Developments in Asymptotic Expansions for Solutions of Navier-Stokes Equations

Joint work with Dat Cao (Texas Tech University)
Luan T. Hoang

Department of Mathematics and Statistics, Texas Tech University

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1. The Navier-Stokes systems

- Foias-Saut asymptotic expansion
- Exponentially decaying forces
- Power-decaying forces

The Navier-Stokes equations

- The Navier-Stokes equations (NSE) in $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f(x,t), \\ \mathrm{div}\ u = 0, \\ u(x,0) = u^0(x), \end{cases}$$

with viscosity $\nu > 0$, velocity field $u(x,t) \in \mathbb{R}^3$, pressure $p(x,t) \in \mathbb{R}$, body force $f(x,t) \in \mathbb{R}^3$, initial velocity $u^0(x)$.

- Let $L > 0$ and $\Omega = (0,L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1,2,3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Functional setting

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued 2π-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

$$H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$
$$V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \quad \mathcal{D}(A) = \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.$$

Norm on H: $|u| = \|u\|_{L^2(\Omega)}$. Norm on V: $\|u\| = |\nabla u|$.
The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u,v) = \mathbb{P}_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}(A).$$

$\mathbb{P}_L$ is the Leray projection from $L^2(\Omega)$ onto H.
WLOG, assume $f(t) = \mathbb{P}_L f(t)$. The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = f(t), \ t > 0,$$

$$u(0) = u^0.$$

Case $f = 0$. Foias-Saut asymptotic expansion

Foias-Saut (1987) for a solution $u(t)$:

$$u(t) \sim \sum_{n=1}^{\infty} q_j(t) e^{-jt},$$

where $q_j(t)$ is a $\mathcal{V}$-valued polynomial in t. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$, the remainder $v_N(t) = u(t) - \sum_{j=1}^{N} q_j(t) e^{-jt}$ satisfies

$$\|v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.

Theorem (H.-Martinez 2017)

The Foias-Saut expansion holds in all Gevrey spaces:

$$\|e^{\sigma A^{1/2}} v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}),$$

for any $\sigma > 0$, $\varepsilon \in (0,1)$.

Gevrey classes

- Spectrum of A is $\{|k|^2 : k \in \mathbb{Z}^3, k \neq 0\}$.
- For $\alpha \geq 0$, $\sigma \geq 0$, define

$$A^\alpha e^{\sigma A^{1/2}} u = \sum_{\mathbf{k}\neq 0} |\mathbf{k}|^{2\alpha} \hat{u}(\mathbf{k}) e^{\sigma|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ for } u = \sum_{\mathbf{k}\neq 0} \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \in H.$$

The domain of $A^\alpha e^{\sigma A^{1/2}}$ is

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) = \{u \in H : |u|_{\alpha,\sigma} \stackrel{\text{def}}{=\!=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty\}.$$

- Compare the Sobolev and Gevrey norms:

$$|A^\alpha u| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} u| \leq \left(\frac{2\alpha}{e\sigma}\right)^{2\alpha} |e^{\sigma A^{1/2}} u|.$$

Notation.

- Denote for $\sigma \in \mathbb{R}$ the space $E^{\infty,\sigma} = \bigcap_{\alpha\geq 0} G_{\alpha,\sigma} = \bigcap_{m\in\mathbb{N}} G_{m,\sigma}$.
- Denote by $\mathcal{P}^{\alpha,\sigma}$ the space of $G_{\alpha,\sigma}$-valued polynomials in case $\alpha \in \mathbb{R}$, and the space of $E^{\infty,\sigma}$-valued polynomials in case $\alpha = \infty$.

