Asymptotic expansions of Foias-Saut type for Navier-Stokes equations with decaying non-potential forces

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Outline

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Introduction

Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = f(x,t), \\ \mathrm{div}\ u = 0, \\ u(x,0) = u^0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
$f(x,t)$ is the body force,
u^0 is the initial velocity.

Let $L > 0$ and $\Omega = (0, L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Functional setting

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued 2π-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

$$H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$
$$V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \quad \mathcal{D}(A) = \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.$$

Norm on H: $|u| = \|u\|_{L^2(\Omega)}$. Norm on V: $\|u\| = |\nabla u|$.
The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u,v) = \mathbb{P}_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}(A).$$

$\mathbb{P}_L$ is the Leray projection from $L^2(\Omega)$ onto H.
Spectrum of A:

$$\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}.$$

Denote by $R_N H$ the eigenspace of A corresponding to N.

Functional form of NSE

WLOG, assume $f(t) = \mathbb{P}_L f(t)$.
The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = f(t), \;\; t > 0,$$

$$u(0) = u^0.$$

Case of potential force: $f = 0$

Foias-Saut (1987) proved that the solution $u(t)$ has an asymptotic expansion:

$$u(t) \sim \sum_{n=1}^{\infty} q_j(t) e^{-jt},$$

where $q_j(t)$ is a $\mathcal{V}$-valued polynomial in t.
This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$, the remainder $v_N(t) = u(t) - \sum_{j=1}^{N} q_j(t) e^{-jt}$ satisfies

$$\|v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.
H.-Martinez (2017) proved that the expansion holds in Gevrey spaces:

$$\|e^{\sigma A^{1/2}} v_N(t)\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t}),$$

for any $\sigma > 0$, $\varepsilon \in (0, 1)$.
They used Gevrey norm techniques (Foias-Temam 1989) to simplify the proof.

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Gevrey classes

For $\alpha \geq 0$, $\sigma \geq 0$, define

$$A^\alpha e^{\sigma A^{1/2}} u = \sum_{\mathbf{k}\neq 0} |\mathbf{k}|^{2\alpha} \hat{u}(\mathbf{k}) e^{\sigma|\mathbf{k}|} e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ for } u = \sum_{\mathbf{k}\neq 0} \hat{u}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \in H.$$

The domain of $A^\alpha e^{\sigma A^{1/2}}$ is

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) = \{u \in H : |u|_{\alpha,\sigma} \stackrel{\text{def}}{=\!=} |A^\alpha e^{\sigma A^{1/2}} u| < \infty\}.$$

- Compare the Sobolev and Gevrey norms:

$$|A^\alpha u| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} u| \leq \left(\frac{2\alpha}{e\sigma}\right)^{2\alpha} |e^{\sigma A^{1/2}} u|.$$

Notation

- Denote for $\sigma \in \mathbb{R}$ the space

$$E^{\infty,\sigma} = \bigcap_{\alpha \geq 0} G_{\alpha,\sigma} = \bigcap_{m \in \mathbb{N}} G_{m,\sigma}.$$

- We will say that an asymptotic expansion holds in $E^{\infty,\sigma}$ if it holds in $G_{\alpha,\sigma}$ for all $\alpha \geq 0$.
- Denote by $\mathcal{P}^{\alpha,\sigma}$ the space of $G_{\alpha,\sigma}$-valued polynomials in case $\alpha \in \mathbb{R}$, and the space of $E^{\infty,\sigma}$-valued polynomials in case $\alpha = \infty$.

Definition

Let X be a real vector space.
(a) An X-valued polynomial is a function $t \in \mathbb{R} \mapsto \sum_{n=1}^{d} a_n t^n$, for some $d \geq 0$, and a_n's belonging to X.
(b) In case $\|\cdot\|$ is a norm on X, a function $g(t)$ from $(0,\infty)$ to X is said to have the asymptotic expansion

$$g(t) \sim \sum_{n=1}^{\infty} g_n(t) e^{-nt} \text{ in } X,$$

where $g_n(t)$'s are X-valued polynomials, if for all $N \geq 1$, there exists $\varepsilon_N > 0$ such that

$$\left\| g(t) - \sum_{n=1}^{N} g_n(t) e^{-nt} \right\| = \mathcal{O}(e^{-(N+\varepsilon_N)t}) \text{ as } t \to \infty.$$

Assumptions.

