Analysis of the Navier-Stokes systems and generalized Forchheimer flows

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Overview of research

1. The Navier-Stokes equations of viscous, incompressible fluids
2. Generalized Forchheimer flows in porous media
3. Regularity theory for partial differential equations.
4. Abstract dynamical systems.

- 29 articles published, 1 in press, 1 accepted, 1 submitted.
- All are mathematical analysis, new results (except for one survey).
- All are in peer-reviewed journals.

Outline

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   - Foias-Saut asymptotic expansion
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   - Power-decaying forces
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   - Generalized Forchheimer flows
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   - Structural stability
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1. The Navier-Stokes systems

- Foias-Saut asymptotic expansion
- Exponentially decaying forces
- Power-decaying forces

The Navier-Stokes equations

- The Navier-Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f(x,t), \\ \text{div } u = 0, \\ u(x,0) = u^0(x), \end{cases}$$

with viscosity $\nu > 0$, velocity field $u(x,t) \in \mathbb{R}^3$, pressure $p(x,t) \in \mathbb{R}$, body force $f(x,t) \in \mathbb{R}^3$, initial velocity $u^0(x)$.

- Let $L > 0$ and $\Omega = (0,L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1,2,3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Functional setting

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued 2π-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

$$H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$

$$V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \quad \mathcal{D}(A) = \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.$$

Norm on H: $|u| = \|u\|_{L^2(\Omega)}$. Norm on V: $\|u\| = |\nabla u|$.
The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u,v) = \mathbb{P}_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}(A).$$

$\mathbb{P}_L$ is the Leray projection from $L^2(\Omega)$ onto H.
WLOG, assume $f(t) = \mathbb{P}_L f(t)$. The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = f(t),\ \ t > 0,$$

$$u(0) = u^0.$$

Case $f = 0$. Foias-Saut asymptotic expansion

Foias-Saut (1987) for a solution $u(t)$:

$$u(t) \sim \sum_{n=1}^{\infty} q_j(t) e^{-jt},$$

where $q_j(t)$ is a $\mathcal{V}$-valued polynomial in t. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$, the remainder $v_N(t) = u(t) - \sum_{j=1}^{N} q_j(t) e^{-jt}$ satisfies

$$\|v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.

Theorem (H.-Martinez 2017)

The Foias-Saut expansion holds in all Gevrey spaces:

$$\|e^{\sigma A^{1/2}} v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}),$$

for any $\sigma > 0$, $\varepsilon \in (0,1)$.

Gevrey classes

- Spectrum of A is $\{|k|^2 : k \in \mathbb{Z}^3, k \neq 0\}$.
- For $\alpha \geq 0$, $\sigma \geq 0$, define

$$A^\alpha e^{\sigma A^{1/2}} u = \sum_{\mathbf{k}\neq 0} |\mathbf{k}|^{2\alpha} \hat{u}(\mathbf{k}) e^{\sigma|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ for } u = \sum_{\mathbf{k}\neq 0} \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \in H.$$

The domain of $A^\alpha e^{\sigma A^{1/2}}$ is

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) = \{u \in H : |u|_{\alpha,\sigma} \stackrel{\text{def}}{=\!=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty\}.$$

- Compare the Sobolev and Gevrey norms:

$$|A^\alpha u| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} u| \leq \left(\frac{2\alpha}{e\sigma}\right)^{2\alpha} |e^{\sigma A^{1/2}} u|.$$

Notation.

- Denote for $\sigma \in \mathbb{R}$ the space $E^{\infty,\sigma} = \bigcap_{\alpha \geq 0} G_{\alpha,\sigma} = \bigcap_{m\in\mathbb{N}} G_{m,\sigma}$.
- Denote by $\mathcal{P}^{\alpha,\sigma}$ the space of $G_{\alpha,\sigma}$-valued polynomials in case $\alpha \in \mathbb{R}$, and the space of $E^{\infty,\sigma}$-valued polynomials in case $\alpha = \infty$.

Definition

Let X be a real vector space.
(a) An X-valued polynomial is a function $t \in \mathbb{R} \mapsto \sum_{n=1}^{d} a_n t^n$, for some $d \geq 0$, and a_n's belonging to X.
(b) In case $\|\cdot\|$ is a norm on X, a function $g(t)$ from $(0,\infty)$ to X is said to have the asymptotic expansion

$$g(t) \sim \sum_{n=1}^{\infty} g_n(t) e^{-nt} \text{ in } X,$$

where $g_n(t)$'s are X-valued polynomials, if for all $N \geq 1$, there exists $\varepsilon_N > 0$ such that

$$\left\| g(t) - \sum_{n=1}^{N} g_n(t) e^{-nt} \right\| = \mathcal{O}(e^{-(N+\varepsilon_N)t}) \text{ as } t \to \infty.$$

• We will say that an asymptotic expansion holds in $E^{\infty,\sigma}$ if it holds in $G_{\alpha,\sigma}$ for all $\alpha \geq 0$.

Exponentially decaying forces

(A1) The function $f(t)$ is continuous from $[0,\infty)$ to H.

(A2) There are a number $\sigma_0 \geq 0$, E^{∞,σ_0}-valued polynomials $f_n(t)$ such that

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-nt} \text{ in } E^{\infty,\sigma_0}.$$

Theorem (H.-Martinez 2018)

Let $u(t)$ be a Leray-Hopf weak solution. Then $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt} \quad \text{in } E^{\infty,\sigma_0}.$$

Moreover, $u_n(t) \stackrel{\text{def}}{=\!=} q_n(t)e^{-nt}$ and $F_n(t) \stackrel{\text{def}}{=\!=} f_n(t)e^{-nt}$ satisfy

$$\frac{d}{dt} u_n(t) + Au_n(t) + \sum_{\substack{k,m\geq 1\\ k+m=n}} B(u_k(t), u_m(t)) = F_n(t), \quad t\in\mathbb{R}, \text{ in } E^{\infty,\sigma_0}.$$

Finite asymptotic approximation

Theorem (H.-Martinez 2018)

Suppose there exist $N_* \geq 1$, $\sigma_0 \geq 0$, $\mu_* \geq \alpha_* \geq N_*/2$, *and* $\delta_n \in (0,1)$ *and* $f_n \in \mathcal{P}^{\mu_n,\sigma_0}$*,for* $1 \leq n \leq N_*$*, such that*

$$\left| f(t) - \sum_{n=1}^{N} f_n(t) e^{-nt} \right|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\delta_N)t}) \quad \text{as } t \to \infty,$$

for $1 \leq N \leq N_*$*, where* $\mu_n = \mu_* - (n-1)/2$, $\alpha_n = \alpha_* - (n-1)/2$.
Let $u(t)$ *be a Leray-Hopf weak . Then there exist polynomials* $q_n \in \mathcal{P}^{\mu_n+1,\sigma_0}$*, for* $1 \leq n \leq N_*$*, such that one has for* $1 \leq N \leq N_*$ *that*

$$\left| u(t) - \sum_{n=1}^{N} q_n(t) e^{-nt} \right|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0, \delta_N^*),$$

where $\delta_N^* = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$.