Definition

Let X be a real vector space.
(a) An X-valued polynomial is a function $t \in \mathbb{R} \mapsto \sum_{n=1}^{d} a_n t^n$, for some $d \geq 0$, and a_n's belonging to X.
(b) In case $\|\cdot\|$ is a norm on X, a function $g(t)$ from $(0,\infty)$ to X is said to have the asymptotic expansion

$$g(t) \sim \sum_{n=1}^{\infty} g_n(t) e^{-nt} \text{ in } X,$$

where $g_n(t)$'s are X-valued polynomials, if for all $N \geq 1$, there exists $\varepsilon_N > 0$ such that

$$\left\| g(t) - \sum_{n=1}^{N} g_n(t) e^{-nt} \right\| = \mathcal{O}(e^{-(N+\varepsilon_N)t}) \text{ as } t \to \infty.$$

• We will say that an asymptotic expansion holds in $E^{\infty,\sigma}$ if it holds in $G_{\alpha,\sigma}$ for all $\alpha \geq 0$.

Exponentially decaying forces

Assumptions.

(A1) The function $f(t)$ is continuous from $[0,\infty)$ to H.

(A2) There are a number $\sigma_0 \ge 0$, E^{∞,σ_0}-valued polynomials $f_n(t)$ for all $n \ge 1$ such that

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-nt} \text{ in } E^{\infty,\sigma_0}.$$

Consequently,

$$f(t) \to 0 \text{ as } t \to \infty, \text{ in } G_{\alpha,\sigma_0} \text{ for all } \alpha > 0.$$

Theorem (H.-Martinez 2018)

Let $u(t)$ be a Leray-Hopf weak solution. Then $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt} \quad \text{in } E^{\infty,\sigma_0}.$$

Moreover, the mappings

$$u_n(t) \stackrel{\text{def}}{=\!=} q_n(t) e^{-nt} \quad \text{and} \quad F_n(t) \stackrel{\text{def}}{=\!=} f_n(t) e^{-nt},$$

satisfy the following ordinary differential equations in the space E^{∞,σ_0}

$$\frac{d}{dt} u_n(t) + A u_n(t) + \sum_{\substack{k,m\geq 1\\ k+m=n}} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R}, \qquad (\star)$$

for all $n \geq 1$.

Finite asymptotic approximation

Theorem (H.-Martinez 2018)

Suppose there exist an integer $N_* \geq 1$*, real numbers* $\sigma_0 \geq 0$*,* $\mu_* \geq \alpha_* \geq N_*/2$*, and, for any* $1 \leq n \leq N_*$*, numbers* $\delta_n \in (0,1)$ *and polynomials* $f_n \in \mathcal{P}^{\mu_n,\sigma_0}$*, such that*

$$\Big| f(t) - \sum_{n=1}^{N} f_n(t) e^{-nt} \Big|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\delta_N)t}) \quad \text{as } t \to \infty,$$

for $1 \leq N \leq N_*$*, where*

$$\mu_n = \mu_* - (n-1)/2, \quad \alpha_n = \alpha_* - (n-1)/2.$$

Theorem (continued)

Let $u(t)$ be a Leray-Hopf weak .

(i) *Then there exist polynomials $q_n \in \mathcal{P}^{\mu_n+1,\sigma_0}$, for $1 \le n \le N_*$, such that one has for $1 \le N \le N_*$ that*

$$\left| u(t) - \sum_{n=1}^{N} q_n(t) e^{-nt} \right|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0, \delta_N^*),$$

where $\delta_N^* = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$.
Moreover, the ODEs

$$\frac{d}{dt} u_n(t) + A u_n(t) + \sum_{\substack{k,m \ge 1 \\ k+m=n}} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R}, \qquad (\star)$$

hold in the corresponding space G_{μ_n,σ_0} for $1 \le n \le N_*$.

(ii) *In particular, if all $f_n(t)$'s belong to $\mathcal{V}$, resp., E^{∞,σ_0}, then so do all $q_n(t)$'s, and the ODEs $(\star)$ hold in $\mathcal{V}$, resp., E^{∞,σ_0}.*

Power-decaying forces

Power asymptotic expansion in $(X, \|\cdot\|)$: $g(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^\infty g_n t^{-n}$ means

$$\|g(t) - \sum_{n=1}^N g_n t^{-n}\| = \mathcal{O}(t^{-(N+\varepsilon)}), \quad \text{for some } \varepsilon > 0, \quad t \to \infty.$$

