Asymptotic expansions for the Lagrangian trajectories from solutions of the Navier-Stokes equations

Luan T. Hoang

Department of Mathematics and Statistics, Texas Tech University

AMS 2020 Fall Central Sectional Meeting
September 12–13, 2020.

Outline

1. Introduction

Lagrangian and Eulerian descriptions.

We study the long-time dynamics of the incompressible, viscous fluid flows in the three-dimensional space.

• Lagrangian description: trajectory $x(t) = x(t, x_0) \in \mathbb{R}^3$ with initial fluid particle (or material point) $x(0, x_0) = x_0$

• Eulerian description: velocity field $u(x, t)$ and pressure $p(x, t)$, where $x \in \mathbb{R}^3$ is the independent spatial variable representing each fixed position in the fluid.

• Relation

$$x' = u(x, t).$$

The solutions $x(t)$ of this system are called the Lagrangian trajectories.

The Navier–Stokes equations

The Eulerian description turns out to be simpler for deriving the set of equations that govern the fluid flows. They are called the Navier–Stokes equations (NSE),

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u = -\nabla p, \\ \operatorname{div} u = 0. \end{cases}$$

where $\nu > 0$ is the kinematic viscosity, and the unknowns are the velocity $u(x,t)$ and pressure $p(x,t)$.
Initial condition $u(x,0) = u_0(x)$, where u_0 is a given initial vector field.

Settings

Dirichlet boundary condition (DBC). Let Ω be an bounded, open, connected set in $\mathbb{R}^3$ with C^∞ boundary.
The boundary condition $u = 0$ on $\partial\Omega \times (0, \infty)$.
Spatial periodicity condition (SPC). Fix a vector $\mathbf{L} = (L_1, L_2, L_3) \in (0, \infty)^3$. We consider $u(\cdot, t)$ and $p(\cdot, t)$ to be **L**-periodic for $t > 0$.
Here, a function g defined on $\mathbb{R}^3$ is called **L**-periodic if

$$g(x + L_i e_i) = g(x) \text{ for } i = 1, 2, 3 \text{ and all } x \in \mathbb{R}^3.$$

Define domain $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$ in this case.
A function g is said to have zero average over Ω if

$$\int_\Omega g(x)\mathrm{d}x = 0.$$

Notation

- $H^m = W^{m,2}$, for $m \in \mathbb{N}$, denotes the standard Sobolev space.
- In the (DBC) case, let $\mathcal{V}$ be the set of divergence-free vector fields in $C_c^\infty(\Omega)^3$.
  Define $\mathcal{X}$ to be the set of functions in $\bigcap_{m=1}^\infty H^m(\Omega)^3$ that are divergence-free and vanish on the boundary $\partial\Omega$, and denote $\Omega^* = \bar{\Omega}$.
- In the (SPC) case, let $\mathcal{V}$ be the set of **L**-periodic trigonometric polynomial vector fields on $\mathbb{R}^3$ which are divergence-free and have zero average over Ω.
  Define $\mathcal{X} = \mathcal{V}$, and denote $\Omega^* = \mathbb{R}^3$.
- In both cases, define space H (respectively, V) to be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega)$ (respectively, $\mathbb{H}^1(\Omega)$).
  The Leray projection $\mathbb{P}$ is the orthogonal projection from $\mathbb{L}^2(\Omega)$ to H.
  The Stokes operator is $(-\mathbb{P}\Delta)$ defined on $V \cap \mathbb{H}^2(\Omega)$.

Exponential decaying rates

• Denote the spectrum of Stokes operator by $\{\Lambda_k : k \in \mathbb{N}\}$, where Λ_k's are positive, strictly increasing to infinity.

• Let $\mathcal{S}$ be the additive semigroup generated by $\nu\Lambda_k$'s, that is,

$$\mathcal{S} = \Big\{\nu \sum_{j=1}^{N} \Lambda_{k_j} : N, k_1, \ldots, k_N \in \mathbb{N}\Big\}.$$

• We arrange the set $\mathcal{S}$ as a sequence $(\mu_n)_{n=1}^{\infty}$ of positive, strictly increasing numbers. Clearly,

$$\lim_{n\to\infty} \mu_n = \infty,$$

$$\mu_n + \mu_k \in \mathcal{S} \quad \forall n, k \in \mathbb{N}.$$

Foias–Saut asymptotic expansions

Assumption

Fix a Leray–Hopf weak solution $u(x,t)$ (with $u(\cdot,t)$ valued in H) and a Lagrangian trajectory $x(t) \in C^1([T,\infty),\Omega)$ in the (DBC) case, or $x(t) \in C^1([T,\infty),\mathbb{R}^3)$ in the (SPC) case.