Power-decaying forces

Power asymptotic expansion in $(X, \|\cdot\|)$: $g(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^\infty g_n t^{-n}$ means

$$\|g(t) - \sum_{n=1}^N g_n t^{-n}\| = \mathcal{O}(t^{-(N+\varepsilon)}), \quad \text{for some } \varepsilon > 0, \quad t \to \infty.$$

Theorem (Cao-H. 2017)

Assume that $f(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^\infty \phi_n t^{-n}$ *in* G_{α,σ_0}, *for some* $\sigma_0 \ge 0$ *and* $\alpha \ge 1/2$, *sequence* $\{\phi_n\}_{n=1}^\infty$ *in* G_{α,σ_0}. *Then any Leray-Hopf weak solution* $u(t)$ *has the asymptotic expansion*

$$u(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^\infty \xi_n t^{-n} \quad \text{in } G_{\alpha,\sigma_0},$$

where $\xi_1 = A^{-1}\phi_1$,
$\xi_n = (n-1)A^{-1}\xi_{n-1} - \sum_{k,m\ge1,k+m=n} A^{-1}B(\xi_k,\xi_m) + A^{-1}\phi_n$ *for* $n \ge 2$.

2. Expansions in a general system of decaying functions

- Continuum systems
- Asymptotic expansions for NSE
- Applications

Expansions in a general system of decaying functions

Definition (Very/Too general)

Let $(\psi_n)_{n=1}^{\infty}$ be a sequence of non-negative functions defined on $[T_*, \infty)$ for some $T_* \in \mathbb{R}$ that satisfies the following two conditions:

1. For each $n \in \mathbb{N}$, $\lim_{t\to\infty} \psi_n(t) = 0$.
2. For $n > m$, $\psi_n(t) = o(\psi_m(t))$.

Let $(X, \|\cdot\|)$ be a normed space, and g be a function from $[T_*, \infty)$ to X.

$$g(t) \sim \sum_{n=1}^{\infty} \xi_n \psi_n(t) \text{ in } X,$$

where $\xi_n \in X$ for all $n \in \mathbb{N}$, if, for any $N \in \mathbb{N}$,

$$\|g(t) - \sum_{n=1}^{N} \xi_n \psi_n(t)\| = o(\psi_N(t)).$$

A continuum system

Definition

Let $\Psi = (\psi_\lambda)_{\lambda>0}$ be a system of functions that satisfies the following two conditions.

(a) There exists $T_* \geq 0$ such that, for each $\lambda > 0$, ψ_λ is a positive function defined on $[T_*, \infty)$, and

$$\lim_{t\to\infty} \psi_\lambda(t) = 0.$$

(b) For any $\lambda > \mu$, there exists $\eta > 0$ such that

$$\psi_\lambda(t) = \mathcal{O}(\psi_\mu(t)\psi_\eta(t)).$$

Asymptotic expansions in a continuum system

Definition

Let $(X, \|\cdot\|)$ be a real normed space, and $g : (0,\infty) \to X$.

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \text{ in } X,$$

where $\xi_n \in X$ for all $n \in \mathbb{N}$, and $(\lambda_n)_{n=1}^{\infty}$ is a strictly increasing, divergent sequence of positive numbers, if it holds, for any $N \geq 1$, that there exists $\varepsilon > 0$ such that

$$\Big\| g(t) - \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \Big\| = \mathcal{O}(\psi_{\lambda_N}(t) \psi_{\varepsilon}(t)).$$

Condition

The system $\Psi = (\psi_\lambda)_{\lambda>0}$ *satisfies* (a) *and* (b) *and the following.*

1. *For any* $\lambda, \mu > 0$, *there exist* $\gamma > \max\{\lambda, \mu\}$ *and a nonzero constant* $d_{\lambda,\mu}$ *such that*
$$\psi_\lambda \psi_\mu = d_{\lambda,\mu}\psi_\gamma.$$
Notation. $\gamma = \lambda \wedge \mu$.

2. *For each* $\lambda > 0$, *the function* ψ_λ *is continuous and differentiable on* $[T_*, \infty)$, *and its derivative* ψ'_λ *has an expansion*
$$\psi'_\lambda(t) \overset{\Psi}{\sim} \sum_{k=1}^{N_\lambda} c_{\lambda,k}\psi_{\lambda^\vee(k)}(t) \text{ in } \mathbb{R},$$
where $N_\lambda \in \mathbb{N} \cup \{0, \infty\}$, *all* $c_{\lambda,k}$ *are constants, all* $\lambda^\vee(k) > \lambda$, *and, for each* $\lambda > 0$, $\lambda^\vee(k)$*'s are strictly increasing in* k.

Condition

The system $\Psi = (\psi_\lambda)_{\lambda>0}$ *satisfies* (a), (b) *and the following.*

1. *For each* $\lambda > 0$*, the function* ψ_λ *is decreasing (in* t*).*
2. *If* $\lambda, \alpha > 0$ *then*
$$e^{-\alpha t} = o(\psi_\lambda(t)).$$
3. *For any number* $a \in (0, 1)$*,*
$$\psi_\lambda(at) = \mathcal{O}(\psi_\lambda(t)).$$

Consequently, for any $T \in \mathbb{R}$,

$$\psi_\lambda(t) = \mathcal{O}(\psi_\lambda(t + T)).$$

Assumption

The function f *belongs to* $L^\infty_{\rm loc}([0, \infty), H)$*.*

Asymptotic expansions for NSE

Assumption

Suppose there exist real numbers $\sigma \geq 0$, $\alpha \geq 1/2$, a strictly increasing, divergent sequence of positive numbers $(\gamma_n)_{n=1}^{\infty}$ and a sequence $(\tilde{\phi}_n)_{n=1}^{\infty}$ in $G_{\alpha,\sigma}$ such that

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \tilde{\phi}_n \psi_{\gamma_n}(t) \textit{ in } G_{\alpha,\sigma}.$$

Assumption

There exists a set S_* that contains $\{\gamma_n : n \in \mathbb{N}\}$, preserves the operations $\vee$ and $\wedge$, and can be ordered so that

$$S_* = \{\lambda_n : n \in \mathbb{N}\}, \textit{ where } \lambda_n\textit{'s are strictly increasing to infinity.}$$

Rewrite

$$f(t) \overset{\Psi}{\sim} \sum^{\infty} \phi_n \psi_{\lambda_n}(t) \quad \text{in } G_{\alpha,\sigma} \quad \text{as } t \to \infty,$$

Theorem (Cao-H. 2018)

Any Leray-Hopf weak solution $u(t)$ has the asymptotic expansion

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

where ξ_n's are defined recursively by

$$\xi_1 = A^{-1}\phi_1,$$
$$\xi_n = A^{-1}\Big(\phi_n - \chi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k\wedge\lambda_m=\lambda_n}} d_{\lambda_k,\lambda_m} B(\xi_k,\xi_m)\Big) \quad \text{for } n \ge 2,$$

where

$$\chi_n = \begin{cases} \displaystyle\sum_{\substack{(p,k)\in[1,n-1]\times\mathbb{N}:\\ \lambda_p^\vee(k)=\lambda_n}} c_{\lambda_p,k}\xi_p, & \text{if } \exists p \in [1,n-1], k\in\mathbb{N} : \lambda_p^\vee(k)=\lambda_n,\\ 0, & \text{otherwise.} \end{cases}$$

Parts of proof (1). Linearized NSE

Theorem (Cao-H. 2018)

Given $\alpha, \sigma \geq 0$, let $\xi \in G_{\alpha,\sigma}$, and f be a function from $(0,\infty)$ to $G_{\alpha,\sigma}$ that satisfies

$$|f(t)|_{\alpha,\sigma} \leq MF(t) \quad \text{a.e. in } (0,\infty),$$

where F is a continuous, decreasing function from $[0,\infty)$ to $[0,\infty)$. Let $w_0 \in G_{\alpha,\sigma}$. Suppose $w \in C([0,\infty), H_{\rm w}) \cap L^1_{\rm loc}([0,\infty), V)$, with $w' \in L^1_{\rm loc}([0,\infty), V')$, is a weak solution of

$$w' = -Aw + \xi + f \text{ in } V' \text{ on } (0,\infty), \quad w(0) = w_0,$$

Then the following statements hold true.