Theorem (Cao-H. 2017)

Assume that $f(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^\infty \phi_n t^{-n}$ *in* G_{α,σ_0}, *for some* $\sigma_0 \ge 0$ *and* $\alpha \ge 1/2$, *sequence* $\{\phi_n\}_{n=1}^\infty$ *in* G_{α,σ_0}. *Then any Leray-Hopf weak solution* $u(t)$ *has the asymptotic expansion*

$$u(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^\infty \xi_n t^{-n} \quad \text{in } G_{\alpha,\sigma_0},$$

where $\xi_1 = A^{-1}\phi_1$,
$\xi_n = (n-1)A^{-1}\xi_{n-1} - \sum_{k,m\ge1,k+m=n} A^{-1}B(\xi_k,\xi_m) + A^{-1}\phi_n$ *for* $n \ge 2$.

2. Expansions in a general system of decaying functions

- Continuum systems
- Asymptotic expansions for NSE
- Finite asymptotic approximations
- Applications

Expansions in a general system of decaying functions

Definition (Very/Too general)

Let $(\psi_n)_{n=1}^{\infty}$ be a sequence of non-negative functions defined on $[T_*, \infty)$ for some $T_* \in \mathbb{R}$ that satisfies the following two conditions:

1. For each $n \in \mathbb{N}$, $\lim_{t\to\infty} \psi_n(t) = 0$.
2. For $n > m$, $\psi_n(t) = o(\psi_m(t))$.

Let $(X, \|\cdot\|)$ be a normed space, and g be a function from $[T_*, \infty)$ to X.

$$g(t) \sim \sum_{n=1}^{\infty} \xi_n \psi_n(t) \text{ in } X,$$

where $\xi_n \in X$ for all $n \in \mathbb{N}$, if, for any $N \in \mathbb{N}$,

$$\|g(t) - \sum_{n=1}^{N} \xi_n \psi_n(t)\| = o(\psi_N(t)).$$

A continuum system

Definition

Let $\Psi = (\psi_\lambda)_{\lambda>0}$ be a system of functions that satisfies the following two conditions.

(a) There exists $T_* \geq 0$ such that, for each $\lambda > 0$, ψ_λ is a positive function defined on $[T_*, \infty)$, and

$$\lim_{t\to\infty} \psi_\lambda(t) = 0.$$

(b) For any $\lambda > \mu$, there exists $\eta > 0$ such that

$$\psi_\lambda(t) = \mathcal{O}(\psi_\mu(t)\psi_\eta(t)).$$

Asymptotic expansions in a continuum system

Definition

Let $(X, \|\cdot\|)$ be a real normed space, and $g : (0,\infty) \to X$.

$$g(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \text{ in } X,$$

where $\xi_n \in X$ for all $n \in \mathbb{N}$, and $(\lambda_n)_{n=1}^{\infty}$ is a strictly increasing, divergent sequence of positive numbers, if it holds, for any $N \geq 1$, that there exists $\varepsilon > 0$ such that

$$\Big\| g(t) - \sum_{n=1}^{N} \xi_n \psi_{\lambda_n}(t) \Big\| = \mathcal{O}(\psi_{\lambda_N}(t) \psi_{\varepsilon}(t)).$$

Condition

The system $\Psi = (\psi_\lambda)_{\lambda>0}$ *satisfies* (a) *and* (b) *and the following.*

1. *For any* $\lambda, \mu > 0$*, there exist* $\gamma > \max\{\lambda, \mu\}$ *and a nonzero constant* $d_{\lambda,\mu}$ *such that*
$$\psi_\lambda \psi_\mu = d_{\lambda,\mu} \psi_\gamma.$$
Notation. $\gamma = \lambda \wedge \mu$.
2. *For each* $\lambda > 0$*, the function* ψ_λ *is continuous and differentiable on* $[T_*, \infty)$*, and its derivative* ψ'_λ *has an expansion*
$$\psi'_\lambda(t) \overset{\Psi}{\sim} \sum_{k=1}^{N_\lambda} c_{\lambda,k} \psi_{\lambda^\vee(k)}(t) \text{ in } \mathbb{R},$$
where $N_\lambda \in \mathbb{N} \cup \{0, \infty\}$*, all* $c_{\lambda,k}$ *are constants, all* $\lambda^\vee(k) > \lambda$*, and, for each* $\lambda > 0$*,* $\lambda^\vee(k)$*'s are strictly increasing in* k*.*