(A1) The function $f(t)$ is continuous from $[0,\infty)$ to H.

(A2) There are a number $\sigma_0 \ge 0$, E^{∞,σ_0}-valued polynomials $f_n(t)$ for all $n \ge 1$, and a sequence of numbers $\delta_n \in (0,1)$ for all $n \ge 1$ such that for each $N \ge 1$

$$\Big| f(t) - \sum_{n=1}^{N} f_n(t) e^{-nt} \Big|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+\delta_N)t}) \quad \text{as } t \to \infty, \quad \text{for all } \alpha \ge 0.$$

That is, the force $f(t)$ admits the following expansion in G_{α,σ_0} for all $\alpha \ge 0$:

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-nt}.$$

Remarks

The followings are direct consequences of the Assumptions.

(a) For each $\alpha > 0$ that $f(t)$ belongs to G_{α,σ_0} for t large.

(b) When $N = 1$,

$$|f(t) - f_1(t)e^{-t}|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(1+\delta_1)t}).$$

Since $f_1(t)$ is a polynomial, it follows that

$$|f(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\lambda t}) \quad \forall \lambda \in (0,1), \forall \alpha > 0.$$

(c) Combining with Assumption (A1), for each $\lambda \in (0,1)$, there is $M_\lambda > 0$ such that

$$|f(t)| \le M_\lambda e^{-\lambda t} \quad \forall t \ge 0.$$

Theorem (Asymptotic expansion, H.-Martinez 2017)

Let $u(t)$ be a Leray-Hopf weak solution. Then there exist polynomials $q_n \in \mathcal{P}^{\infty,\sigma_0}$, for all $n \geq 1$, such that $u(t)$ has the asymptotic expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t)e^{-nt} \quad \text{in } E^{\infty,\sigma_0}.$$

Moreover, the mappings

$$u_n(t) \overset{\text{def}}{=\!=} q_n(t)e^{-nt} \quad \text{and} \quad F_n(t) \overset{\text{def}}{=\!=} f_n(t)e^{-nt},$$

satisfy the following ordinary differential equations in the space E^{∞,σ_0}

$$\frac{d}{dt}u_n(t) + Au_n(t) + \sum_{\substack{k,m\geq 1\\ k+m=n}} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R}, \qquad (\star)$$

for all $n \geq 1$.

Finite asymptotic approximation

Theorem (Finite asymptotic approximation, H.-Martinez 2017)

Suppose there exist an integer $N_* \geq 1$*, real numbers* $\sigma_0 \geq 0$*,* $\mu_* \geq \alpha_* \geq N_*/2$*, and, for any* $1 \leq n \leq N_*$*, numbers* $\delta_n \in (0,1)$ *and polynomials* $f_n \in \mathcal{P}^{\mu_n,\sigma_0}$*, such that*

$$\Big| f(t) - \sum_{n=1}^{N} f_n(t) e^{-nt} \Big|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\delta_N)t}) \quad \text{as } t \to \infty,$$

for $1 \leq N \leq N_*$*, where*

$$\mu_n = \mu_* - (n-1)/2, \quad \alpha_n = \alpha_* - (n-1)/2.$$

Theorem (continued)

Let $u(t)$ be a Leray-Hopf weak .
(i) Then there exist polynomials $q_n \in \mathcal{P}^{\mu_n+1,\sigma_0}$, for $1 \le n \le N_*$, such that one has for $1 \le N \le N_*$ that

$$\Big| u(t) - \sum_{n=1}^{N} q_n(t) e^{-nt} \Big|_{\alpha_N,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad \text{as } t \to \infty, \quad \forall \varepsilon \in (0, \delta_N^*),$$

where $\delta_N^* = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$.
Moreover, the ODEs

$$\frac{d}{dt} u_n(t) + Au_n(t) + \sum_{\substack{k,m \ge 1 \\ k+m=n}} B(u_k(t), u_m(t)) = F_n(t), \quad t \in \mathbb{R}, \qquad (\star)$$

hold in the corresponding space G_{μ_n,σ_0} for $1 \le n \le N_*$.
(ii) In particular, if all $f_n(t)$'s belong to $\mathcal{V}$, resp., E^{∞,σ_0}, then so do all $q_n(t)$'s, and the ODEs $(\star)$ hold in $\mathcal{V}$, resp., E^{∞,σ_0}.