1. $w(t) \in G_{\alpha+1-\varepsilon,\sigma}$ *for all* $\varepsilon \in (0,1)$ *and* $t > 0$.

Theorem (continued)

2. *For any numbers* $a, a_0 \in (0,1)$ *with* $a + a_0 < 1$ *and any* $\varepsilon \in (0,1)$*, there exists a positive constant* C *depending on* a_0*,* a*,* ε*,* M*,* $F(0)$*,* $|\xi|_{\alpha,\sigma}$ *and* $|w_0|_{\alpha,\sigma}$ *such that*

$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le C\big(e^{-a_0 t} + F(at)\big) \quad \forall t \ge 1.$$

3. *Assume, in addition, that*
   - *There exist* $k_0 > 0$ *and* $D_1 > 0$ *such that*

$$e^{-k_0 t} \le D_1 F(t) \quad \forall t \ge 0, \text{ and} \tag{F1}$$

   - *For any* $a \in (0,1)$*, there exists* $D_2 = D_{2,a} > 0$ *such that*

$$F(at) \le D_2 F(t) \quad \forall t \ge 0. \tag{F2}$$

   Then there exists $C > 0$ *such that*

$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le CF(t) \quad \forall t \ge 1.$$

Parts of proof (2). Small data Gevrey results

Theorem (Cao-H. 2018)

Let F be a continuous, decreasing, non-negative function on $[0,\infty)$. Given $\alpha \ge 1/2$ and numbers $\theta_0, \theta \in (0,1)$ such that $\theta_0 + \theta < 1$. Then there exist positive numbers $c_k = c_k(\alpha, \theta_0, \theta, F)$, for $k = 0,1,2,3$, such that the following holds true. If

$$|A^\alpha u^0| \le c_0,$$
$$|f(t)|_{\alpha-1/2,\sigma} \le c_1 F(t) \quad \text{a.e. in } (0,\infty) \text{ for some } \sigma \ge 0,$$

then there exists a unique regular solution $u(t)$, which also belongs to $C([0,\infty), \mathcal{D}(A^\alpha))$ and satisfies, for all $t \ge 8\sigma(1-\theta)/(1-\theta-\theta_0)$,

$$|u(t)|_{\alpha,\sigma} \le c_2 (e^{-2\theta_0 t} + F^2(\theta t))^{1/2},$$
$$\int_t^{t+1} |u(\tau)|^2_{\alpha+1/2,\sigma} d\tau \le c_3^2 (e^{-2\theta_0 t} + F^2(\theta t)).$$

Parts of proof (3). Estimates for Leray-Hopf weak solutions

Theorem (Cao-H. 2018)

Let F be a continuous, decreasing, non-negative function such that

$$\lim_{t\to\infty} F(t) = 0,$$

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(F(t)), \text{ for some } \sigma \geq 0,\ \alpha \geq 1/2.$$

Let $u(t)$ be a Leray-Hopf weak solution. Then there exists $\hat{T} > 0$ such that $u(t)$ is a regular solution on $[\hat{T}, \infty)$, and for any $\varepsilon, \lambda \in (0,1)$, and $a_0, a, \theta_0, \theta \in (0,1)$ with $a_0 + a < 1$, $\theta_0 + \theta < 1$,

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon,\sigma} \leq C(e^{-a_0 t} + e^{-2\theta_0 a t} + F^{2\lambda}(\theta a t) + F(at)) \quad \forall t \geq 0.$$

If, in addition, F satisfies (F1) and (F2), then

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon,\sigma} \leq CF(t) \quad \forall t \geq 0.$$

Application: iterated logarithmic decaying functions

For $k, m \in \mathbb{N}$, let

$$L_k(t) = \underbrace{\ln(\ln(\cdots \ln(t)))}_{k\text{-times}} \quad \text{and} \quad \mathcal{L}_m(t) = (L_1(t), L_2(t), \cdots, L_m(t)).$$

• Let $Q_0 : \mathbb{R}^m \to R$ be a polynomial in m variables with positive degree and positive leading coefficient:

$$Q_0(z) = \sum_\alpha c_\alpha z^\alpha \text{ for } z \in \mathbb{R}^m.$$

We use the lexicographic order for the multi-indices.
• Let Q_1 be a polynomial in one variable of positive degree with positive leading coefficient.
Given a number $\beta > 0$, we define

$$\omega(t) = (Q_0 \circ \mathcal{L}_m \circ Q_1)(t^\beta)) \text{ with } t \in \mathbb{R}.$$

Let $\psi_\lambda(t) = \omega(t)^{-\lambda}$ and $\Psi = (\psi_\lambda(t))_{\lambda>0}$. Note $\psi'_\lambda \overset{\Psi}{\sim} 0$.

Theorem (Cao-H. 2018)

Assume

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \omega(t)^{-\lambda_n} \quad \textit{in } G_{\alpha,\sigma},$$

for some $\sigma \geq 0$, $\alpha \geq 1/2$. Then any Leray-Hopf weak solution $u(t)$ of the NSE has the asymptotic expansion

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \omega(t)^{-\lambda_n} \quad \textit{in } G_{\alpha+1-\rho,\sigma} \textit{ for all } \rho \in (0,1),$$

where

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\Big(\phi_n - \sum_{\substack{1\leq k,m\leq n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m)\Big) \quad \textit{for } n \geq 2.$$

Corollary (Cao-H. 2018)

Given $m \in \mathbb{N}$, *define* $\Psi = (L_m(t)^{-\lambda})_{\lambda>0}$. *Suppose* $(\lambda_n)_{n=1}^{\infty}$ *is a strictly increasing, divergent sequence of positive numbers such that the set* $\{\lambda_n : n \in \mathbb{N}\}$ *preserves the addition. If*

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n L_m(t)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then any Leray-Hopf weak solution $u(t)$ *of the NSE admits the asymptotic expansion*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n L_m(t)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1).$$

Expansions with trigonometric functions

Example. If

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \left[\sin(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \left[\sin(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

Example. If

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \left[\tan(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \left[\tan(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

Infinite expansions for the derivatives

Consider $\Psi = (\psi_\lambda)_{\lambda>0}$ with $\psi_\lambda = (\sqrt{t}+1)^{-\lambda}$. Then

$$\psi_\lambda'(t) = -\lambda(\sqrt{t}+1)^{-\lambda-1}\frac{1}{2}\frac{1}{\sqrt{t}} = -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-1}\frac{1}{\sqrt{t}+1}\cdot\frac{1}{1-\frac{1}{\sqrt{t}+1}}$$

$$= \sum_{k=1}^{\infty} -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-k-1}.$$

Proposition (Cao-H. 2018)

Assume $f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n(\sqrt{t}+1)^{-\lambda_n}$ *in* $G_{\alpha,\sigma}$. *Then*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n(\sqrt{t}+1)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho\in(0,1),$$

where $\xi_1 = A^{-1}\phi_1$, $\xi_n = A^{-1}(\phi_n + \frac{1}{2}\sum_{p\in\mathcal{Z}_n}\lambda_p\xi_p - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m))$ *for* $n\ge 2$, *with* $\mathcal{Z}_n = \{p\in\mathbb{N}\cap[1,n-1] : \exists k\in\mathbb{N}, \lambda_p+1+k=\lambda_n\}$.

