further that

$$\lim_{t\to\infty} (e^{(\bar{\lambda} - \lambda)t} |y(t)|) = 0, \ \text{where } \bar{\lambda} = \max\{\lambda_j : 1 \le j \le d_*, \lambda_j < \lambda\}.$$

Then there exists a unique $\mathbb{R}^d$-valued polynomial $q(t)$ such that

$$q'(t) = -(A - \lambda I_d) q(t) + p(t) \ \text{for } t \in \mathbb{R}, \quad |y(t) - q(t)| = \mathcal{O}(e^{-\varepsilon t}).$$

(Note. The special case we used before is when $p = 0$.)
By the Approximation Lemma, there exists polynomial $q_{N+1} : \mathbb{R} \to \mathbb{R}^d$ and a number $\delta_{N+1} > 0$ such that

$$|w_N(t) - q_{N+1}(t)| = \mathcal{O}(e^{-\delta_{N+1} t}).$$

Moreover $q_{N+1}(t)$ solves

$$q'_{N+1} + (A - \mu_{N+1} I_d) q_{N+1} = \mathcal{J}_{N+1},$$

Multiplying estimate of $w_N - q_{N+1}$ by $e^{-\mu_{N+1} t}$ gives

$$\left| y(t) - \sum_{n=1}^{N+1} q_n(t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_{N+1} + \delta_{N+1}) t}),$$

which proves the statement for $N := N + 1$.

4. Extended results and examples

Extended results and examples

- Expansion exists even when all F_k's belong to $C^\infty(V)$ for some open set V in $\mathbb{R}^d$, and all eigenvectors of A also belong to V.
- Assume

$$\left|F(x) - \sum_{k=1}^{N_*} F_k(x)\right| = \mathcal{O}(|x|^{\beta_{N_*}+\bar\varepsilon}) \text{ as } x \to 0, \text{ for some number } \bar\varepsilon > 0.$$

Same as before,

$$\tilde S = \Big\{ \sum_{k=n_0}^{d_*} m_k(\lambda_k - \lambda_*) + \sum_{j=1}^{N_*} z_j\alpha_j\lambda_* : m_k, z_j \in \mathbb{Z}_+ \Big\}.$$

Then $\tilde S = (\tilde\mu_n)_{n=1}^\infty$, and define $S = \{\mu_n = \tilde\mu_n + \lambda_* : n \in \mathbb{N}\}$.
Let $\bar N \in \mathbb{N}$ be defined by

$$\bar N = \max\{N \in \mathbb{N} : \lambda_*(\beta_{N_*} + \bar\varepsilon) > \mu_N\}.$$

Theorem

There exist $\mathbb{R}^d$-valued polynomials $q_n(t)$'s, for $1 \le n \le \bar{N}$, and a number $\delta > 0$ such that

$$\left| y(t) - \sum_{n=1}^{\bar{N}} q_n(t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_{\bar{N}}+\delta)t}),$$

where each polynomial $q_n(t)$, for $1 \le n \le \bar{N}$, satisfies equation

$$q_n' + (A - \mu_n I_d) q_n = \sum_{r=1}^{N_*} \sum_{\substack{m \ge 0, k_1, k_2, \dots, k_m \ge 2, \\ \sum_{j=1}^m \tilde{\mu}_{k_j} + \alpha_r \lambda_* = \tilde{\mu}_n}} \mathcal{F}_{r,m}(q_{k_1}, q_{k_2}, \dots, q_{k_m}) \text{ in } \mathbb{R}.$$

Remarks:

- Solution $y(t)$ has a finite expansion.
- Even in this case, the expansion is not limited by the singularity of F_k's. Our result provides many terms for the expansion of $y(t)$.

For $n \in \mathbb{N}$, $p \in [1, \infty)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the ℓ^p-norm of x is

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}.$$

We prove specific theorems to dealt with functions F_k's which are sums/products of $\|x\|_p^\alpha$, x_j^α, $|x_j|^\alpha$.

