Studying nonlinear fluid flows in heterogeneous porous media

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1. Introduction

- Darcy's and Forchheimer's flows
- PDE for compressible Forchheimer flows

Darcy's and Forchheimer's flows

Fluid flows in porous media with velocity v and pressure p:

- Darcy's Law:

$$\alpha v = -\nabla p,$$

- Forchheimer's "two term" law

$$\alpha v + \beta |v|\, v = -\nabla p,$$

- Forchheimer's "three term" law

$$\mathcal{A} v + \mathcal{B}\, |v|\, v + \mathcal{C} |v|^2\, v = -\nabla p.$$

- Forchheimer's "power" law

$$a v + c^n |v|^{n-1}\, v = -\nabla p,$$

Here $\alpha, \beta, a, c, n, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

Generalized Forchheimer equations

[Aulisa-Bloshanskaya-H.-Ibragimov 2009]
Generalizing the above equations as follows

$$g(|v|)v = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|v|) = |\nabla p| \Rightarrow |v| = G^{-1}(|\nabla p|)$. Hence

$$v = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow v = -K(|\nabla p|)\nabla p,$$

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

Class $FP(N, \vec{\alpha})$. Let $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$,

$$FP(N, \vec{\alpha}) = \left\{ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} \right\},$$

where $a_0, a_N > 0$, $a_1, \ldots, a_{N-1} \geq 0$. Notation: $\alpha_N = \deg(g)$,

$$a = \frac{\alpha_N}{\alpha_N + 1} \in (0, 1).$$

Works on Forchheimer flows

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990's, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)

Works on generalized Forchheimer flows

- 1990's Numerical study
- L^2-theory: Aulisa-Bloshanskaya-H.-Ibragimov (2009), H.-Ibragimov: Dirichlet B.C. (2011), H.-Ibragimov Flux B.C. (2012), Aulisa-Bloshanskaya-Ibragimov total flux, productivity index (2011, 2012).
- L^α-theory: H.-Ibragimov-Kieu-Sobol (2015)
- L^∞, $W^{1,p}$-theory: H.-Kieu-Phan (2014).
- $W^{1,\infty}$-theory: interior H.-Kieu (2017), global H.-Kieu (2015-accepted).
- Heterogeneous porous media: Celik-H.(2016,2017).
- Isentropic gases: Celik-H.-Kieu (2017-in press, 2017-submitted).
- Mixed pre-Darcy, Darcy, Forchheimer flows: Celik-H.-Ibragimov-Kieu (2017)
- Two-phase flows: H.-Ibragimov-Kieu (2013,2014)
- Numericals: Kieu (2016,2017) Ibragimov-Kieu (2016)

Note: there are more works on Forchheimer flows (2-term or 3-term).

PDE for compressible Forchheimer flows

Let ρ be the density. Continuity equation

$$\phi\frac{d\rho}{dt} + \nabla\cdot(\rho v) = 0.$$

For slightly compressible fluid:

$$\frac{1}{\rho}\frac{d\rho}{dp} = \frac{1}{\kappa},$$

where $\kappa \gg 1$. Then

$$\phi\frac{dp}{dt} = \kappa\nabla\cdot\Big(K(|\nabla p|)\nabla p\Big) + K(|\nabla p|)|\nabla p|^2.$$

Since $\kappa \gg 1$, we neglect the last terms

$$\phi\frac{dp}{dt} = \nabla\cdot\Big(K(|\nabla p|)\nabla p\Big).$$

Degeneracy

Lemma

Let $g(s,\vec{a})$ *be in class* $FP(N,\vec{\alpha})$. *One has for any* $\xi \geq 0$ *that*

$$\frac{C_1(\vec{a})}{(1+\xi)^a} \leq K(\xi,\vec{a}) \leq \frac{C_2(\vec{a})}{(1+\xi)^a},$$

$$C_3(\vec{a})(\xi^{2-a}-1) \leq K(\xi,\vec{a})\xi^2 \leq C_2(\vec{a})\xi^{2-a}.$$