3. Generalized Forchheimer gas flows in porous media

- Generalized Forchheimer flows
- Gas flows: mathematical model
- Estimates of the Lebesgue norms
- Maximum estimates
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Darcy's and Forchheimer's flows

Fluid flows in porous media with velocity v and pressure p:

- Darcy's Law:

$$\alpha v = -\nabla p,$$

- Forchheimer's "two term" law

$$\alpha v + \beta |v| \, v = -\nabla p,$$

- Forchheimer's "three term" law

$$\mathcal{A} v + \mathcal{B} \, |v| \, v + \mathcal{C} |v|^2 \, v = -\nabla p.$$

- Forchheimer's "power" law

$$a v + c^n |v|^{n-1} \, v = -\nabla p,$$

Here $\alpha, \beta, a, c, n, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

Generalized Forchheimer equations

[Aulisa-Bloshanskaya-H.-Ibragimov 2009]: $g(|v|)v = -\nabla p$.
Let $G(s) = sg(s)$. Then $G(|v|) = |\nabla p| \Rightarrow |v| = G^{-1}(|\nabla p|)$. Hence

$$v = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow v = -K(|\nabla p|)\nabla p,$$

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

Class $FP(N, \vec{\alpha})$. Let $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$,

$$FP(N, \vec{\alpha}) = \left\{ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} \right\}.$$

Lemma (Degeneracy)

$$\frac{C_1(\vec{a})}{(1+\xi)^a} \le K(\xi, \vec{a}) \le \frac{C_2(\vec{a})}{(1+\xi)^a},$$

$$C_3(\vec{a})(\xi^{2-a} - 1) \le K(\xi, \vec{a})\xi^2 \le C_2(\vec{a})\xi^{2-a}.$$

Works on Forchheimer flows

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990's, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)

Works on generalized Forchheimer flows

- 1990's Numerical study
- L^2-theory: Aulisa-Bloshanskaya-H.-Ibragimov (2009), H.-Ibragimov: Dirichlet B.C. (2011), H.-Ibragimov Flux B.C. (2012), Aulisa-Bloshanskaya-Ibragimov total flux, productivity index (2011, 2012).
- L^α-theory: H.-Ibragimov-Kieu-Sobol (2015)
- L^∞, $W^{1,p}$-theory: H.-Kieu-Phan (2014).
- $W^{1,\infty}$-theory: interior H.-Kieu (2017), global H.-Kieu (2018-in press).
- Heterogeneous porous media: Celik-H.(2016, 2017).
- Isentropic gases: Celik-H.-Kieu (2018a, 2018b).
- Mixed pre-Darcy, Darcy, Forchheimer flows: Celik-H.-Ibragimov-Kieu (2017)
- Two-phase flows: H.-Ibragimov-Kieu (2013, 2014)
- Numericals: Kieu (2016, 2017, 2018) Ibragimov-Kieu (2016)

Gas flows: mathematical model [Celik-H.-Kieu 2018]

Based on dimensional analysis by Muskat and Ward, we consider

$$\sum_{i=0}^{N} a_i \rho^{\alpha_i} |v|^{\alpha_i} v = -\nabla p + \rho \vec{\mathbf{g}}.$$

Multiplying both sides by ρ, we obtain

$$g(|\rho v|)\rho v = -\rho \nabla p + \rho^2 \vec{\mathbf{g}}.$$

Solving for ρv gives

$$\rho v = -K(|\rho \nabla p - \rho^2 \vec{\mathbf{g}}|)(\rho \nabla p - \rho^2 \vec{\mathbf{g}}),$$

where the function $K : \mathbb{R}^+ \to \mathbb{R}^+$ is defined for $\xi \geq 0$ by

$$K(\xi) = \frac{1}{g(s(\xi))},$$

with $s(\xi) = s$ being the unique non-negative solution of $sg(s) = \xi$.

Conservation of mass:

$$\phi\rho_t + \mathrm{div}(\rho v) = F,$$

where the porosity $\phi \in (0,1)$, F is the *source term*. Then

$$\phi\rho_t = \mathrm{div}(K(|\rho\nabla p - \rho^2\vec{\mathbf{g}}|)(\rho\nabla p - \rho^2\vec{\mathbf{g}})) + F.$$

Isentropic gas flows. In this case

$$p = \bar{c}\rho^\gamma \quad \text{for some constants } \bar{c}, \gamma > 0.$$

Here, γ is the specific heat ratio. Note that $\rho\nabla p = \nabla(\bar{c}\gamma\rho^{\gamma+1}/(\gamma+1))$. Let (pseudo-pressure) $u = \frac{\bar{c}\gamma\rho^{\gamma+1}}{\gamma+1} = \frac{\gamma p^{\frac{\gamma+1}{\gamma}}}{\bar{c}^{\frac{1}{\gamma}}(\gamma+1)}$, we have

$$\phi c^{1/2}(u^\lambda)_t = \nabla\cdot(K(|\nabla u - cu^\ell\vec{\mathbf{g}}|)(\nabla u - cu^\ell\vec{\mathbf{g}})) + F,$$

where $\lambda = \frac{1}{\gamma+1} \in (0,1)$, $\quad \ell = 2\lambda \quad$ and $\quad c = \left(\frac{\gamma+1}{\bar{c}\gamma}\right)^\ell$.

Ideal gases. The equation of state is

$$p = \bar{c}\rho \quad \text{for some constant } \bar{c} > 0.$$

Same equation with $\gamma = 1$, $\lambda = 1/2$, the pseudo-pressure $u \sim p^2$.

