Asymptotic expansions about infinity for solutions of nonlinear differential equations with coherently decaying forcing functions

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Outline

1. Introduction

Foias–Saut result for Navier–Stokes equations

Functional form of the Navier–Stokes equations:

$$u_t + Au + B(u,u) = f, \quad u(0) = u_0.$$

• If $f = const. \neq 0$, turbulence.
• If $f = 0$ or $f = f(t) \to 0$ as $t \to \infty$, turbulence for short time, then the flows settle (to zero) eventually.
• Consider $f = 0$. Foias–Saut (1987) proved that any Leray–Hopf weak solution $u(t)$ has an asymptotic expansion,

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-\mu_n t},$$

where $q_j(t)$'s are polynomials in t with values in functional spaces.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative numbers. A function $f : [T, \infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X-valued polynomial, if one has, for any $N \geq 1$, that

$$\Big\| f(t) - \sum_{n=1}^{N} f_n(t) e^{-\alpha_n t} \Big\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N)t}) \quad \text{as } t \to \infty,$$

for some $\varepsilon_N > 0$.

Other NSE and PDE results

- H.–Martinez (2017, 2018) prove that the Foias–Saut expansion holds in Gevrey spaces with non-potential force

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-\gamma_n t}.$$

- Cao–H. (2020), Cao–H. (2020)

$$u_t + Au + B(u,u) = f(t) \sim \sum_{n=1}^{\infty} \chi_n \phi(t)^{-\gamma_n},$$

  where $\phi(t) = t, \ln t, \ln\ln t$, etc.

- H.–Titi (2021): Rotating fluids

$$u_t - \nu\Delta u + (u\cdot\nabla)u + Re_3 \times u = -\nabla p.$$

- Dissipative wave equations: Shi (2000)

ODE results

A. With analytic nonlinear terms, no forcing.

$$y' + Ay = F(y).$$

- Normal forms: Poincaré, Dulac, Lyapunov (first method), Bruno.
- Power geometry: Bruno (1960s–present).
- Foias–Saut approach: Minea (1998).

B. Lagrangian trajectories. H. (2021): For a Leray–Hopf weak solution $u(x,t)$ of NSE,

$$y' = u(y,t).$$

C. With forcing.

$$y' + Ay = F(y) + f(t).$$

Cao–H. (2021). For $\mu > 0$ and $r \in \mathbb{R}$:

$$f(t) \sim \sum t^{-\mu}, (\ln t)^r, (\ln\ln t)^r(\ln\ln\ln t)^r, \ldots.$$

2. Problem and main assumptions

Consider the following system of nonlinear ODEs in $\mathbb{C}^n$:

$$y' = -Ay + G(y) + f(t),$$

where A is an $n \times n$ constant matrix of complex numbers, G is a vector field on $\mathbb{C}^n$, and f is a function from $(0, \infty)$ to $\mathbb{C}^n$.

Assumption

All eigenvalues of the matrix A have positive real parts.

Denote by Λ_k, for $1 \le k \le n$, the eigenvalues of A counting the multiplicities. The spectrum of A is

$$\sigma(A) = \{\Lambda_k : 1 \le k \le n\} \subset \mathbb{C}.$$

We order the set $\mathbf{Re}\sigma(A)$ by strictly increasing numbers λ_j's, with $1 \le j \le d$ for some $d \le n$. Of course,

$$0 < \lambda_1 \le \mathbf{Re}\Lambda_k \le \lambda_d \quad \text{for } k = 1, 2, \dots, n.$$

Nonlinear part

Assumption

Function $G:\mathbb{C}^n\to\mathbb{C}^n$ has the the following properties.