II. Power-decaying forces [Cao-H. 2017]

Power asymptotic expansion in $(X, \|\cdot\|)$:

$$g(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^{\infty} g_n t^{-n}$$

means

$$\|g(t) - \sum_{n=1}^{N} g_n t^{-n}\| = \mathcal{O}(t^{-(N+\varepsilon)}), \quad \text{for some } \varepsilon > 0, \quad t \to \infty.$$

Theorem (Power asymptotic expansion, Cao-H. 2017)

Assume that $f(t)$ has the asymptotic expansion

$$f(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^{\infty} \phi_n t^{-n} \quad \textit{in } G_{\alpha,\sigma_0},$$

for some fixed numbers $\sigma_0 \geq 0$ and $\alpha \geq 1/2$, where $\{\phi_n\}_{n=1}^{\infty}$ is a sequence in G_{α,σ_0}. Then any Leray-Hopf weak solution $u(t)$ has the asymptotic expansion

$$u(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^{\infty} \xi_n t^{-n} \quad \textit{in } G_{\alpha,\sigma_0},$$

where $\{\xi_n\}_{n=1}^{\infty}$ is explicitly defined as follows

$$\xi_1 = A^{-1}\phi_1,$$

$$\xi_n = (n-1)A^{-1}\xi_{n-1} - \sum_{k,m\geq 1, k+m=n} A^{-1}B(\xi_k, \xi_m) + A^{-1}\phi_n \quad \textit{for } n \geq 2.$$

Next decay form is exponential

Theorem

If $\bar{u} = \sum_{n=1}^{\infty} \xi_n t^{-n}$ *converges absolutely and uniformly in* $G_{\alpha+1,\sigma_0}$ *on* $[T,\infty)$, *and* $f = \sum_{n=1}^{\infty} \phi_n t^{-n}$ *converges absolutely and uniformly in* G_{α,σ_0} *on* $[T,\infty)$, *then*

$$|u(t) - \bar{u}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(1-\varepsilon)t}).$$

for any $\varepsilon > 0$.

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Estimates for the bilinear form

Lemma

If $\alpha \geq 1/2$ *then*

$$|B(u,v)|_{\alpha,\sigma} \leq K^{\alpha}|u|_{\alpha+1/2,\sigma}\,|v|_{\alpha+1/2,\sigma},$$

for all $u, v \in G_{\alpha+1/2,\sigma}$.

I. Case of exponential decay

Recall

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-nt} = \sum_{n=1}^{\infty} F_n(t).$$

Need to prove

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt}.$$

Small data

Proposition

Let $\delta \in (0,1), \lambda \in (1-\delta, 1]$ *and* $\sigma \geq 0, \alpha \geq 1/2$. *There are* $C_0, C_1 > 0$ *such that if*

$$|A^\alpha u^0| \leq C_0, \quad |f(t)|_{\alpha-1/2,\sigma} \leq C_1 e^{-\lambda t}, \quad \forall t \geq 0,$$

then there exists a unique solution $u \in C([0,\infty), \mathcal{D}(A^\alpha))$ *that satisfies and*

$$|u(t)|_{\alpha,\sigma} \leq \sqrt{2} C_0 e^{-(1-\delta)t}, \quad \forall t \geq t_*,$$

where $t_* = 6\sigma/\delta$. *Moreover, one has for all* $t \geq t_*$ *that*

$$\int_t^{t+1} |A^{\alpha+1/2} u(\tau)|^2 d\tau \leq \frac{2C_0^2}{1-\delta} e^{-2(1-\delta)t}.$$

Long-time estimates for the solution

Theorem

For $\alpha \in [0,\infty)$ and $\delta \in (0,1)$, there exists a positive number T_0 such that

$$|u(T_0+t)|_{\alpha,\sigma_0} \le e^{-(1-\delta)t} \quad \forall t \ge 0,$$

and

$$|B(u(T_0+t),u(T_0+t))|_{\alpha,\sigma_0} \le e^{-2(1-\delta)t} \quad \forall t \ge 0.$$

Note: Can use different bootstrapping procedures for $\sigma_0 > 0$ (faster) and $\sigma_0 = 0$ (gradually).