2. Flows in heterogeneous media

- Mathematical model
- Energy estimates
- Gradient estimates
- Continuous dependence

Mathematical model

[Celik-H. 2016, 2017]
Generalized Forchheimer equation for heterogeneous porous media

$$g(x,s) = a_0(x)s^{\alpha_0} + a_1(x)s^{\alpha_1} + \cdots + a_N(x)s^{\alpha_N}, \quad s \geq 0,$$

where $a_1(x)$, $a_2(x)$, $\ldots$, $a_{N-1}(x) \geq 0$, and $a_0(x), a_N(x) > 0$.
Then

$$v = -K(x, |\nabla p|)\nabla p,$$

where the function $K : \bar{U} \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$K(x,\xi) = \frac{1}{g(x, s(x,\xi))} \quad \text{for } x \in \bar{U},\ \xi \geq 0,$$

with $s = s(x,\xi)$ being non-negative solution of the equation $sg(x,s) = \xi$.

Degenerate/Singular parabolic equation and IBVP

The initial boundary value problem (IBVP) of our interest is

$$\begin{cases} \phi(x)\dfrac{\partial p}{\partial t} = \nabla \cdot (K(x,|\nabla p|)\nabla p) \quad \text{on } U \times (0,\infty), \\ p = \psi \quad \text{on} \quad \partial U \times (0,\infty), \\ p(x,0) = p_0(x) \quad \text{on} \quad U, \end{cases}$$

where $p_0(x)$ and $\psi(x,t)$ are given initial and boundary data.

Let $\Psi(x,t)$ be an extension of $\psi(x,t)$ from ∂U to $\bar{U}$.

Weight functions

$$M(x) = \max\{a_j(x) : j = 0, \ldots, N\}, \quad m(x) = \min\{a_0(x), a_N(x)\},$$

$$W_1(x) = \frac{a_N(x)^a}{2NM(x)}, \quad \text{and} \quad W_2(x) = \frac{NM(x)}{m(x)a_N(x)^{1-a}}.$$

Lemma

For $\xi \geq 0$, one has

$$\frac{2W_1(x)}{\xi^a + a_N(x)^a} \leq K(x,\xi) \leq \frac{W_2(x)}{\xi^a}$$

and, consequently,

$$W_1(x)\xi^{2-a} - \frac{a_N(x)}{2} \leq K(x,\xi)\xi^2 \leq W_2(x)\xi^{2-a}.$$

Two-weight Poincaré-Sobolev inequality

Let $\bar{p} = p - \Psi$, then we have

$$\phi(x)\frac{\partial \bar{p}}{\partial t} = \nabla \cdot (K(x, |\nabla p|)\nabla p) - \phi(x)\Psi_t \quad \text{on } U \times (0, \infty),$$
$$\bar{p} = 0 \quad \text{on } \Gamma \times (0, \infty).$$

Assume the following two-weight Poincaré-Sobolev inequality

$$\left(\int_U |u|^2 \phi(x) dx\right)^{\frac{1}{2}} \le c_P \left(\int_U W_1(x) |\nabla u|^{2-a} dx\right)^{\frac{1}{2-a}}$$

for functions u in certain classes that satisfy $u = 0$ on Γ.
Example, under Strict Degree Condition

$$\deg(g) < \frac{4}{n-2}. \qquad \text{(SDC)}$$

Energy estimates

Let $B_* = \max\left\{1, \int_U a_N(x)dx\right\}$,

$$G(t) = G[\Psi](t) \stackrel{\text{def}}{=\!=} B_* + \int_U a_0(x)^{-1}|\nabla\Psi(x,t)|^2 dx + \int_U W_1(x)|\nabla\Psi(x,t)|^{2-a}dx + \left(\int_U |\Psi_t(x,t)|^2\phi(x)dx\right)^{\frac{2-a}{2(1-a)}}.$$