1. *G is locally Lipschitz.*
2. *There exist functions $G_m:\mathbb{C}^n\to\mathbb{C}^n$, for $m\ge 2$, each is a homogeneous polynomial of degree m, such that, for any $N\ge 2$, there exists $\delta>0$ so that*

$$\left|G(x)-\sum_{m=2}^{N} G_m(x)\right| = \mathcal{O}(|x|^{N+\delta}) \text{ as } x\to 0.$$

For each $m\ge 2$, there exists an m-linear mapping $\mathcal{G}_m:(\mathbb{C}^n)^m\to\mathbb{C}^n$ such that

$$G_m(x)=\mathcal{G}_m(x,x,\dots,x) \text{ for } x\in\mathbb{C}^n.$$

General consideration

Assumption

There exists a number $T_f \geq 0$ such that f is continuous on $[T_f, \infty)$.

Assumption

There exists a number $T_0 \geq 0$ such that $y \in C^1((T_0, \infty))$ is a solution on (T_0, ∞), and $y(t) \to 0$ as $t \to \infty$.

3. Asymptotic estimates

Theorem

Assume there is $T \geq 0$ *such that* $f \in C((T,\infty))$. *Let* $y \in C^1((T,\infty))$ *be a solution on* (T,∞) *that satisfies* $\liminf_{t\to\infty} |y(t)| = 0$.

1. *If there is a number* $\alpha \in (0,\lambda_1)$ *such that*

$$f(t) = \mathcal{O}(e^{-\alpha t}),$$

then

$$y(t) = \mathcal{O}(e^{-\alpha t}).$$

2. *If there are numbers* $m \in \mathbb{Z}_+$ *and* $\alpha > 0$ *such that*

$$f(t) = o(L_m(t)^{-\alpha}),$$

then

$$y(t) = \mathcal{O}(L_m(t)^{-\alpha}).$$

4. Exponential decay

For $z \in \mathbb{C}$ and $t > 0$, the exponential and power functions are defined by

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ and } t^z = \exp(z \ln t).$$

If $z = a + ib$ with $a, b \in \mathbb{R}$, then

$$t^z = t^a(\cos(b \ln t) + i \sin(b \ln t)) \text{ and } |t^z| = t^a.$$

Definition

Let X be a linear space over $\mathbb{C}$.

1. Define $\mathcal{F}_E(X)$ to be the collection of functions $g : \mathbb{R} \to X$ of the form

$$g(t) = \sum_{\lambda \in S} p_\lambda(t) e^{\lambda t} \text{ for } t \in \mathbb{R},$$

   where S is some finite subset of $\mathbb{C}$, and each p_λ is a polynomial from $\mathbb{R}$ to X.

2. For $\mu \in \mathbb{R}$, define

$$\mathcal{F}_E(\mu, X) = \Big\{ g(t) = \sum_{\lambda \in S} p_\lambda(t) e^{\lambda t} \in \mathcal{F}_E : \mathbf{Re}\lambda = \mu \text{ for all } \lambda \in S \Big\}.$$

Assumption

The function $f(t)$ admits the asymptotic expansion in $\mathbb{C}^n$

$$f(t) \sim \sum_{k=1}^{\infty} f_k(t) = \sum_{k=1}^{\infty} \hat{f}_k(t) e^{-\mu_k t}, \text{ where } f_k \in \mathcal{F}_E(-\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N},$$

with $(\mu_k)_{k=1}^{\infty}$ being a divergent, strictly increasing sequence of positive numbers. Moreover, the set $\mathcal{S} \stackrel{\text{def}}{=\!=} \{\mu_k : k \in \mathbb{N}\}$ preserves the addition and contains $\mathbf{Re}\,\sigma(A)$.

Main result (I)

Theorem

There exist functions

$$y_k \in \mathcal{F}_E(-\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N},$$

such that the solution $y(t)$ admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^{\infty} y_k(t) = \sum_{k=1}^{\infty} \hat{y}_k(t) e^{-\mu_k t}.$$

Moreover, for each $k \in \mathbb{N}$, the functions $y_k(t)$ solves the following equation

$$y_k' + Ay_k = \sum_{m\geq 2} \sum_{\mu_{j_1}+\mu_{j_2}+\dots\mu_{j_m}=\mu_k} \mathcal{G}_m(y_{j_1}, y_{j_2}, \dots, y_{j_m}) + f_k, \text{ for } t \in \mathbb{R}.$$

Main tool for the proof

Asymptotic approximations for the linear system.