Foias–Saut (1987) proved that the solution $u(x,t)$ has an asymptotic expansion,

$$u(\cdot,t) \sim \sum_{n=1}^{\infty} q_n(\cdot,t)e^{-\mu_n t} \text{ in } \mathbb{H}^m(\Omega),$$

for any $m \in \mathbb{N}$, where $q_j(\cdot,t)$'s are polynomials in t with values in $\mathcal{X} \subset C^\infty(\Omega^*)^3$.

Asymptotic expansions

Let $(X, \|\cdot\|)$ be a normed space and $(\alpha_n)_{n=1}^\infty$ be a sequence of strictly increasing non-negative numbers. A function $f : [T, \infty) \to X$, for some $T \in \mathbb{R}$, is said to have an asymptotic expansion

$$f(t) \sim \sum_{n=1}^\infty f_n(t) e^{-\alpha_n t} \quad \text{in } X,$$

where $f_n(t)$ is an X-valued polynomial, if one has, for any $N \geq 1$, that

$$\left\| f(t) - \sum_{n=1}^N f_n(t) e^{-\alpha_n t} \right\| = \mathcal{O}(e^{-(\alpha_N + \varepsilon_N) t}) \quad \text{as } t \to \infty,$$

for some $\varepsilon_N > 0$.

In fact, $q_1(x,t)$ is independent of t, hence we write

$$q_1(x,t) = q_1(x) \in \mathcal{X}.$$

According to the Foias–Saut expansion with $m = 2$, we have

$$\Big\| u(\cdot,t) - \sum_{n=1}^{N} q_n(\cdot,t) e^{-\mu_n t} \Big\|_{H^2(\Omega)^3} = \mathcal{O}(e^{-(\mu_N+\delta_N)t}),$$

for any $N \in \mathbb{N}$, and some $\delta_N > 0$.
By Morrey's embedding theorem, it follows that

$$\sup_{x\in\Omega^*} \Big| u(x,t) - \sum_{n=1}^{N} q_n(x,t) e^{-\mu_n t} \Big| = \mathcal{O}(e^{-(\mu_N+\delta_N)t}).$$

In particular, letting $N = 1$, we infer

$$\sup_{x\in\Omega^*} |u(x,t)| \le \sup_{x\in\Omega^*} |q_1(x)|e^{-\mu_1 t} + \mathcal{O}(e^{-(\mu_1+\delta_1)t}) = \mathcal{O}(e^{-\mu_1 t}).$$

Therefore, there is $C_0 > 0$ such that

$$\sup_{x\in\Omega^*} |u(x,t)| \le C_0 e^{-\mu_1 t} \text{ for all } t \ge T.$$

Taking $x = x(t)$ gives

$$\left| u(x(t),t) - \sum_{n=1}^{N} q_n(x(t),t) e^{-\mu_n t} \right| = \mathcal{O}(e^{-(\mu_N+\delta_N)t}),$$

$$|u(x(t),t)| \le C_0 e^{-\mu_1 t} \text{ for all } t \ge T.$$

2. Results and proofs

Convergence of the Lagrangian trajectories

$$x'(t) = u(x(t), t).$$

Proposition (H. 2020)

The limit $x_* \overset{\text{def}}{=\!=} \lim_{t\to\infty} x(t)$ *exists and belongs to* Ω^**, and*

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu_1 t}).$$

Proof. For $t \ge T$, we have $x(t) = x(T) + \int_T^t u(x(\tau), \tau)\mathrm{d}\tau$.
Since $|u(x(t), t)| \le Ce^{-\mu_1 t}$ for $t \ge T$,

$$x_* = \lim_{t\to\infty} x(t) = x(T) + \int_T^\infty u(x(\tau), \tau)\mathrm{d}\tau \text{ which exists in } \mathbb{R}^3.$$

Obviously, $x_* \in \Omega^*$. Error estimate:

$$|x(t) - x_*| = \Big|\int_t^\infty u(x(\tau), \tau)\mathrm{d}\tau\Big| \le \int_t^\infty C_0 e^{-\mu_1 \tau}\mathrm{d}\tau = C_0\mu_1^{-1}e^{-\mu_1 t}.$$

Consideration I. (SPC) or $x_* \in \Omega$ for (DBC).