Let $\mathcal{M}(t) = \mathcal{M}[\Psi](t)$ be a continuous function on $[0,\infty)$ that satisfies

$$\mathcal{M}(t) \text{ is increasing and } \mathcal{M}(t) \geq G(t)\ \forall t \geq 0.$$

Denote

$$\mathcal{A} = \mathcal{A}[\Psi] \stackrel{\text{def}}{=\!=} \limsup_{t\to\infty} G(t) \quad \text{and} \quad \mathcal{B} = \mathcal{B}[\Psi] \stackrel{\text{def}}{=\!=} \limsup_{t\to\infty} [G'(t)]^-.$$

Theorem (Celik-H. 2016)

(i) *If $t > 0$ then*

$$\int_U \bar{p}^2(x,t)\phi(x)dx \le \int_U \bar{p}^2(x,0)\phi(x)dx + C\mathcal{M}(t)^{\frac{2}{2-a}}.$$

(ii) *If $\mathcal{A} < \infty$ then*

$$\limsup_{t\to\infty} \int_U \bar{p}^2(x,t)\phi(x)dx \le C\mathcal{A}^{\frac{2}{2-a}}.$$

(iii) *If $\mathcal{B} < \infty$ then there is $T > 0$ such that for all $t > T$*

$$\int_U \bar{p}^2(x,t)\phi(x)dx \le C(\mathcal{B}^{\frac{1}{1-a}} + G(t)^{\frac{2}{2-a}}).$$

Idea: Differential inequalities

Let $y(t) = \int_U \bar{p}^2(x,t)\phi(x)dx$ and $\gamma = (2-a)/2$. Then

$$y'(t) \le -c_1 y(t)^\gamma + c_2 G(t).$$

Lemma

Assume $y(t), f(t) \ge 0$, $h(t) > 0$, and $\gamma > 0$ satisfy

$$y'(t) \le -h(t)y(t)^\gamma + f(t), \quad t > 0.$$

Then

$$y(t) \le y(0) + \Big[Env\big(f(t)/h(t)\big)\Big]^{1/\gamma}.$$

Also, if $\int_0^\infty h(t)dt = \infty$, then

$$\limsup_{t\to\infty} y(t) \le \Big[\limsup_{t\to\infty} \big(f(t)/h(t)\big)\Big]^{1/\gamma}.$$

Gradient estimates

We use of the function

$$H(x,\xi) = \int_0^{\xi^2} K(x,\sqrt{s})ds \quad \text{for } x \in U,\ \xi \geq 0.$$

We have the comparison

$$K(x,\xi)\xi^2 \leq H(x,\xi) \leq 2K(x,\xi)\xi^2.$$

Then

$$W_1(x)\xi^{2-a} - \frac{a_N(x)}{2} \leq H(x,\xi) \leq 2W_2(x)\xi^{2-a}.$$

Key relation:

$$K(x,|\nabla p(x,t)|)\nabla p(x,t) \cdot \nabla p_t(x,t) = \frac{1}{2}\frac{\partial}{\partial t}H(x,|\nabla p(x,t)|).$$

We define

$$G_1(t) = G_1[\Psi](t) \stackrel{\text{def}}{=\!=} \int_U a_0(x)^{-1} |\nabla \Psi_t(x,t)|^2 dx.$$

Theorem (Celik-H. 2016)

For $t > 0$,

$$\int_U W_1(x) |\nabla p(x,t)|^{2-a} dx \le e^{-\frac{1}{4}t} \int_U H(x, |\nabla p(x,0)|) dx + C\Big(\int_U \bar{p}^2(x,0)\phi(x) dx + \mathcal{M}^{\frac{2}{2-a}}(t) + \int_0^t e^{-\frac{1}{4}(t-\tau)} G_1(\tau) d\tau \Big).$$

For $t \ge 1$,

$$\int_U W_1(x) |\nabla p(x,t)|^{2-a} dx \le C\Big(\int_U \bar{p}^2(x,0)\phi(x) dx + \mathcal{M}(t)^{\frac{2}{2-a}} + \int_{t-1}^t G_1(\tau) d\tau \Big).$$