Condition

The system $\Psi = (\psi_\lambda)_{\lambda>0}$ *satisfies* (a), (b) *and the following.*

1. *For each* $\lambda > 0$, *the function* ψ_λ *is decreasing (in* t*).*
2. *If* $\lambda, \alpha > 0$ *then*
$$e^{-\alpha t} = o(\psi_\lambda(t)).$$
3. *For any number* $a \in (0,1)$,
$$\psi_\lambda(at) = \mathcal{O}(\psi_\lambda(t)).$$

Consequently, for any $T \in \mathbb{R}$,

$$\psi_\lambda(t+T) = \mathcal{O}(\psi_\lambda(t)).$$

Assumption

The function f *belongs to* $L^\infty_{\rm loc}([0,\infty), H)$.

Asymptotic expansions for NSE

Assumption

(1) *Suppose there exist real numbers* $\sigma \geq 0$, $\alpha \geq 1/2$, *a strictly increasing, divergent sequence of positive numbers* $(\gamma_n)_{n=1}^\infty$ *and a sequence* $(\tilde{\phi}_n)_{n=1}^\infty$ *in* $G_{\alpha,\sigma}$ *such that*

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \tilde{\phi}_n \psi_{\gamma_n}(t) \text{ in } G_{\alpha,\sigma}.$$

(2) *There exists a set* S_* *that contains* $\{\gamma_n : n \in \mathbb{N}\}$, *preserves the operations* $\vee$ *and* $\wedge$, *and can be ordered so that*

$$S_* = \{\lambda_n : n \in \mathbb{N}\}, \text{ where } \lambda_n\text{'s are strictly increasing to infinity.}$$

We rewrite

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n \psi_{\lambda_n}(t) \quad \text{in } G_{\alpha,\sigma} \quad \text{as } t \to \infty,$$

Theorem (Cao-H. 2018)

Any Leray-Hopf weak solution $u(t)$ has the asymptotic expansion

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \psi_{\lambda_n}(t) \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

where ξ_n's are defined recursively by

$$\xi_1 = A^{-1}\phi_1,$$
$$\xi_n = A^{-1}\Big(\phi_n - \chi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k \wedge \lambda_m = \lambda_n}} d_{\lambda_k,\lambda_m} B(\xi_k,\xi_m)\Big) \quad \text{for } n \ge 2,$$

where

$$\chi_n = \begin{cases} \displaystyle\sum_{\substack{(p,k)\in[1,n-1]\times\mathbb{N}:\\ \lambda_p^\vee(k)=\lambda_n}} c_{\lambda_p,k}\xi_p, & \text{if } \exists p \in [1,n-1], k\in\mathbb{N} : \lambda_p^\vee(k) = \lambda_n,\\ 0, & \text{otherwise.} \end{cases}$$

Parts of proof (1). Linearized NSE

Theorem (Cao-H. 2018)

Given $\alpha, \sigma \geq 0$, *let* $\xi \in G_{\alpha,\sigma}$, *and* f *be a function from* $(0,\infty)$ *to* $G_{\alpha,\sigma}$ *that satisfies*

$$|f(t)|_{\alpha,\sigma} \leq MF(t) \quad a.e.\ in\ (0,\infty),$$

where F *is a continuous, decreasing function from* $[0,\infty)$ *to* $[0,\infty)$. *Let* $w_0 \in G_{\alpha,\sigma}$. *Suppose* $w \in C([0,\infty), H_{\rm w}) \cap L^1_{\rm loc}([0,\infty), V)$, *with* $w' \in L^1_{\rm loc}([0,\infty), V')$, *is a weak solution of*

$$w' = -Aw + \xi + f \ \ in\ V'\ on\ (0,\infty), \quad w(0) = w_0,$$

Then the following statements hold true.