Slightly compressible fluids. The equation of state is

$$\frac{1}{\rho}\frac{d\rho}{dp} = \frac{1}{\kappa} = const. > 0.$$

Then $\rho\nabla p = \kappa\nabla\rho$. Same equation with

$$\lambda = 1, \quad u = \kappa\rho, \quad \ell = 2 \quad \text{and} \quad c = 1/\kappa^2.$$

Boundary condition. Volumetric flux condition

$$v \cdot \vec{\nu} = \psi \text{ on } \partial U.$$

This gives $\rho v \cdot \vec{\nu} = \psi\rho$, hence,

$$-K(|\nabla u - cu^\ell \vec{\mathbf{g}}|)(\nabla u - cu^\ell \vec{\mathbf{g}}) \cdot \vec{\nu} = c^{1/2}\psi u^\lambda.$$

$$\begin{cases} \dfrac{\partial(u^\lambda)}{\partial t} = \nabla\cdot\big(K(|\nabla u + Z(u)|)(\nabla u + Z(u))\big) + f(x,t,u) & \text{on } U\times(0,\infty),\\ u(x,0) = u_0(x) & \text{on } U,\\ K(|\nabla u + Z(u)|)(\nabla u + Z(u))\cdot\vec{\nu} = B(x,t,u) & \text{on } \Gamma\times(0,\infty), \end{cases}$$

where $u(x,t) \ge 0$

Assumption (A1). Assume $Z(u) : [0,\infty) \to \mathbb{R}^n$, $B(x,t,u) : \Gamma\times[0,\infty)\times[0,\infty) \to \mathbb{R}$, and $f(x,t,u) : U\times[0,\infty)\times[0,\infty)\to\mathbb{R}$ satisfy

$$|Z(u)| \le d_0 u^{\ell_Z},$$

$$B(x,t,u) \le \varphi_1(x,t) + \varphi_2(x,t)u^{\ell_B},$$

$$f(x,t,u) \le f_1(x,t) + f_2(x,t)u^{\ell_f}$$

with constants $d_0, \ell_Z > 0$, $\ell_f, \ell_B \ge 0$, and functions $\varphi_1, \varphi_2, f_1, f_2 \ge 0$.

Trace and Sobolev inequalities

Lemma

Assume $1 > a > \delta \ge 0$, $\alpha \ge 2 - \delta$, *and* $|u|^\alpha \in W^{1,1}(U)$. *Let* $r > 0$.

(i) *For any* $\varepsilon > 0$ *one has*

$$\int_\Gamma |u|^{\alpha+r} d\sigma \le \varepsilon \int_U |u|^{\alpha-2+\delta} |\nabla u|^{2-a} dx + c_1 \int_U |u|^{\alpha+r} dx + (c_2(\alpha+r))^{\frac{2-a}{1-a}} \varepsilon^{-\frac{1}{1-a}} \int_U |u|^{\alpha + \frac{(2-a)r+a-\delta}{1-a}} dx.$$

(ii) *If* $\alpha > \frac{n(r+a-\delta)}{2-a}$, *then for any* $\varepsilon > 0$ *one has*

$$\int_U |u|^{\alpha+r} dx \le \varepsilon \int_U |u|^{\alpha-2+\delta} |\nabla u|^{2-a} dx + D_1 \|u\|_{L^\alpha}^{\alpha+r} + D_2 \varepsilon^{-\frac{\theta}{1-\theta}} \|u\|_{L^\alpha}^{\alpha\left(1+\frac{2-a}{n}\cdot\frac{\theta}{1-\theta}\right)}.$$

Estimates of the Lebesgue norms

If α is large and $t > 0$ then

$$\frac{d}{dt}\int_U u(x,t)^\alpha dx + \int_U |\nabla u(x,t)|^{2-a} u(x,t)^{\alpha-\lambda-1} dx \le C_0 \cdot \Big(\|u(t)\|_{L^\alpha(U)}^{\nu_1} + \|u(t)\|_{L^\alpha(U)}^{\nu_2} + \Upsilon(t) \Big),$$

where $\Upsilon(t) = \|\varphi_1(t)\|_{L^{q_1}(\Gamma)}^{q_1} + \|\varphi_2(t)\|_{L^{q_2}(\Gamma)}^{q_2} + \|f_1(t)\|_{L^{q_3}(U)}^{q_3} + \|f_2(t)\|_{L^{q_4}(U)}^{q_4}$.

Theorem (Celik-H.-Kieu 2018)

If $\int_0^T (1+\Upsilon(t))dt \le C_1 \cdot (1-2^{-\nu_4})\Big(1+\int_U u_0^\alpha(x)dx\Big)^{-\nu_4}$, *then*

$$\int_U u^\alpha(x,t)dx \le 1 + 2\int_U u_0^\alpha(x)dx \quad \textit{for all} \quad t\in[0,T],$$

$$\int_0^T \int_U |\nabla u|^{2-a} u^{\alpha-\lambda-1} dxdt \le 2^{\nu_4+1}(1+1/\nu_4)\Big(1+\int_U u_0^\alpha(x)dx\Big).$$

Aiming at maximum estimates

Lemma

If α is large and $T > T_2 > T_1 \geq 0$ then

$$\sup_{t\in[T_2,T]} \int_U u^\alpha(x,t)dx + \int_{T_2}^T \int_U |\nabla u(x,t)|^{2-a} u(x,t)^{\alpha-\lambda-1} dxdt$$

$$\leq c_6(1+T)\Big(1+\frac{1}{T_2-T_1}\Big)\alpha^2 \mathcal{M}_0\Big(\|u\|^{\nu_5}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))} + \|u\|^{\nu_6}_{L^{\tilde\kappa\alpha}(U\times(T_1,T))}\Big),$$

where $\mathcal{M}_0 = 1 + \|\varphi_1\|^{\frac{2-a}{1-a}}_{L^{q_1}(\Gamma_T)} + \|\varphi_2\|^{\frac{2-a}{1-a}}_{L^{q_2}(\Gamma_T)} + \|f_1\|_{L^{q_3}(Q_T)} + \|f_2\|_{L^{q_4}(Q_T)}$.

Parabolic Sobolev embedding

Lemma

Assume $1 > a > \delta \geq 0$*,* $\alpha > 0$ *is large, and* $T > 0$*, then*

$$\Big(\int_0^T \int_U |u|^{\kappa\alpha} dxdt\Big)^{\frac{1}{\kappa\alpha}}$$

$$\leq (c_5\alpha^{2-a})^{\frac{1}{\kappa\alpha}}\Big(\int_0^T \int_U |u|^{\alpha+\delta-a} dxdt + \int_0^T \int_U |u|^{\alpha+\delta-2}|\nabla u|^{2-a} dxdt\Big)^{\frac{\tilde\theta}{\alpha+\delta-a}}$$

$$\cdot \sup_{t\in[0,T]}\Big(\int_U |u(x,t)|^{\alpha} dx\Big)^{\frac{1-\tilde\theta}{\alpha}},$$

where $c_5 \geq 1$ *is independent of* α *and* T*, and*

$$\tilde\theta = \tilde\theta_\alpha \overset{\text{def}}{=\!=} \frac{1}{1+\frac{\alpha(2-a)}{n(\alpha+\delta-a)}}, \quad \kappa = \kappa(\alpha) \overset{\text{def}}{=\!=} 1 + \frac{2-a}{n} - \frac{a-\delta}{\alpha} = 1 + (a-\delta)\Big(\frac{1}{\alpha_*} - \frac{1}{\alpha}\Big)$$

Basic inequality

Lemma

If $T > T_2 > T_1 \geq 0$ *then*

$$\|u\|_{L^{\kappa\alpha}(U\times(T_2,T))} \leq A_\alpha^{\frac{1}{\alpha}} \Big(\|u\|^{\nu_5}_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))} + \|u\|^{\nu_7}_{L^{\tilde{\kappa}\alpha}(U\times(T_1,T))} \Big)^{\frac{1}{\alpha}},$$

where $A_\alpha = c_7(1+T)^2\Big(1+\frac{1}{T_2-T_1}\Big)^2 \alpha^{6-a}\mathcal{M}_0^2.$

Fix $\tilde{\kappa}$.
Set up $\kappa = \kappa(\alpha) > \tilde{\kappa}$.
Then we modify Moser's iterations.