Theorem

Given $\mu > 0$, $f \in \mathcal{F}_E(-\mu, \mathbb{C}^n)$ and a function $g \in C([T,\infty),\mathbb{C}^n)$, for some $T \geq 0$, that satisfies

$$g(t) = \mathcal{O}(e^{-(\mu+\delta)t}) \text{ for some } \delta > 0.$$

Assume $y \in C([T,\infty),\mathbb{C}^n)$ is a solution of

$$y'(t) + Ay(t) = f(t) + g(t), \qquad \text{for } t > T,$$

and it holds for any $\lambda \in \mathbf{Re}\sigma(A)$ with $\lambda < \mu$ and any number $m \in \mathbb{N}$ that

$$\lim_{t\to\infty} t^m e^{\lambda t}|y(t)| = 0.$$

Theorem (continued)

Then there exists a function $z \in \mathcal{F}_E(-\mu, \mathbb{C}^n)$ *and a number* $\varepsilon > 0$ *such that*

$$z'(t) + Az(t) = f(t) \quad \text{for } t \in \mathbb{R},$$

and

$$|y(t) - z(t)| = \mathcal{O}(e^{-(\mu+\varepsilon)t}).$$

Proof of Main result (I). By induction. The remainder $v_N(t) = y(t) - \sum_{k=1}^N y_k(t)$ satisfies

$$v_N' + Av_N = p(t) + g(t), \quad g(t) = \mathcal{O}(e^{-(\mu_{N+1}+\delta)t}).$$

Then approximate $v_N(t)$ by $y_{N+1}(t)$ with

$$y_{N+1}' + Ay_{N+1} = p(t), \quad |v_N(t) - y_{N+1}(t)| = \mathcal{O}(e^{-(\mu_{N+1}+\varepsilon)t}).$$

Thus,

$$|y(t) - \sum_{k=1}^{N+1} y_k(t)| = \mathcal{O}(e^{-(\mu_{N+1}+\varepsilon)t}).$$

5. Power decay

Definition

Define the iterated exponential and logarithmic functions as follows:

$$E_0(t) = t \text{ for } t \in \mathbb{R}, \text{ and } E_{m+1}(t) = e^{E_m(t)} \text{ for } m \in \mathbb{Z}_+, \ t \in \mathbb{R},$$
$$L_{-1}(t) = e^t, \quad L_0(t) = t \text{ for } t \in \mathbb{R}, \text{ and}$$
$$L_{m+1}(t) = \ln(L_m(t)) \text{ for } m \in \mathbb{Z}_+, \ t > E_m(0).$$

For $k \in \mathbb{Z}_+$, define

$$\mathcal{L}_k = (L_1, L_2, \ldots, L_k) \quad \text{and} \quad \widehat{\mathcal{L}}_k = (L_{-1}, L_0, L_1, \ldots, L_k).$$

Explicitly,

$$\widehat{\mathcal{L}}_k(t) = (e^t, t, \ln t, \ln\ln t, \ldots, L_k(t)).$$

For $z = (z_{-1}, z_0, z_1, \ldots, z_k) \in (0, \infty)^{k+2}$ and $\alpha = (\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_k) \in \mathbb{C}^{k+2}$, define

$$z^\alpha = \prod_{j=-1}^{k} z_j^{\alpha_j}.$$

For $\mu \in \mathbb{R}$, $m, k \in \mathbb{Z}$ with $k \geq m \geq -1$, denote by $\mathcal{E}(m,k,\mu)$ the set of vectors α such that $\mathbf{Re}(\alpha_j) = 0$ for $-1 \leq j < m$ and $\mathbf{Re}(\alpha_m) = \mu$.

Definition

Let $\mathbb{K}$ be $\mathbb{C}$ or $\mathbb{R}$, and X be a linear space over $\mathbb{K}$.