Proof of Asymptotic Expansion. First step $N = 1$

Let $w_0(t) = e^t u(t)$ and $w_{0,k}(t) = R_k w_0(t)$. We have

$$\frac{d}{dt} w_0 + (A - 1) w_0 = f_1 + H_1(t),$$

where

$$H_1(t) = e^t (f - F_1 - B(u, u)).$$

Taking the projection R_k gives

$$\frac{d}{dt} w_{0,k} + (k - 1) w_{0,k} = R_k f_1 + R_k H_1(t).$$

Note that $R_k f_1(t)$ is a polynomial in $R_k H$.

Fact: there are $T_0 > 0$ and $M \geq 1$ such that for $t \geq 0$,

$$e^t |f(T_0 + t) - F_1(T_0 + t)|_{\alpha,\sigma_0} \leq M e^{-\delta_1 t}$$

$$e^t |B(u(T_0 + t), u(T_0 + t))|_{\alpha,\sigma_0} \leq e^{-2(1-\delta)t + t} \leq e^{-\delta_1 t}.$$

Then, by setting $M_1 = M + 1$, we have

$$|H_1(T_0 + t)|_{\alpha,\sigma_0} \leq M_1 e^{-\delta_1 t} \quad \forall t \geq 0.$$

Lemma

Let $(X, \|\cdot\|)$ be a Banach space. Suppose $y(t)$ is in $C([0,\infty),X)$ and $C^1((0,\infty),X)$ that solves the following ODE

$$\frac{dy}{dt} + \alpha y = p(t) + g(t) \quad \text{for } t > 0,$$

where constant $\alpha \in \mathbb{R}$, $p(t)$ is a X-valued polynomial in t, and $g(t) \in C([0,\infty),X)$ satisfies

$$\|g(t)\| \le Me^{-\delta t} \quad \forall t \ge 0, \quad \text{for some } M, \delta > 0.$$

Define $q(t)$ for $t \in \mathbb{R}$ by

$$q(t) = \begin{cases} e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha\tau} p(\tau) d\tau & \text{if } \alpha > 0, \\ y(0) + \int_0^\infty g(\tau) d\tau + \int_0^t p(\tau) d\tau & \text{if } \alpha = 0, \\ -e^{-\alpha t} \int_t^\infty e^{\alpha\tau} p(\tau) d\tau & \text{if } \alpha < 0. \end{cases}$$

Then $q(t)$ is an X-valued polynomial that satisfies

$$\frac{dq(t)}{dt} + \alpha q(t) = p(t) \quad \forall t \in \mathbb{R},$$

and the following estimates hold.

(i) If $\alpha > 0$ then

$$\|y(t) - q(t)\| \le \left(\|y(0) - q(0)\| + \frac{M}{|\alpha - \delta|} \right) e^{-\min\{\delta,\alpha\}t}, t \ge 0, \text{ for } \alpha \ne \delta,$$

and

$$\|y(t) - q(t)\| \le (\|y(0) - q(0)\| + Mt)e^{-\delta t}, t \ge 0, \text{ for } \alpha = \delta.$$

(ii) If $(\alpha = 0)$ or $(\alpha < 0$ and $\lim_{t\to\infty} e^{\alpha t} y(t) = 0)$ then

$$\|y(t) - q(t)\| \le \frac{Me^{-\delta t}}{|\alpha - \delta|} \quad \forall t \ge 0.$$

For the Lemma, we just use the following elementary identities: for $\beta > 0$, integer $d \geq 0$, and any $t \in \mathbb{R}$,

$$\int_{-\infty}^{t} \tau^d e^{\beta\tau} \, d\tau = e^{\beta t} \sum_{n=0}^{d} \frac{(-1)^{d-n} d!}{n! \beta^{d+1-n}} t^n,$$

$$\int_{t}^{\infty} \tau^d e^{-\beta\tau} \, d\tau = e^{-\beta t} \sum_{n=0}^{d} \frac{d!}{n! \beta^{d+1-n}} t^n.$$