Large time estimates

Theorem (Celik-H. 2016)

If $\mathcal{A} < \infty$ then

$$\limsup_{t\to\infty} \int_U W_1(x)|\nabla p(x,t)|^{2-a} dx \le C\Big(\mathcal{A}^{\frac{2}{2-a}} + \limsup_{t\to\infty} \int_{t-1}^t G_1(\tau) d\tau\Big).$$

If $\mathcal{B} < \infty$ then there is $T > 1$ such that for all $t > T$,

$$\int_U W_1(x)|\nabla p(x,t)|^{2-a} dx \le C\Big(\mathcal{B}^{\frac{1}{1-a}} + G(t)^{\frac{2}{2-a}} + \int_{t-1}^t G_1(\tau) d\tau\Big).$$

Continuous dependence

Let $p_1(x,t)$ and $p_2(x,t)$ be two solutions with boundary data $\psi_1(x,t)$ and $\psi_2(x,t)$, respectively. Denote

$$P = p_1 - p_2,\ \Phi = \Psi_1 - \Psi_2 \quad \text{and} \quad \bar{P} = \bar{p}_1 - \bar{p}_2 = P - \Phi.$$

Then

$$\phi(x)\frac{\partial \bar{P}}{\partial t} = \nabla \cdot (K(x,|\nabla p_1|)\nabla p_1 - K(x,|\nabla p_2|)\nabla p_2) - \phi(x)\Phi_t,$$
$$\bar{P} = 0 \quad \text{on } \Gamma \times (0,\infty).$$

Lemma (Monotonicity)

For any $y, y' \in \mathbb{R}^n$, one has

$$(K(x,|y|)y - K(x,|y'|)y') \cdot (y' - y) \geq (1-a)K(x,\max\{|y|,|y'|\})|y-y'|^2.$$

Let

$$D(t) = \int_U a_0(x)^{-1} |\nabla\Phi(x,t)|^2 dx + \Big(\int_U a_0(x)^{-1} |\nabla\Phi(x,t)|^2 dx\Big)^{\frac{1}{2}} + \Big(\int_U |\Phi_t(x,t)|^2 \phi(x) dx\Big)^{\frac{1}{2}}.$$

Theorem (Celik-H. 2016)

For $t \geq 0$,

$$\|\bar{P}(t)\|_{L^2_\phi}^2 \leq e^{-d_4 \int_0^t \mathcal{M}_1(\tau)^{-\frac{a}{2-a}} d\tau} \|\bar{P}(0)\|_{L^2_\phi}^2 + C \int_0^t e^{-d_4 \int_s^t \mathcal{M}_1(\tau)^{-\frac{a}{2-a}} d\tau} \mathcal{M}_1(s)^{\frac{1}{2}} D(s)\, ds.$$

In particular, for any $T > 0$,

$$\sup_{t\in[0,T]} \|\bar{P}(t)\|_{L^2_\phi}^2 \leq \|\bar{P}(0)\|_{L^2_\phi}^2 + C\mathcal{M}_1(T)^{\frac{1}{2}} \int_0^T D(t) dt.$$

Asymptotic dependence

$$\tilde{\mathcal{A}} = \sum_{i=1}^{2} \mathcal{A}[\Psi_i] = \sum_{i=1}^{2} \limsup_{t\to\infty} G[\Psi_i](t),$$

$$\mathcal{G}_1 = \sum_{i=1}^{2} \limsup_{t\to\infty} \int_{t-1}^{t} G_1[\Psi_i](\tau)d\tau.$$

The asymptotic behavior of $\Phi(x,t)$ as $t\to\infty$ will be characterized by

$$\mathcal{D} = \limsup_{t\to\infty} D(t).$$

Theorem (Celik-H. 2016)

If $\tilde{\mathcal{A}}$ and $\mathcal{G}_1$ are finite, then

$$\limsup_{t\to\infty} \|\bar{P}(t)\|^2_{L^2_\phi} \le C(\tilde{\mathcal{A}}^{\frac{2}{2-a}} + \mathcal{G}_1)^{\kappa_0}\mathcal{D}.$$