1. $w(t) \in G_{\alpha+1-\varepsilon,\sigma}$ *for all* $\varepsilon \in (0,1)$ *and* $t > 0$.

Theorem (continued)

2. *For any numbers* $a, a_0 \in (0,1)$ *with* $a + a_0 < 1$ *and any* $\varepsilon \in (0,1)$, *there exists a positive constant* C *depending on* a_0, a, ε, M, $F(0)$, $|\xi|_{\alpha,\sigma}$ *and* $|w_0|_{\alpha,\sigma}$ *such that*

$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le C\big(e^{-a_0 t} + F(at)\big) \quad \forall t \ge 1.$$

3. *Assume, in addition, that*
   - *There exist* $k_0 > 0$ *and* $D_1 > 0$ *such that*

$$e^{-k_0 t} \le D_1 F(t) \quad \forall t \ge 0, \text{ and} \tag{F1}$$

   - *For any* $a \in (0,1)$, *there exists* $D_2 = D_{2,a} > 0$ *such that*

$$F(at) \le D_2 F(t) \quad \forall t \ge 0. \tag{F2}$$

   Then there exists $C > 0$ *such that*

$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le CF(t) \quad \forall t \ge 1.$$

Parts of proof (2). Small data Gevrey results

Theorem (Cao-H. 2018)

Let F be a continuous, decreasing, non-negative function on $[0,\infty)$. Given $\alpha \ge 1/2$ and numbers $\theta_0, \theta \in (0,1)$ such that $\theta_0 + \theta < 1$. Then there exist positive numbers $c_k = c_k(\alpha, \theta_0, \theta, F)$, for $k = 0, 1, 2, 3$, such that the following holds true. If

$$|A^\alpha u^0| \le c_0,$$
$$|f(t)|_{\alpha-1/2,\sigma} \le c_1 F(t) \quad \text{a.e. in } (0,\infty) \text{ for some } \sigma \ge 0,$$

then there exists a unique regular solution $u(t)$, which also belongs to $C([0,\infty), \mathcal{D}(A^\alpha))$ and satisfies, for all $t \ge 8\sigma(1-\theta)/(1-\theta-\theta_0)$,

$$|u(t)|_{\alpha,\sigma} \le c_2 (e^{-2\theta_0 t} + F^2(\theta t))^{1/2},$$
$$\int_t^{t+1} |u(\tau)|^2_{\alpha+1/2,\sigma} d\tau \le c_3^2 (e^{-2\theta_0 t} + F^2(\theta t)).$$

Parts of proof (3). Estimates for Leray-Hopf weak solutions

Theorem (Cao-H. 2018)

Let F be a continuous, decreasing, non-negative function such that

$$\lim_{t\to\infty} F(t) = 0,$$

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(F(t)), \text{ for some } \sigma \geq 0,\ \alpha \geq 1/2.$$

Let $u(t)$ be a Leray-Hopf weak solution. Then there exists $\hat{T} > 0$ such that $u(t)$ is a regular solution on $[\hat{T}, \infty)$, and for any $\varepsilon, \lambda \in (0,1)$, and $a_0, a, \theta_0, \theta \in (0,1)$ with $a_0 + a < 1$, $\theta_0 + \theta < 1$,

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon,\sigma} \leq C(e^{-a_0 t} + e^{-2\theta_0 a t} + F^{2\lambda}(\theta a t) + F(at)) \quad \forall t \geq 0.$$

If, in addition, F satisfies (F1) and (F2), then

$$|u(\hat{T} + t)|_{\alpha+1-\varepsilon,\sigma} \leq CF(t) \quad \forall t \geq 0.$$

Finite asymptotic approximations

Assumption

Suppose there exist numbers $\sigma \geq 0$, $\alpha \geq 1/2$, *an integer* $N_0 \geq 1$, *strictly increasing, positive numbers* γ_n *and functions* $\tilde{\phi}_n \in G_{\alpha,\sigma}$ *for* $1 \leq n \leq N_0$ *such that*

$$\left| f(t) - \sum_{n=1}^{N_0} \tilde{\phi}_n \psi_{\gamma_n}(t) \right|_{\alpha,\sigma} = \mathcal{O}(\psi_\lambda(t)) \text{ for some } \lambda > \gamma_{N_0}.$$