N=1 (continued). Then there exists a polynomial $q_1(t)$ such that

$$|w_0(t) - q_1(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\delta t}).$$

Hence

$$|u(t) - q_1(t)e^{-t}|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(1+\delta)t}).$$

Induction step

Denote $\varepsilon_* \in (0, \delta^*_{N+1})$ and $\bar{u}_N(t) = \sum_{n=1}^N u_n(t)$.
Remainder $v_N(t) = u(t) - \bar{u}_N(t)$ satisfies for any $\beta > 0$ that

$$|v_N(t)|_{\beta,\sigma_0} = \mathcal{O}(e^{-(N+\varepsilon_*)t}) \text{ as } t \to \infty.$$

Evolution of v_N:

$$\frac{d}{dt} v_N + A v_N + \sum_{m+j=N+1} B(u_m, u_j) = F_{N+1}(t) + h_N(t),$$

where

$$h_N(t) = -B(v_N, u) - B(\bar{u}_N, v_N) - \sum_{\substack{1 \le m,j \le N \\ m+j \ge N+2}} B(u_m, u_j) + \tilde{F}_{N+1}(t),$$

$$\tilde{F}_{N+1}(t) = f(t) - \sum_{n=1}^{N+1} F_n(t).$$

Fact:

$$h_N(t) = \mathcal{O}_{\alpha,\sigma_0}(e^{-(N+1+\varepsilon_*)t}).$$

Let $w_N(t) = e^{(N+1)t} v_N(t)$ and $w_{N,k} = R_k w_N(t)$. The ODE for $w_{N,k}$:

$$\frac{d}{dt} w_{N,k} + (k - (N+1)) w_{N,k} + \sum_{m+j=N+1} R_k B(q_m, q_j) = R_k f_{N+1} + H_{N,k},$$

with $H_{N,k} = e^{(N+1)t} R_k h_N(t)$.
Fact:

$$|H_{N,k}|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_* t}).$$

Then there are $T > T_0$ and $M > 0$ such that for $t \geq 0$

$$|H_{N,k}(T+t)|_{\alpha,\sigma_0} \leq M e^{-\varepsilon_* t}.$$

Case $k = N + 1$

By Lemma(ii), there is a polynomial $q_{N+1,N+1}(t)$ valued in $R_{N+1}H$ such that

$$|w_{N,N+1}(T+t) - q_{N+1,N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_* t}),$$

thus,

$$|R_{N+1}w_N(t) - q_{N+1,N+1}(t-T)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_* t}).$$

Case $k \le N$

Note

$$\lim_{t\to\infty} e^{(k-(N+1))t} w_{N,k}(t) = \lim_{t\to\infty} e^{kt} R_k v_N(t) = 0.$$

Applying Lemma(ii) with $\alpha = k - N - 1 < 0$, there is a polynomial $q_{N+1,k}(t)$ valued in $R_k H$ such that

$$|w_{N,k}(T+t) - q_{N+1,k}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_* t}),$$

$$|R_k w_N(t) - q_{N+1,k}(t-T)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_* t}).$$

Case $k \geq N + 2$

Similarly, applying Lemma(i), there is a polynomial $q_{N+1,k}(t)$ valued in $R_k H$ such that

$$|w_{N,k}(T+t) - q_{N+1,k}(t)|_{\alpha,\sigma_0} \leq \Big(|R_k v_N(T)|_{\alpha,\sigma_0} + |q_{N+1,k}(0)|_{\alpha,\sigma_0} + \frac{M}{k-(N+1)} \Big) e^{-\varepsilon_* t}.$$

Thus

$$|R_k w_N(t) - q_{N+1,k}(t-T)|_{\alpha,\sigma_0} \leq e^{\varepsilon_* T} \Big(|R_k v_N(T)|_{\alpha,\sigma_0} + |q_{N+1,k}(0)|_{\alpha,\sigma_0} + \frac{M}{k-(N+1)} \Big) e^{-\varepsilon_* t}.$$