1. For $k \geq -1$, define $\mathcal{P}(k, X)$ to be the set of functions of the form

$$p(z) = \sum_{\alpha \in S} z^\alpha \xi_\alpha \text{ for } z \in (0,\infty)^{k+2},$$

   where S is some finite subset of $\mathbb{K}^{k+2}$, and each ξ_α belongs to X.
2. Let $\mathbb{K} = \mathbb{C}$, $k \geq m \geq -1$ and $\mu \in \mathbb{R}$. Define $\mathcal{P}_m(k,\mu,X)$ to be set of functions of the above form, where S is a finite subset of $\mathcal{E}(m,k,\mu)$ and each ξ_α belongs to X. Define

$$\mathcal{F}_m(k,\mu,X) = \left\{ p \circ \widehat{\mathcal{L}}_k : p \in \mathcal{P}_m(k,\mu,X) \right\}.$$

Definition

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, and $(X, \|\cdot\|_X)$ be a normed space over $\mathbb{K}$. Suppose g is a function from (T, ∞) to X for some $T \in \mathbb{R}$, and $m_* \in \mathbb{Z}_+$.
Let $(\gamma_k)_{k=1}^{\infty}$ be a divergent, strictly increasing sequence of positive numbers, and $(n_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{N} \cap [m_*, \infty)$. We say

$$g(t) \sim \sum_{k=1}^{\infty} g_k(t) = \sum_{k=1}^{\infty} \hat{g}_k(t) L_{m_*}(t)^{-\gamma_k}, \text{ where } g_k \in \mathcal{F}_{m_*}(n_k, -\gamma_k, X),$$

if, for each $N \in \mathbb{N}$, there is some $\mu > \gamma_N$ such that

$$\left\| g(t) - \sum_{k=1}^{N} g_k(t) \right\|_X = \mathcal{O}(L_{m_*}(t)^{-\mu}).$$

Operators

Given an integer $k \geq -1$, let $p = \sum_{\alpha \in S} z^\alpha \xi_\alpha \in \mathcal{P}(k, \mathbb{C}^n)$.
Define, for $j = -1, 0, \ldots, k$, the function $\mathcal{M}_j p : (0, \infty)^{k+2} \to \mathbb{C}^n$ by

$$(\mathcal{M}_j p)(z) = \sum_{\alpha \in S} \alpha_j z^\alpha \xi_\alpha.$$

In the case $k \geq 0$, define the function $\mathcal{R}p : (0, \infty)^{k+2} \to \mathbb{C}^n$ by

$$(\mathcal{R}p)(z) = \sum_{j=0}^{k} z_0^{-1} z_1^{-1} \ldots z_j^{-1} (\mathcal{M}_j p)(z).$$

In the case $p \in \mathcal{P}_{-1}(k, 0, \mathbb{C}^n)$, define the function $\mathcal{Z}_A p : (0, \infty)^{k+2} \to \mathbb{C}^n$ by

$$(\mathcal{Z}_A p)(z) = \sum_{\alpha \in S} z^\alpha (A + \alpha_{-1} I_n)^{-1} \xi_\alpha.$$

Assumption

The function $f(t)$ admits the asymptotic expansion with $m_* = 0$,

$$f(t) \sim \sum_{k=1}^{\infty} f_k(t) = \sum_{k=1}^{\infty} \hat{f}_k(t) t^{-\mu_k}, \text{ where } f_k \in \mathcal{F}_0(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N},$$

with $(\mu_k)_{k=1}^{\infty}$ being a divergent, strictly increasing sequence of positive numbers, and $(n_k)_{k=1}^{\infty}$ being an increasing sequence in $\mathbb{Z}_+$. Moreover, the set $\mathcal{S} \stackrel{\text{def}}{=\!=} \{\mu_k : k \in \mathbb{N}\}$ preserves the addition and the unit increment.