3. Flows of mixed regimes

- Models
- Estimates for solutions
- Continuous dependence on the boundary data
- Structural stability

Different regimes

- Darcy:

$$v = -k\nabla p.$$

- Pre-Darcy: When $|v|$ is small,

$$|v|^{-\alpha} v = -k\nabla p, \alpha \in (0,1).$$

- Post-Darcy:

$$(a_0 + a_1|v|^{\alpha_1} + \ldots + a_N|v|^{\alpha_N})v = -\nabla p.$$

Unified form

[Celik-H.-Ibragimov-Kieu 2017]

$$\mathbf{G}(v) = -\nabla p,$$

where

$$\mathbf{G}(v) = \begin{cases} g(|v|)v & \text{if } v \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } v = 0, \end{cases}$$

where $g(s)$ is a continuous function from $(0,\infty)$ to $(0,\infty)$ that satisfies

$$\lim_{s \searrow 0} sg(s) = 0.$$

Solve for v

Taking the modulus both sides, we have

$$G(|v|) = |\nabla p|,$$

where

$$G(s) = \begin{cases} sg(s) & \text{if } s > 0, \\ 0 & \text{if } s = 0. \end{cases}$$

We assume

- $G(s)$ is strictly increasing on $[0, \infty)$,
- $G(s) \to \infty$ as $s \to \infty$, and
- the function $1/g(s)$ on $(0, \infty)$ can be extended to a continuous function $k_g(s)$ on $[0, \infty)$.

Then

$$v = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = k_g(G^{-1}(\xi)) \quad \text{for } \xi \geq 0.$$

Two models of g

Model 1. Function $g(s)$ is piece-wise defined:

$$g(s) = \bar{g}(s) \stackrel{\text{def}}{=\!=} c_1 s^{-\alpha} \mathbf{1}_{(0,s_1)}(s) + c_2 \mathbf{1}_{[s_1,s_2]}(s) + g_F(s) \mathbf{1}_{(s_2,\infty)}(s) \quad \text{for } s > 0,$$

Continuity condition:

$$c_1 s_1^{-\alpha} = c_2 = g_F(s_2).$$

$$K(\xi) = \bar{K}(\xi) \stackrel{\text{def}}{=\!=} M_1 \xi^{\beta_1} \mathbf{1}_{[0,Z_1)}(\xi) + M_2 \mathbf{1}_{[Z_1,Z_2]}(\xi) + K_F(\xi) \mathbf{1}_{(Z_2,\infty)}(\xi).$$

Model 2. Function $g(s)$ is smooth on $(0,\infty)$:

$$g(s) = g_I(s) \stackrel{\text{def}}{=\!=} a_{-1} s^{-\alpha} + a_0 + a_1 s^{\alpha_1} + \cdots + a_N s^{\alpha_N} \quad \text{for } s > 0, \qquad (1)$$

where $N \geq 1$, $\alpha \in (0,1)$, $\alpha_N > 0$,

$$a_{-1}, a_N > 0 \text{ and } a_i \geq 0 \quad \forall i = 0, 1, \ldots, N-1.$$

$$K(\xi) = K_I(\xi) \stackrel{\text{def}}{=\!=} \frac{s(\xi)^\alpha}{a_{-1} + a_0 s(\xi)^\alpha + a_1 s(\xi)^{\alpha+\alpha_1} + \ldots + a_N s(\xi)^{\alpha+\alpha_N}},$$

with $G(s(\xi)) = \xi$.