Polynomial $q_{N+1}(t)$

Define $q_{N+1}(t) = \sum_{k=1}^{\infty} q_{N+1,k}(t-T)$. Then squaring and summing in k, we obtain

$$\sum_{k=N+2}^{\infty} |R_k w_N(t) - q_{N+1,k}(t-T))|_{\alpha,\sigma_0}^2$$
$$\leq 3e^{2\varepsilon_* T}\Big(\sum_{k=N+2}^{\infty} |R_k v_N(T)|_{\alpha,\sigma_0}^2 + \sum_{k=N+2}^{\infty} |R_k q_{N+1}(T)|_{\alpha,\sigma_0}^2$$
$$+ \sum_{k=N+2}^{\infty} \frac{M^2}{(k-(N+1))^2}\Big) e^{-2\varepsilon_* t}$$
$$= \mathcal{O}(e^{-2\varepsilon_* t}).$$

Thus,

$$|w_N(t) - q_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-\varepsilon_* t}),$$

therefore,

$$|v_N(t) - e^{-(N+1)t} q_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(e^{-(N+1+\varepsilon_*)t}).$$

Check ODE for $u_{N+1}(t)$

The polynomial $q_{N+1}(t)$ satisfies

$$\frac{d}{dt}R_k q_{N+1}(t)+(k-(N+1))R_k q_{N+1}(t)+\sum_{m+j=N+1} R_k B(q_m,q_j) = R_k f_{N+1}(t),$$

$$\frac{d}{dt}R_k u_{N+1}(t)+kR_k u_{N+1}(t)+\sum_{m+j=N+1} R_k B(u_m,u_j) = R_k F_{N+1}(t) \quad \forall k\ge 1,$$

which we rewrite as

$$\frac{d}{dt}u_{N+1}(t)+Au_{N+1}(t)+\sum_{m+j=N+1} B(u_m,u_j) = F_{N+1}(t).$$

II. Case of power decay

Recall

$$f(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^{\infty} \phi_n t^{-n}.$$

Need to prove

$$u(t) \overset{\text{pow.}}{\sim} \sum_{n=1}^{\infty} \xi_n t^{-n},$$

$$\xi_1 = A^{-1}\phi_1,$$

$$\xi_n = (n-1)A^{-1}\xi_{n-1} - \sum_{k,m\ge 1, k+m=n} A^{-1}B(\xi_k,\xi_m) + A^{-1}\phi_n \quad \text{for } n \ge 2.$$

Proposition (Higher regularity)

If $\phi_n \in G_{\alpha,\sigma_0}$ for all n, some $\alpha \ge 1/2$, then

$$\xi_n \in G_{\alpha+1,\sigma_0} \quad \forall n.$$

Power-decay for weak solutions

Proposition

Assume there are numbers $M_*, \lambda_* > 0$ *such that*

$$|f(t)| \le M_*(1+t)^{-\lambda_*}, \quad \forall t \ge 0,$$

and, additionally, that there are $\sigma \ge 0$, $\alpha \ge 1/2$ *and* $\lambda_0 > 0$ *such that*

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\lambda_0}) \quad \text{as } t \to \infty.$$

Let $u(t)$ *be a Leray-Hopf weak solution. Then for any* $\lambda \in (0, \lambda_0)$, *there exists* $T_* > 0$ *such that* $u(t)$ *is a regular solution on* $[T_*, \infty)$, *and one has for all* $t \ge 0$ *that*

$$|u(T_* + t)|_{\alpha+1/2,\sigma} \le Ct^{-\lambda},$$

$$|B(u(T_*+t), u(T_*+t))|_{\alpha,\sigma} \le Ct^{-2\lambda}.$$

Main ideas of the induction step

Define

$$\bar{u}_N = \sum_{n=1}^{N} \xi_n t^{-n}, \quad v_N = u - \bar{u}_N, \text{and} \quad \tilde{F}_{N+1}(t) = f(t) - \sum_{n=1}^{N+1} \phi_n t^{-n}.$$