Foias–Saut expansion: $u(x,t) \sim \sum q_n(x,t) e^{-\mu_n t}$. Write

$$q_n(x,t) = \sum_{k=0}^{d_n} t^k q_{n,k}(x), \text{ where } d_n \geq 0, \text{ and } q_{n,k} \in \mathcal{X}.$$

The Taylor expansion: for any $s \geq 0$,

$$q_{n,k}(x) = \sum_{m=0}^{s} \frac{1}{m!} D_x^m q_{n,k}(x_*)(x - x_*)^{(m)} + g_{n,k,s}(x),$$

where $D_x^m q_{n,k}$ denotes the m-th order derivative of $q_{n,k}$ (m-linear mapping), and $g_{n,k,s} \in C(\Omega^*)^3$ satisfying

$$g_{n,k,s}(x) = \mathcal{O}(|x - x_*|^{s+1}) \text{ as } x \to x_*.$$

Then

$$q_n(x,t) = \sum_{k=0}^{d_n} t^k \Big[\sum_{m=0}^{s} \frac{1}{m!} D_x^m q_{n,k}(x_*)(x - x_*)^{(m)} + g_{n,k,s}(x) \Big].$$

Rewrite

$$q_n(x,t) = \sum_{m=0}^{s} \mathcal{Q}_{n,m}(x_*,t)(x-x_*)^{(m)} + \sum_{k=0}^{d_n} t^k g_{n,k,s}(x),$$

where

$$\mathcal{Q}_{n,m}(x_*,t) = \sum_{k=0}^{d_n} \frac{t^k}{m!} D_x^m q_{n,k}(x_*) = \frac{1}{m!} D_x^m q_n(x_*,t).$$

In particular,

$$\mathcal{Q}_{n,0}(x_*,t) = q_n(x_*,t), \quad \mathcal{Q}_{n,1}(x_*,t) = D_x q_n(x_*,t),$$
$$\mathcal{Q}_{n,2}(x_*,t) = \frac{1}{2} D_x^2 q_n(x_*,t).$$

Note that $\mathcal{Q}_{n,m}(x_*,t)$ is a polynomial in t valued in the space of m-linear mappings from $(\mathbb{R}^3)^m$ to $\mathbb{R}^3$.

Above, $x(t) \to x_*$ as $t \to \infty$. Denote $z(t) = x(t) - x_*$. Then

$$|z(t)| = \mathcal{O}(e^{-\mu_1 t}).$$

We have

$$q_n(x(t), t) = \sum_{m=0}^{s} \mathcal{Q}_{n,m}(x_*, t) z(t)^{(m)} + \sum_{k=0}^{d_n} t^k \mathcal{O}(|z(t)|^{s+1}),$$

thus

$$q_n(x(t), t) = \sum_{m=0}^{s} \mathcal{Q}_{n,m}(x_*, t) z(t)^{(m)} + \mathcal{O}(e^{-(\mu_1(s+1)-\delta)t}) \quad \forall \delta > 0.$$

Heuristic arguments.

Assume $z(t) \sim \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t}$.

$$z'(t) = x'(t) = u(x(t),t) = u(x(t),t) \sim \sum_{k=1}^{\infty} q_k(x(t),t) e^{-\mu_k t},$$

$$\sum_{n=1}^{\infty} (\zeta_n'(t) - \mu_n \zeta_n(t)) e^{-\mu_n t} \sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{Q}_{k,m}(x_*,t) z(t)^{(m)} e^{-\mu_k t}$$

$$\sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \mathcal{Q}_{k,m}(x_*,t) (\sum_{j_1} \zeta_{j_1}(t) e^{-\mu_{j_1} t}, \ldots, \sum_{j_m} \zeta_{j_m}(t) e^{-\mu_{j_m} t}) e^{-\mu_k t}$$

$$\sim \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{j_1, j_2, \ldots, \mu_{j_m}} \mathcal{Q}_{k,m}(x_*,t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)) e^{-(\mu_{j_1} + \ldots + \mu_{j_m})t} e^{-\mu_k t}.$$

Then

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*,t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)).$$

Theorem (H. 2020)

Under Consideration I, there exist polynomials $\zeta_n : \mathbb{R} \to \mathbb{R}^3$, for $n \geq 0$, such that solution $x(t)$ has an asymptotic expansion,

$$x(t) \sim x_* + \sum_{n=1}^{\infty} \zeta_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3,$$

where each ζ_n, for $n \geq 1$, is the unique polynomial solution of the following differential equation

$$\zeta_n'(t) - \mu_n \zeta_n(t) = \sum_{\mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n} \mathcal{Q}_{k,m}(x_*, t)(\zeta_{j_1}(t), \ldots, \zeta_{j_m}(t)).$$

for all $t \in \mathbb{R}$.