Main result (II)

Theorem

There exist functions

$$y_k \in \mathcal{F}_0(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N},$$

such that the solution $y(t)$ admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^{\infty} y_k(t) = \sum_{k=1}^{\infty} \hat{y}_k(t) t^{-\mu_k}.$$

More specifically, assume, for all $k \in \mathbb{N}$,

$$f_k(t) = p_k(\widehat{\mathcal{L}}_{n_k}(t)) \text{ for some } p_k \in \mathcal{P}_0(n_k, -\mu_k, \mathbb{C}^n).$$

Theorem (continued)

Then the functions y_k's can be constructed recursively as follows. For each $k \in \mathbb{N}$,

$$y_k(t) = q_k(\widehat{\mathcal{L}}_{n_k}(t)),$$

where

$$q_k = \mathcal{Z}_A \left(\sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \ldots \mu_{j_m} = \mu_k} \mathcal{G}_m(q_{j_1}, q_{j_2}, \ldots, q_{j_m}) + p_k - \chi_k \right),$$

with

$$\chi_k = \begin{cases} \mathcal{R} q_\lambda & \text{if there exists } \lambda \leq k-1 \text{ such that } \mu_\lambda + 1 = \mu_k, \\ 0 & \text{otherwise.} \end{cases}$$

Main tools

Theorem

Given integers $m, k \in \mathbb{Z}_+$ *with* $k \geq m$, *and a number* $t_0 > E_k(0)$. *Let* $\mu > 0$, $p \in \mathcal{P}_m(k, -\mu, \mathbb{C}^n)$, *and let function* $g \in C([t_0, \infty), \mathbb{C}^n)$ *satisfy*

$$|g(t)| = \mathcal{O}(L_m(t)^{-\alpha}) \text{ for some } \alpha > \mu.$$

Suppose $y \in C([t_0, \infty), \mathbb{C}^n)$ *is a solution of*

$$y' = -Ay + p(\widehat{\mathcal{L}}_k(t)) + g(t) \text{ on } (t_0, \infty).$$

Then there exists $\delta > 0$ *such that*

$$|y(t) - (\mathcal{Z}_A p)(\widehat{\mathcal{L}}_k(t))| = \mathcal{O}(L_m(t)^{-\mu-\delta}).$$

Lemma

If $k \in \mathbb{Z}_+$ *and* $q \in \mathcal{P}(k, \mathbb{C}^n)$*, then*

$$\frac{\mathrm{d}}{\mathrm{d}t} q(\widehat{\mathcal{L}}_k(t)) = \mathcal{M}_{-1} q(\widehat{\mathcal{L}}_k(t)) + \mathcal{R} q(\widehat{\mathcal{L}}_k(t)) \text{ for } t > E_k(0).$$

In particular, when $k \geq m \geq 1$, $\mu \in \mathbb{R}$, *and* $q \in \mathcal{P}_m(k, \mu, \mathbb{C}^n)$*, one has*

$$\frac{\mathrm{d}}{\mathrm{d}t} q(\widehat{\mathcal{L}}_k(t)) = \mathcal{M}_{-1} q(\widehat{\mathcal{L}}_k(t)) + \mathcal{O}(t^{-\gamma}) \quad \text{for all } \gamma \in (0, 1).$$

6. Logarithmic and iterated logarithmic decay

Assumption

There exist a number $m_* \in \mathbb{N}$*, a divergent, strictly increasing sequence* $(\mu_k)_{k=1}^\infty \subset (0,\infty)$*, and an increasing sequence* $(n_k)_{k=1}^\infty \subset \mathbb{N}\cap[m_*,\infty)$ *such that the function* $f(t)$ *admits the asymptotic expansion*

$$f(t) \sim \sum_{k=1}^\infty f_k(t) = \sum_{k=1}^\infty \hat{f}_k(t) L_{m_*}(t)^{-\mu_k}, \text{ where } f_k \in \mathcal{F}_{m_*}(n_k, -\mu_k, \mathbb{C}^n).$$

Moreover, the set $\mathcal{S} \stackrel{\text{def}}{=\!=} \{\mu_k : k \in \mathbb{N}\}$ *preserves the addition.*

Main result (III)