Then

$$|v_N(t)|_{\alpha,\sigma_0} = \mathcal{O}(t^{-(N+\varepsilon)}).$$

$$|\tilde{F}_{N+1}(t)|_{\alpha,\sigma_0} = \mathcal{O}(t^{-(N+\varepsilon)}).$$

Let $w_N(t) = t^{N+1} v_N(t)$ then

$$
\begin{aligned}
w_N' &= -Aw_N \\
&\quad + t^{N+1}\Big\{ -\sum_{n=1}^{N} \frac{1}{t^n}\Big(A\xi_n + \sum_{n=1}^{N}\sum_{k+j=n} B(\xi_m,\xi_j) - (n-1)\xi_{n-1} - \phi_n\Big)\Big\} \\
&\quad - \sum_{k+j=N+1} B(\xi_m,\xi_j) + \phi_{N+1} + N\xi_N \\
&\quad - t^{N+1} \sum_{\substack{1\le m,j\le N\\ m+j\ge N+2}} t^{-m-j} B(\xi_m,\xi_j) \\
&\quad + t^{N+1}(-B(\bar u_N, v_N) - B(v_N,\bar u_N) - B(v_N,v_N) \\
&\quad + \tilde F_{N+1}) + (N+1)t^N v_N.
\end{aligned}
$$

Since

$$A\xi_n + \sum_{n=1}^{N} \sum_{k+j=n} B(\xi_m, \xi_j) - (n-1)\xi_{n-1} - \phi_n = 0 \text{ for } 1 \le n \le N,$$

and

$$-\sum_{k+j=N+1} B(\xi_m, \xi_j) + \phi_{N+1} + N\xi_N = A\xi_{N+1},$$

we obtain

$$w_N' = -Aw_N + A\xi_{N+1} + H_N(t),$$

where

$$H_N(t) = -t^{N+1} \sum_{\substack{1\le m,j\le N\\ m+j\ge N+2}} t^{-(m+j)} B(\xi_m, \xi_j)$$
$$+ t^{N+1}(-B(\bar{u}_N, v_N) - B(v_N, \bar{u}_N) - B(v_N, v_N) + \tilde{F}_{N+1}) + (N+1)t^N v_N$$

We carefully control norms of $H_N(t)$. Term by term:

$$|B(\xi_m,\xi_j)|_{\alpha+1/2,\sigma_0}, |v_N|_{\alpha,\sigma_0}, |\tilde{F}_{N+1}|_{\alpha,\sigma_0},$$
$$|B(\bar{u}_N,v_N)|_{\alpha-1/2,\sigma_0}, |B(v_N,\bar{u}_N)|_{\alpha-1/2,\sigma_0}, |B(v_N,v_N)|_{\alpha-1/2,\sigma_0}.$$

Therefore,

$$|H_N(t)|_{\alpha-1/2,\sigma_0} = \mathcal{O}(t^{-\delta}).$$

Lemma

If $\xi \in G_{\alpha,\sigma}$, and $|f(t)|_{\alpha,\sigma} \le M(1+t)^{-\lambda}$ and

$$y' = -Ay + \xi + f(t).$$

For $\varepsilon \in (0,1)$, there exist $C_\varepsilon > 0$ and $T > 0$ such that

$$|y(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \le C_\varepsilon (1+t)^{-\lambda+\varepsilon} \quad t \ge T.$$

Note that $A\xi_{N+1} \in G_{\alpha,\sigma_0} \subset G_{\alpha-1/2,\sigma_0}$.
Hence applying ODE lemma with power-decay forcing gives

$$|w_N(t) - A^{-1}(A\xi_{N+1})|_{\alpha-1/2+1-\varepsilon',\sigma_0} = \mathcal{O}(t^{-\delta+\varepsilon'}),$$

that is

$$|w_N(t) - \xi_{N+1}|_{\alpha+1/2-\varepsilon',\sigma_0} = \mathcal{O}(t^{-\delta+\varepsilon'}),$$

for sufficiently small $\varepsilon' > 0$.
This shows

$$\begin{aligned} |v_{N+1}(t)|_{G_{\alpha+1/2-\varepsilon',\sigma_0}} &= |v_N(t) - \xi_{N+1}t^{-N-1}|_{\alpha+1/2-\varepsilon',\sigma_0} \\ &= |t^{-N-1}(w_N(t) - \xi_{N+1})|_{\alpha+1/2-\varepsilon',\sigma_0} \\ &= \mathcal{O}(t^{-N-1-\delta+\varepsilon',\sigma_0}). \end{aligned}$$

Taking small ε' gives what we desire for the induction step.

THANK YOU FOR YOUR ATTENTION.