Example

Let α be any number in $(0, \infty)$ that is not an even integer, and

$$F(x) = |x|^\alpha Mx \text{ for } x \in \mathbb{R}^d.$$

Example

Given a constant $d \times d$ matrix M_0, even numbers $p_1, p_2 \geq 2$, and real numbers $\alpha, \beta > 0$, let

$$F(x) = \frac{\|x\|_{p_1}^\alpha M_0 x}{1 + \|x\|_{p_2}^\beta} \text{ for } x \in \mathbb{R}^d.$$

For $x \in \mathbb{R}^d$ with $\|x\|_{p_2} < 1$, we expand $1/(1+\|x\|_{p_2}^\beta)$, using the geometric series, and can verify that

$$F(x) \sim \sum_{k=1}^{\infty} (-1)^{k-1} \|x\|_{p_1}^{\alpha} \|x\|_{p_2}^{(k-1)\beta} M_0 x.$$

When $\|\cdot\|_{p_1} = \|\cdot\|_{p_2} = |\cdot|$, function F has form

$$F(x) \sim \sum_{k=1}^{\infty} c_k |x|^{\alpha+(k-1)\beta} M_0 x.$$

Others:

$$F(x) \sim \sum_{k=1}^{\infty} \|x\|_{p_k}^{\alpha_k} M_k x,$$

$$\left| F(x) - \sum_{k=1}^{N_*} \|x\|_{p_k}^{\alpha_k} M_k x \right| = \mathcal{O}(|x|^{\alpha_{N_*}+1+\bar{\varepsilon}}) \text{ as } x \to 0.$$

The last example, in general, only yields finite expansion for $y(t)$.

Example

Consider $d = 2$ and let

$$F(x_1, x_2) = (|x_1^3 - x_2^3|^{p_1} + |x_1^3 + x_2^3|^{p_1})^{\alpha/p_1} \cdot (|x_1 x_2|^{p_2} + |3x_1^2 - 2x_2^2|^{p_2})^{\beta/p_2} M_0(x_1, x_2)$$

where $p_1, p_2 \geq 2$ are even numbers, M_0 is a $\mathbb{R}^2$-valued homogeneous polynomials of degree $m_0 \in \mathbb{Z}_+$, and $\alpha, \beta > 0$.

Example

Consider the following system of ODEs in $\mathbb{R}^2$:

$$y_1' + 2y_1 + y_2 = |y|^{2/3} |y_1|^{1/2} y_2^3,$$
$$y_2' + y_1 + 2y_2 = \|y\|_{5/2}^{1/3} y_1 |y_2|^{1/4} \mathrm{sign}(y_2).$$

Eigenvalues and bases of A: $\lambda_1 = 1$, basis $\{(-1, 1)\}$, and $\lambda_2 = 3$, basis $\{(1, 1)\}$. Then any eigenvector of A belongs to $V = (\mathbb{R}_*)^2$. Note: function $F \in C^\infty(V) \cap C(\mathbb{R}^d)$. We obtain infinite series expansions for any *eventually* non-trivial, decaying solution $y(t)$.

Example

Let $d = 3$, and assume 3×3 matrix A has the following eigenvalues and bases of the corresponding eigenspaces

$$\lambda_1 = \lambda_2 = 1, \text{ basis } \{\xi_1 = (1,0,1), \xi_2 = (0,1,0)\},$$

$$\lambda_3 = 2, \text{ basis } \{\xi_3 = (1,1,-1)\}.$$

Let ξ be an eigenvector of A. One can verify that

$$\xi \in V = (\mathbb{R} \times \mathbb{R}_* \times \mathbb{R}) \cup (\mathbb{R}_* \times \mathbb{R} \times \mathbb{R}_*) = (\mathbb{R}_0^2 \times \mathbb{R}) \cap (\mathbb{R} \times \mathbb{R}_0^2).$$

Let $F(x) = (x_1^2 + x_2^2)^{1/3} \cdot (x_2^6 + x_3^6)^{1/5} P(x)$, where P is a polynomial vector field on $\mathbb{R}^3$ of degree $m_0 \in \mathbb{N}$ without the constant term, i.e., $P(0) = 0$. Then, any non-trivial decaying solution $y(t)$ has an infinite series expansion.

THANK YOU!