Maximum estimates

Theorem (Celik-H.-Kieu 2018)

Let α_0 *be sufficiently large. If* $T > 0$ *and* $\sigma \in (0,1)$ *then*

$$\|u\|_{L^\infty(U\times(\sigma T,T))} \le C\left(1+\frac{1}{\sigma T}\right)^{\omega_1}(1+T)^{\omega_2}\mathcal{M}_0^{\omega_3} \cdot \max\left\{\|u\|^{\tilde\mu}_{L^{\tilde\kappa\alpha_0}(U\times(0,T))}, \|u\|^{\tilde\nu}_{L^{\tilde\kappa\alpha_0}(U\times(0,T))}\right\}.$$

Theorem (Celik-H.-Kieu 2018)

Let α_0 *be sufficiently large. If* $T > 0$ *is small, then for* $0 < \varepsilon < \min\{1, T\}$, *one has*

$$\|u\|_{L^\infty(U\times(\varepsilon,T))} \le C\varepsilon^{-\omega_1}(1+T)^{\omega_2+\frac{\tilde\nu}{\beta_1}}(1+\|u_0(x)\|_{L^{\beta_1}(U)})^{\tilde\nu}\mathcal{M}_0^{\omega_3}.$$

Gradient estimates

Assumption (A2).

(i) The function $Z(u)$ satisfies

$$|Z'(u)| \le d_4 u^{\ell_Z - 1} \quad \forall u \in (0,\infty),$$

for some constant $d_4 > 0$.

(ii) There are non-negative functions $\varphi_3(x,t)$ and $\varphi_4(x,t)$ defined on $\Gamma \times (0,\infty)$ such that

$$\left|\frac{\partial B(x,t,u)}{\partial t}\right| \le \varphi_3(x,t) + \varphi_4(x,t) u^{\ell_B} \quad \forall (x,t,u) \in \Gamma \times [0,\infty) \times [0,\infty).$$

(iii) We also assume

$$|B(x,t,u)| \le \varphi_1(x,t) + \varphi_2(x,t) u^{\ell_B} \quad \forall (x,t,u) \in \Gamma \times [0,\infty) \times [0,\infty),$$

$$|f(x,t,u)| \le f_1(x,t) + f_2(x,t) u^{\ell_f} \quad \forall (x,t,u) \in U \times [0,\infty) \times [0,\infty).$$

Assumption (A3).

$$2\ell_Z > \lambda + 1.$$

Remark. For our original problem, $\ell_Z = 2\lambda$, then Assumption (A3) becomes $\lambda > 1/3$.

- For slightly compressible fluids, $\lambda = 1$.
- For ideal gases, $\lambda = 1/2$.
- For isentropic gas flows, all values of the specific heat ratio γ found belong to the interval $(1, 2)$, therefore $\lambda = 1/(1+\gamma)$, satisfies

$$1/3 < \lambda < 1/2.$$

Thus Assumption (A3) is naturally met in all cases.

Let

$$\mathcal{I}(t) = \int_U H\Big(|\nabla u(x,t) + Z(u(x,t))|\Big)dx,$$

$$\mathcal{Z}_0 = \int_U u_0^{\lambda+1}(x)dx + \mathcal{I}(0) + \int_\Gamma \Big(\varphi_1(x,0)u_0(x) + \varphi_2(x,0)u_0^{\ell_B+1}(x)\Big)d\sigma,$$

$$N_1(t) = \int_\Gamma \Big(\varphi_1^{\eta_5}(x,t) + \varphi_2^2(x,t)\Big)d\sigma,$$

$$N_2(t) = N_1(t) + \int_\Gamma \Big(\varphi_3^{\eta_5}(x,t) + \varphi_4^2(x,t)\Big)d\sigma + \int_U \Big(f_1^{\eta_6}(x,t) + f_2^4(x,t)\Big)dx.$$

Theorem

If $T > 0$ is small, then for all $t \in (0, T]$

$$\int_U |\nabla u(x,t)|^{2-a}dx \le C\Big\{\mathcal{Z}_0 + (t+1)\Big(1 + \int_U u_0^{\eta_7}(x)dx\Big) + N_1(t) + \int_0^t N_2(\tau)d\tau\Big\}.$$

4. Flows of mixed regimes

- Unified models
- Estimates for solutions
- Continuous dependence on the boundary data
- Structural stability

Different regimes

- Darcy:

$$v = -k\nabla p.$$

- Pre-Darcy: When $|v|$ is small,

$$|v|^{-\alpha} v = -k\nabla p, \alpha \in (0,1).$$

- Post-Darcy:

$$(a_0 + a_1|v|^{\alpha_1} + \ldots + a_N|v|^{\alpha_N})v = -\nabla p.$$

Unified form [Celik-H.-Ibragimov-Kieu 2017]

$$-\nabla p = \mathbf{G}(v) = \begin{cases} g(|v|)v & \text{if } v \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } v = 0, \end{cases}$$

where $g(s) > 0$ on $(0, \infty)$ and $\lim_{s \searrow 0} sg(s) = 0$.

Solve for v. Taking the modulus both sides, we have $G(|v|) = |\nabla p|$, where

$$G(s) = \begin{cases} sg(s) & \text{if } s > 0, \\ 0 & \text{if } s = 0. \end{cases}$$

We assume

- $G(s)$ is strictly increasing on $[0, \infty)$,
- $G(s) \to \infty$ as $s \to \infty$, and
- the function $1/g(s)$ on $(0, \infty)$ can be extended to a continuous function $k_g(s)$ on $[0, \infty)$.

Then

$$v = -K(|\nabla p|)\nabla p,$$

where $K(\xi) = k_g(G^{-1}(\xi)) \quad \text{for } \xi \geq 0.$

Two models of g

Model 1. Function $g(s)$ is piece-wise defined:

$$g(s) = \bar{g}(s) \stackrel{\text{def}}{=\!=} c_1 s^{-\alpha} \mathbf{1}_{(0,s_1)}(s) + c_2 \mathbf{1}_{[s_1,s_2]}(s) + g_F(s) \mathbf{1}_{(s_2,\infty)}(s) \quad \text{for } s > 0,$$

Continuity condition:

$$c_1 s_1^{-\alpha} = c_2 = g_F(s_2).$$

$$K(\xi) = \bar{K}(\xi) \stackrel{\text{def}}{=\!=} M_1 \xi^{\beta_1} \mathbf{1}_{[0,Z_1)}(\xi) + M_2 \mathbf{1}_{[Z_1,Z_2]}(\xi) + K_F(\xi) \mathbf{1}_{(Z_2,\infty)}(\xi).$$

Model 2. Function $g(s)$ is smooth on $(0,\infty)$:

$$g(s) = g_I(s) \stackrel{\text{def}}{=\!=} a_{-1} s^{-\alpha} + a_0 + a_1 s^{\alpha_1} + \cdots + a_N s^{\alpha_N} \quad \text{for } s > 0, \tag{4.1}$$

where $N \geq 1$, $\alpha \in (0,1)$, $\alpha_N > 0$,

$$a_{-1}, a_N > 0 \text{ and } a_i \geq 0 \quad \forall i = 0, 1, \ldots, N-1.$$

$$K(\xi) = K_I(\xi) \stackrel{\text{def}}{=\!=} \frac{s(\xi)^\alpha}{a_{-1} + a_0 s(\xi)^\alpha + a_1 s(\xi)^{\alpha+\alpha_1} + \ldots + a_N s(\xi)^{\alpha+\alpha_N}},$$

with $G(s(\xi)) = \xi$.