Assume further that there exists a set S_∞ *that contains* $\{\gamma_n : 1 \leq n \leq N_0\}$ *and preserves the operations* $\vee$ *and* $\wedge$, *so that the set* $S_* \stackrel{\text{def}}{=\!=} S_\infty \cap [\gamma_1, \gamma_{N_0}]$ *is finite.*

We rewrite $S_* = \{\lambda_n : 1 \le n \le N_*\}$ for some integer $N_* \ge N_0$, where λ_n's are strictly increasing. Note that $\lambda_{N_*} = \gamma_{N_0}$. Then

$$\left| f(t) - \sum_{n=1}^{N_*} \phi_n \psi_{\lambda_n}(t) \right|_{\alpha,\sigma} = \mathcal{O}(\psi_\lambda(t)) \text{ for some } \lambda > \lambda_{N_*}.$$

where $\phi_n \in G_{\alpha,\sigma}$ for all $1 \le n \le N_*$.

Theorem (Cao-H. 2018)

For any Leray-Hopf weak solution $u(t)$, it holds that

$$\left| u(t) - \sum_{n=1}^{N_*} \xi_n \psi_{\lambda_n}(t) \right|_{\alpha,\sigma} = \mathcal{O}(\psi_\lambda(t)) \textit{ for some } \lambda > \lambda_{N_*}.$$

Application: iterated logarithmic decaying functions

For $k, m \in \mathbb{N}$, let

$$L_k(t) = \underbrace{\ln(\ln(\cdots \ln(t)))}_{k\text{-times}} \quad \text{and} \quad \mathcal{L}_m(t) = (L_1(t), L_2(t), \cdots, L_m(t)).$$

• Let $Q_0 : \mathbb{R}^m \to R$ be a polynomial in m variables with positive degree and positive leading coefficient:

$$Q_0(z) = \sum_\alpha c_\alpha z^\alpha \text{ for } z \in \mathbb{R}^m.$$

We use the lexicographic order for the multi-indices.
• Let Q_1 be a polynomial in one variable of positive degree with positive leading coefficient.
Given a number $\beta > 0$, we define

$$\omega(t) = (Q_0 \circ \mathcal{L}_m \circ Q_1)(t^\beta)) \text{ with } t \in \mathbb{R}.$$

Let $\psi_\lambda(t) = \omega(t)^{-\lambda}$ and $\Psi = (\psi_\lambda(t))_{\lambda>0}$. Note $\psi'_\lambda \overset{\Psi}{\sim} 0$.

Theorem (Cao-H. 2018)

Assume

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \phi_n \omega(t)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

for some $\sigma \ge 0$, $\alpha \ge 1/2$. Then any Leray-Hopf weak solution $u(t)$ of the NSE has the asymptotic expansion

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^\infty \xi_n \omega(t)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1),$$

where

$$\xi_1 = A^{-1}\phi_1, \quad \xi_n = A^{-1}\Big(\phi_n - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m)\Big) \quad \text{for } n \ge 2.$$

Corollary (Cao-H. 2018)

Given $m \in \mathbb{N}$, *define* $\Psi = (L_m(t)^{-\lambda})_{\lambda>0}$. *Suppose* $(\lambda_n)_{n=1}^{\infty}$ *is a strictly increasing, divergent sequence of positive numbers such that the set* $\{\lambda_n : n \in \mathbb{N}\}$ *preserves the addition. If*

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n L_m(t)^{-\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then any Leray-Hopf weak solution $u(t)$ *of the NSE admits the asymptotic expansion*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n L_m(t)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1).$$

Expansions with trigonometric functions

Example. If

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \left[\sin(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \left[\sin(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1).$$

Example. If

$$f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n \left[\tan(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha,\sigma},$$

then

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n \left[\tan(L_m^{-1}(t))\right]^{\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho \in (0,1).$$