Proof I. Approximation lemma

Let $(X, \|\cdot\|_X)$ be a Banach space. Let $p : \mathbb{R} \to X$ be a polynomial, and $\|g(t)\|_X \le Me^{-\delta t}$ for $t \ge t_*$, for some $M, \delta > 0$.
Let $\gamma > 0$. Suppose that $y : [t_*, \infty) \to X$ solves

$$y'(t) - \gamma y(t) = p(t) + g(t) \quad \text{for } t > t_*,$$

and satisfies

$$\lim_{t\to\infty} (e^{-\gamma t}\|y(t)\|_X) = 0.$$

Then there exists a unique polynomial $q : \mathbb{R} \to X$ such that

$$\|y(t) - q(t)\|_X \le \frac{M}{\gamma + \delta} e^{-\delta t} \quad \text{for all } t \ge t_*.$$

More precisely, $q(t)$ is the unique polynomial solution of

$$q'(t) - \gamma q(t) = p(t) \quad \text{for } t \in \mathbb{R},$$

and can be explicitly defined by

$$q(t) = -\int_t^\infty e^{\gamma(t-\tau)} p(\tau) \mathrm{d}\tau.$$

Proof II. Sketch of the induction step

Let $z_N(t) = \sum_{n=1}^N \zeta_n(t)e^{-\mu_n t}$ and $\tilde{z}_N(t) = z(t) - z_N(t)$.
Induction hypothesis gives

$$|\tilde{z}_N(t)| = \mathcal{O}(e^{-(\mu_N+\varepsilon_N)t}).$$

Define $w_N(t) = e^{\mu_{N+1}t}\tilde{z}_N(t)$. Long calculations give

$$\begin{aligned} & w_N' - \mu_{N+1}w_N \\ & = \sum_{\mu_k+\mu_{j_1}+\mu_{j_2}+\ldots+\mu_{j_m}=\mu_{N+1}} \mathcal{Q}_{k,m}(x_*,t)(\zeta_{j_1}(t),\ldots,\zeta_{j_m}(t)) + \mathcal{O}(e^{-\varepsilon_{N+1}t}). \end{aligned}$$

Applying Approximation Lemma, one has

$$\Big|w_N(t) - \zeta_{N+1}(t)\Big| = \mathcal{O}(e^{-\varepsilon_{N+1}t}).$$

Multiplying by $e^{-\mu_{N+1}t}$ gives

$$\Big|\tilde{z}_N(t) - \zeta_{N+1}(t)e^{-\mu_{N+1}t}\Big| = \mathcal{O}(e^{-(\mu_{N+1}+\varepsilon_{N+1})t}).$$

Consideration II. (DBC) with $x_* \in \partial\Omega$

Theorem (H. 2020)

Under Consideration II, one has

$$|x(t) - x_*| = \mathcal{O}(e^{-\mu t}) \text{ for all } \mu > 0.$$

Proof. Note $q_n(x_*, t) = 0$ for all n. Explicit formulas:

$$\zeta_n(t) = -\int_t^\infty e^{\mu_n(t-\tau)} \Big\{ q_n(x_*, \tau) + \sum_{m=1}^{s_n} \sum_{\substack{k, j_1, \ldots, j_m = 1, \\ \mu_k + \mu_{j_1} + \mu_{j_2} + \ldots + \mu_{j_m} = \mu_n}}^{n-1} \mathcal{Q}_{k,m}(x_*, \tau)(\zeta_{j_1}(\tau), \ldots, \zeta_{j_m}(\tau)) \Big\} \mathrm{d}\tau.$$

In particular, when $n = 1$, one has $\zeta_1(t) = -q_1(x_*)/\mu_1$.
Check: $\zeta_1(t) = 0$. Then, recursively, $\zeta_2(t) = 0$, $\zeta_3(t) = 0$, etc.

(SPC) without the zero average condition

Let $(u(x,t), p(x,t))$ be a **L**-periodic, classical solution the NSE on $\mathbb{R}^3 \times (0,\infty)$.
Let $x(t) \in \mathbb{R}^3$ be a Lagrangian trajectory corresponding to $u(x,t)$.

Theorem (H. 2020)

There exist $x_* \in \mathbb{R}^3$ and polynomials $X_n : \mathbb{R} \to \mathbb{R}^3$, for $n \in \mathbb{N}$, such that

$$x(t) \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t} \text{ in } \mathbb{R}^3,$$

where $U_0 = (L_1 L_2 L_3)^{-1} \int_\Omega u(x,0)\mathrm{d}x$.

Proof.

Galilean transformation. Set

$$v(X,t) = u(X + U_0 t, t) - U_0 \text{ and } P(X,t) = p(X + U_0 t, t).$$

Then (v, P) is a solution, $\mathbf{L}$-periodic, and $v(\cdot, t)$ has zero average.
Let $X(t) = x(t) - U_0 t$. We have

$$X'(t) = x'(t) - U_0 = u(x(t), t) - U_0 = v(x(t) - U_0 t, t) + U_0 - U_0 = v(X(t), t).$$

Applying above result (for zero average solutions) to $v(X,t)$ and $X(t)$ yields

$$X(t) \sim x_* + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t}.$$

Consequently, we obtain

$$x(t) = X(t) + U_0 t \sim (x_* + U_0 t) + \sum_{n=1}^{\infty} X_n(t) e^{-\mu_n t}.$$

THANK YOU!