Two direct models of K

$$v = -K(|\nabla p|)\nabla p.$$

Note: $K(\xi)$ behaves like ξ^{β_1} for small ξ, and like $(1+\xi)^{-\beta_2}$ for large ξ,

$$\beta_1 = \frac{\alpha}{1-\alpha}, \quad \beta_2 = \frac{\alpha_N}{\alpha_N + 1}.$$

Model 3.

$$K(\xi) = \hat{K}(\xi) \stackrel{\text{def}}{=\!=} \frac{a\xi^{\beta_1}}{(1+b\xi^{\beta_1})(1+c\xi^{\beta_2})}.$$

Model 4. More precisely, $K(\xi)$ is close to $M_1\xi^{\beta_1}$ when $\xi \to 0$, and to $K_F(\xi)$ when $\xi \to \infty$. Then we choose

$$K(\xi) = K_M(\xi) \stackrel{\text{def}}{=\!=} K_F(\xi) \cdot \frac{\bar{k}\xi^{\beta_1}}{1+\bar{k}\xi^{\beta_1}}.$$

where $\bar{k} = M_1/K_F(0) > 0$.

Initial Boundary Value Problem

Let $K(\xi)$ be one of the functions $\bar{K}(\xi)$, $K_I(\xi)$, $\hat{K}(\xi)$, $K_M(\xi)$.
Slightly compressible fluids (with simplification.)
After scaling the time variable (to simplify ϕ), we obtain the IBVP:

$$\begin{cases} p_t = \nabla \cdot (K(|\nabla p|)\nabla p) & \text{in } U \times (0,\infty), \\ p(x,0) = p_0(x), & \text{in } U \\ p = \psi(x,t), & \text{on } \partial U \times (0,\infty). \end{cases}$$

Lemma (Degeneracy)

Then there exist $d_2, d_3 > 0$ such that

$$\frac{d_2 \xi^{\beta_1}}{(1+\xi)^{\beta_1+\beta_2}} \le K(\xi) \le \frac{d_3 \xi^{\beta_1}}{(1+\xi)^{\beta_1+\beta_2}} \quad \forall \xi \ge 0.$$

Energy estimates

Let $\bar{p} = p - \Psi$, where $\Psi(x,t)$ is an extension of ψ from $x \in \partial U$ to $x \in \bar{U}$.

Theorem (Celik-H.-Ibragimov-Kieu 2017)

(i) There exists a positive constant C such that for all $t \ge 0$,

$$\|\bar{p}(t)\|^2 \le \|\bar{p}(0)\|^2 + C\left[1 + Env(f(t))\right]^{\frac{2}{2-\beta_2}},$$

where $f(t) = f[\Psi](t) \stackrel{\rm def}{=\!=} \|\nabla\Psi(t)\|^2 + \|\Psi_t(t)\|^{\frac{2-\beta_2}{1-\beta_2}}$.
(ii) Furthermore,

$$\limsup_{t\to\infty} \|\bar{p}(t)\|^2 \le C(1 + \limsup_{t\to\infty} f(t))^{\frac{2}{2-\beta_2}}.$$

(iii) If $\lim_{t\to\infty} \|\nabla\Psi(t)\| = \lim_{t\to\infty} \|\Psi_t(t)\| = 0$, *then*

$$\lim_{t\to\infty} \|\bar{p}(t)\| = 0.$$

Gradient Estimates

Theorem (Celik-H.-Ibragimov-Kieu 2017)

(i) For all $t \geq 0$,

$$\int_U |\nabla p(x,t)|^{2-\beta_2} dx \leq C\Big(1 + \|\bar{p}(0)\|^2 + e^{-\frac{t}{2}} \int_U |\nabla p(x,0)|^{2-\beta_2} dx + \big[Env(f(t))\big]^{\frac{2}{2-\beta_2}} + \int_0^t e^{-\frac{1}{2}(t-\tau)} \|\nabla \Psi_t(\tau)\|^2 d\tau\Big).$$

(ii) $\limsup_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx \leq C(1 + \limsup_{t\to\infty} G_1(t))$,
where $G_1(t) = G_1[\Psi](t) \stackrel{\text{def}}{=\!=} f(t)^{\frac{2}{2-\beta_2}} + \|\nabla \Psi_t(t)\|^2$.
(iii) If $\lim_{t\to\infty} \|\nabla \Psi(t)\| = \lim_{t\to\infty} \|\Psi_t(t)\| = \lim_{t\to\infty} \|\nabla \Psi_t(t)\| = 0$ *then*

$$\lim_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx = 0.$$

Improvements for large t

Theorem (Celik-H.-Ibragimov-Kieu 2017)

(i) If $t \ge 1$ then

$$\int_U |\nabla p(x,t)|^{2-\beta_2} dx \le C\Big(1 + \|\bar{p}(0)\|^2 + [Env(f(t))]^{\frac{2}{2-\beta_2}} + \int_{t-1}^t \|\nabla \Psi_t(\tau)\|^2 d\tau\Big)$$

(ii) One has

$$\limsup_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx \le C\Big(1 + \limsup_{t\to\infty} f(t)^{\frac{2}{2-\beta_2}} + \limsup_{t\to\infty} \int_{t-1}^t \|\nabla \Psi_t(\tau)\|^2 d\tau\Big).$$

(iii) Moreover, $\lim_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx = 0$ *provided* $\lim_{t\to\infty} \|\nabla \Psi(t)\| = \lim_{t\to\infty} \|\Psi_t(t)\| = \lim_{t\to\infty} \int_{t-1}^t \|\nabla \Psi_t(\tau)\|^2 d\tau = 0.$

Continuous dependence on the boundary data

- We consider $K(\xi) = \bar{K}(\xi)$, $K_I(\xi)$, $\hat{K}(\xi)$, $K_M(\xi)$.
- For $i = 1, 2$, let $p_i(x,t)$ be a solution with boundary data $\psi_i(x,t)$, with extensions $\Psi_i(x,t)$. Let $\bar{p}_i = p_i - \Psi_i$.
- Denote $\Phi = \Psi_1 - \Psi_2$ and $\bar{P} = \bar{p}_1 - \bar{p}_2 = p_1 - p_2 - \Phi$.
- Set $\mathcal{Y}_0 = 1 + \sum_{i=1,2} \left(\|\bar{p}_i(0)\|^2 + \|\nabla p_i(0)\|_{L^{2-\beta_2}}^{2-\beta_2} \right)$,

$$\widetilde{\mathcal{Y}}(t) = \mathcal{Y}_0 + \sum_{i=1,2} [Env(f[\Psi_i](t))]^{\frac{2}{2-\beta_2}}$$
$$+ \begin{cases} \int_0^t e^{-\frac{1}{2}(t-\tau)} \sum_{i=1,2} \|\nabla \Psi_{i,t}(\tau)\|^2 d\tau & \text{if } 0 \le t < 1, \\ \int_{t-1}^t \sum_{i=1,2} \|\nabla \Psi_{i,t}(\tau)\|^2 d\tau & \text{if } t \ge 1. \end{cases}$$

- Let

$$D(t) = \|\Phi_t(t)\| + \|\nabla \Phi(t)\|_{L^{2-\beta_2}} + \|\nabla \Phi(t)\|_{L^{2+\beta_1}}^{2+\beta_1}.$$

and $\mathcal{D} = \limsup_{t\to\infty} D(t)$.