Infinite expansions for the derivatives

Consider $\Psi = (\psi_\lambda)_{\lambda>0}$ with $\psi_\lambda = (\sqrt{t}+1)^{-\lambda}$. Then

$$\psi'_\lambda(t) = -\lambda(\sqrt{t}+1)^{-\lambda-1}\frac{1}{2}\frac{1}{\sqrt{t}} = -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-1}\frac{1}{\sqrt{t}+1}\cdot\frac{1}{1-\frac{1}{\sqrt{t}+1}}$$

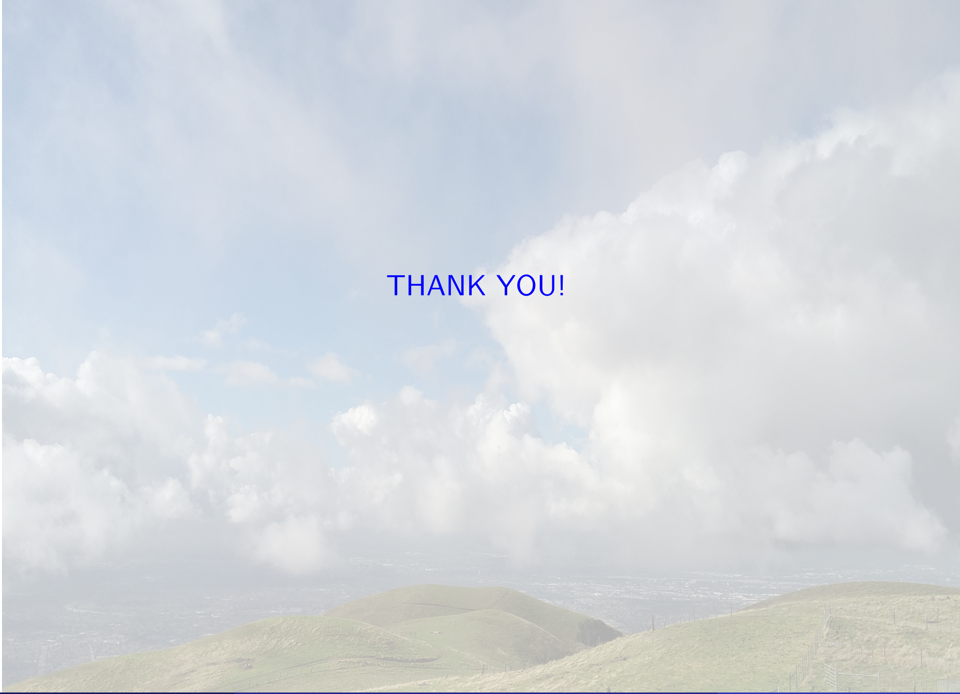
$$= \sum_{k=1}^{\infty} -\frac{\lambda}{2}(\sqrt{t}+1)^{-\lambda-k-1}.$$

Proposition (Cao-H. 2018)

Assume $f(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \phi_n(\sqrt{t}+1)^{-\lambda_n}$ *in* $G_{\alpha,\sigma}$. *Then*

$$u(t) \overset{\Psi}{\sim} \sum_{n=1}^{\infty} \xi_n(\sqrt{t}+1)^{-\lambda_n} \quad \text{in } G_{\alpha+1-\rho,\sigma} \text{ for all } \rho\in(0,1),$$

where $\xi_1 = A^{-1}\phi_1$, $\xi_n = A^{-1}\big(\phi_n + \frac{1}{2}\sum_{p\in\mathcal{Z}_n}\lambda_p\xi_p - \sum_{\substack{1\le k,m\le n-1,\\ \lambda_k+\lambda_m=\lambda_n}} B(\xi_k,\xi_m)\big)$ *for* $n\ge 2$, *with* $\mathcal{Z}_n = \{p\in\mathbb{N}\cap[1,n-1] : \exists k\in\mathbb{N}, \lambda_p+1+k=\lambda_n\}$.



THANK YOU!