Theorem

There exist functions

$$y_k \in \mathcal{F}_{m_*}(n_k, -\mu_k, \mathbb{C}^n) \text{ for } k \in \mathbb{N},$$

such that the solution $y(t)$ admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^{\infty} y_k(t) = \sum_{k=1}^{\infty} \hat{y}_k(t) L_{m_*}(t)^{-\mu_k}.$$

More specifically, suppose $f_k(t) = p_k(\widehat{\mathcal{L}}_{n_k}(t))$ with $p_k \in \mathcal{P}_{m_*}(n_k, -\mu_k, \mathbb{C}^n)$ for all $k \in \mathbb{N}$. Then $y_k(t) = q_k(\widehat{\mathcal{L}}_{n_k}(t))$, where

$$q_k = \mathcal{Z}_A \left(\sum_{m \geq 2} \sum_{\mu_{j_1} + \mu_{j_2} + \dots \mu_{j_m} = \mu_k} \mathcal{G}_m(q_{j_1}, q_{j_2}, \dots, q_{j_m}) + p_k \right).$$

7. Problems in real linear spaces

Problems in real linear spaces

Consider the following system of nonlinear ODEs in $\mathbb{R}^n$:

$$y' = -Ay + G(y) + f(t).$$

1. The matrix A is an $n \times n$ matrix of real numbers. All eigenvalues have positive real parts.
2. The function G is from $\mathbb{R}^n$ to $\mathbb{R}^n$,

$$G(x) \sim \sum_{m=2}^{\infty} G_m(x), \quad G_m(x) = \mathcal{G}_m(x, x, \ldots, x),$$

   with $G_m : \mathbb{R}^n$ to $\mathbb{R}^n$, and the multi-linear mappings $\mathcal{G}_m : (\mathbb{R}^n)^m \to \mathbb{R}^n$.
3. The forcing function $f(t)$ and solution $y(t)$ are $\mathbb{R}^n$-valued.

Theorem (Summary of three results)

If

$$f(t) \sim \sum_{k=1}^{\infty} f_k(t),$$

with real-valued functions f_k's, then

$$y(t) \sim \sum_{k=1}^{\infty} y_k(t),$$

with real-valued functions y_k's.

Ideas:

- Complexification of the multi-linear mappings $\mathcal{G}_m$'s.
- Exponential decay: complex expansion of a real function is a real expansion. (By the uniqueness of the expansions + complex conjugation.)
- Power and iterated logarithmic decay: real approximation comes from the explicit construction at each step.

8. Examples

Example

If

$$f(t)=\frac{\cos(\alpha t)(\ln t)(\ln\ln t)^{-1/3}}{t^m}\xi \text{ for some } m\in\mathbb{N} \text{ and } \xi\in\mathbb{R}^n,$$

then the solution $y(t)$ admits the asymptotic expansion

$$y(t)\sim\sum_{k=0}^{\infty}\frac{q_k(t)}{t^{m+k}},$$

where $q_k(t)=\widehat{q}_k(\widehat{\mathcal{L}}_2(t))$ with $\widehat{q}_k\in\mathcal{P}_0^1(2,\mathbb{R}^n)$. Roughly speaking, the functions $q_k(t)$'s are composed by

$$\cos(\omega L_j(t)),\ \sin(\omega L_j(t)),\ L_\ell(t)^\alpha,$$

for $j=0,1,2$ and $\ell=1,2$, with some real numbers ω's and α's.

Example

If

$$f(t) = \frac{\cos(2t)\sin(3\ln\ln t)(\ln\ln\ln t)^2\sin(5\ln\ln\ln t)}{(\ln t)^{1/2}}\xi \text{ for some } \xi \in \mathbb{R}^n,$$

then the solution $y(t)$ admits the asymptotic expansion

$$y(t) \sim \sum_{k=1}^{\infty} \frac{q_k(t)}{(\ln t)^{k/2}},$$

where, roughly speaking, $q_k(t)$'s are functions composed by the functions

$$\cos(\omega L_j(t)),\ \sin(\omega L_j(t)),\ L_\ell(t)^\alpha,$$

for $j = 0, 1, 2, 3$ and $\ell = 2, 3$.

THANK YOU!