- For asymptotic estimates, we use

$$\widetilde{\mathcal{A}} = \Big(\sum_{i=1,2} \limsup_{t\to\infty} f[\Psi_i](t) \Big)^{\frac{1}{2-\beta_2}},$$

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{A}}^2 + \sum_{i=1,2} \limsup_{t\to\infty} \int_{t-1}^t \|\nabla \Psi_{i,t}(\tau)\|^2 d\tau.$$

Theorem (Celik-H.-Ibragimov-Kieu 2017)

For $t \ge 0$,

$$\|\bar{P}(t)\|^2 \le \|\bar{P}(0)\|^2 + C\Big\{ Env\Big[\widetilde{\mathcal{Y}}(t)^{\frac{\beta_1+\beta_2}{2-\beta_2}+\frac12} D(t) \Big] \Big\}^{\frac{2}{2+\beta_1}}.$$

If $\widetilde{\mathcal{K}} < \infty$ *then*

$$\limsup_{t\to\infty} \|\bar{P}(t)\|^2 \le C\Big\{ (1+\widetilde{\mathcal{K}})^{\frac{\beta_1+\beta_2}{2-\beta_2}+\frac12} \mathcal{D} \Big\}^{\frac{2}{2+\beta_1}}.$$

Structural stability

- Consider $K(\xi) = K_I(\xi, \vec{a})$ and study the dependence of the solutions on the coefficient vector $\vec{a}$.
- Let $N \geq 1$ and the exponent vector $\vec{\alpha} = (-\alpha, 0, \alpha_1, \ldots, \alpha_N)$ be fixed.
- Denote the set of admissible $\vec{a}$

$$S = \{\vec{a} = (a_{-1}, a_0, \ldots, a_N) : a_{-1}, a_N > 0, a_0, a_1, \ldots, a_{N-1} \geq 0\}.$$

Lemma (Perturbed Monotonicity)

For any coefficient vectors $\vec{a}^{(1)}$, $\vec{a}^{(2)} \in S$, and any $y, y' \in \mathbb{R}^n$, one has

$$(K_I(|y'|, \vec{a}^{(1)})y' - K_I(|y|, \vec{a}^{(2)})y) \cdot (y' - y) \geq \frac{d_6 |y - y'|^{2+\beta_1}}{(1 + |y| + |y'|)^{\beta_1+\beta_2}} - d_7 K\left(|y| \vee |y'|, \vec{a}^{(1)} \wedge \vec{a}^{(2)}\right) (|y| \vee |y'|) \, |\vec{a}^{(1)} - \vec{a}^{(2)}| \, |y - y'|.$$

$$\mathcal{Y}(t) = \mathcal{Y}_0 + [Env(f(t))]^{\frac{2}{2-\beta_2}} + \begin{cases} \int_0^t \|\nabla\Psi_t(\tau)\|^2 d\tau & \text{if } 0 \le t < 1, \\ \int_{t-1}^t \|\nabla\Psi_t(\tau)\|^2 d\tau & \text{if } t \ge 1, \end{cases}$$

$$\mathcal{A} = \limsup_{t\to\infty} f(t)^{\frac{1}{2-\beta_2}} \quad \text{and} \quad \mathcal{K} = \mathcal{A}^2 + \limsup_{t\to\infty} \int_{t-1}^t \|\nabla\Psi_t(\tau)\|^2 d\tau.$$

Theorem (Celik-H.-Ibragimov-Kieu 2017)

(i) *For $t \ge 0$, one has*

$$\int_U |P(x,t)|^2 dx \le \int_U |P(x,0)|^2 dx + C[Env(\mathcal{Y}(t))]^{\frac{2}{2-\beta_2}} |\vec{a}^{(1)} - \vec{a}^{(2)}|^{\frac{2}{2+\beta_1}}.$$

(ii) *If $\mathcal{K} < \infty$ then*

$$\limsup_{t\to\infty} \int_U |P(x,t)|^2 dx \le C(1+\mathcal{K})^{\frac{2}{2-\beta_2}} |\vec{a}^{(1)} - \vec{a}^{(2)}|^{\frac{2}{2+\beta_1}}.$$

5. Publications and Funding

Publications

1. D. Cao, L. Hoang, *Asymptotic expansions in a general system of decaying functions for solutions of the Navier-Stokes equations,* 54 pp, submitted.
2. D. Cao, L. Hoang, *Long-time asymptotic expansions for Navier-Stokes equations with power-decaying forces,* Proceedings A of the Royal Society of Edinburgh, 35 pp, accepted.
3. L. Hoang, T. Kieu, *Global estimates for generalized Forchheimer flows of slightly compressible fluids,* Journal d'Analyse Mathematique, (2018), 54 pp, in press.
4. E. Celik, L. Hoang, T. Kieu, *Doubly nonlinear parabolic equations for a general class of Forchheimer gas flows in porous media,* Nonlinearity, Vol. 31, No. 8 (2018) 3617–3650.
5. C. Foias, L. Hoang, J.-C. Saut, *Navier and Stokes meet Poincaré and Dulac,* J. Appl. Anal. Comput., Volume 8, Number 3, (June 2018) 727–763. (survey)
6. L. Hoang, V. Martinez, *Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces,* J. Math. Anal. Appl., Volume 462, Issue 1, (1 June 2018) 84–113.
7. E. Celik, L. Hoang, T. Kieu, *Generalized Forchheimer flows of isentropic gases,* J. Math. Fluid Mech., (March 2018) Volume 20, Issue 1, 83–115.
8. L. Hoang, E. Olson, J. Robinson, *Continuity of pullback and uniform attractors,* J. Differential Equations, Volume 264, Issue 6, (15 March 2018) 4067–4093.

9. L. Hoang, T. Kieu, *Interior estimates for generalized Forchheimer flows of slightly compressible fluids,* Advanced Nonlinear Studies, 17(4), (October 2017) 739–767.
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11. Emine Celik, L. Hoang, A. Ibragimov, T. Kieu, *Fluid flows of mixed regimes in porous media,* J. Math. Phys., Volume 58 (2017), No. 2, 023102, 30 pp.
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13. L. Hoang, T. Nguyen, T. Phan, *Local gradient estimates for degenerate elliptic equations,* Advanced Nonlinear Studies, Volume 16, Issue 3 (Aug 2016), 479–489.
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32. C. Foias, L. Hoang, E. Olson, M. Ziane, *On the solutions to the normal form of*

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1. **Nonlinear Couplings for Flows in Fractured Porous Media: Analysis and Numerical Algorithms**
   National Science Foundation - Applied Mathematics (DMS 1412796)
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