Two direct models of K

$$v = -K(|\nabla p|)\nabla p.$$

Note: $K(\xi)$ behaves like ξ^{β_1} for small ξ, and like $(1+\xi)^{-\beta_2}$ for large ξ,

$$\beta_1 = \frac{\alpha}{1-\alpha}, \quad \beta_2 = \frac{\alpha_N}{\alpha_N+1}.$$

Model 3.

$$K(\xi) = \hat{K}(\xi) \stackrel{\text{def}}{=\!=} \frac{a\xi^{\beta_1}}{(1+b\xi^{\beta_1})(1+c\xi^{\beta_2})}.$$

Model 4. More precisely, $K(\xi)$ is close to $M_1\xi^{\beta_1}$ when $\xi \to 0$, and to $K_F(\xi)$ when $\xi \to \infty$. Then we choose

$$K(\xi) = K_M(\xi) \stackrel{\text{def}}{=\!=} K_F(\xi) \cdot \frac{\bar{k}\xi^{\beta_1}}{1+\bar{k}\xi^{\beta_1}}.$$

where $\bar{k} = M_1/K_F(0) > 0$.

Initial Boundary Value Problem

Let $K(\xi)$ be one of the functions $\bar{K}(\xi)$, $K_I(\xi)$, $\hat{K}(\xi)$, $K_M(\xi)$.
After scaling the time variable (to simplify ϕ), we obtain the IBVP:

$$\begin{cases} p_t = \nabla \cdot (K(|\nabla p|)\nabla p) & \text{in } U \times (0,\infty), \\ p(x,0) = p_0(x), & \text{in } U \\ p = \psi(x,t), & \text{on } \partial U \times (0,\infty). \end{cases}$$

Let $\Psi(x,t)$ be an extension of ψ from $x \in \partial U$ to $x \in \bar{U}$.
Let $\bar{p} = p - \Psi$. Then

$$\begin{cases} \bar{p}_t = \nabla \cdot (K(|\nabla p|)\nabla p) - \Psi_t & \text{in } U \times (0,\infty), \\ \bar{p}(x,0) = p_0(x) - \Psi(x,0), & \text{in } U \\ \bar{p} = 0, & \text{on } \partial U \times (0,\infty). \end{cases}$$

Degeneracy

Lemma

Then there exist $d_2, d_3 > 0$ *such that*

$$\frac{d_2\xi^{\beta_1}}{(1+\xi)^{\beta_1+\beta_2}} \le K(\xi) \le \frac{d_3\xi^{\beta_1}}{(1+\xi)^{\beta_1+\beta_2}} \quad \forall \xi \ge 0.$$

Consequently, for all $m \ge \beta_2$ *and* $\delta > 0$,

$$d_2\Big(\frac{\delta}{1+\delta}\Big)^{\beta_1+\beta_2}(\xi^{m-\beta_2} - \delta^{m-\beta_2}) \le K(\xi)\xi^m \le d_3\xi^{m-\beta_2} \quad \forall \xi \ge 0.$$

Energy estimates

Theorem (Celik-H.-Ibragimov-Kieu 2016)

(i) There exists a positive constant C such that for all $t \geq 0$,

$$\|\bar{p}(t)\|^2 \leq \|\bar{p}(0)\|^2 + C\left[1 + Env(f(t))\right]^{\frac{2}{2-\beta_2}},$$

where

$$f(t) = f[\Psi](t) \stackrel{\text{def}}{=\!=} \|\nabla\Psi(t)\|^2 + \|\Psi_t(t)\|^{\frac{2-\beta_2}{1-\beta_2}}.$$

(ii) Furthermore,

$$\limsup_{t\to\infty} \|\bar{p}(t)\|^2 \leq C(1 + \limsup_{t\to\infty} f(t))^{\frac{2}{2-\beta_2}}.$$

(iii) If $\lim_{t\to\infty} \|\nabla\Psi(t)\| = \lim_{t\to\infty} \|\Psi_t(t)\| = 0,$ *then*

$$\lim_{t\to\infty} \|\bar{p}(t)\| = 0.$$

Gradient Estimates

Theorem (Celik-H.-Ibragimov-Kieu 2016)

For all $t \geq 0$,

$$\int_U |\nabla p(x,t)|^{2-\beta_2} dx \leq C\Big(1 + \|\bar{p}(0)\|^2 + e^{-\frac{t}{2}} \int_U |\nabla p(x,0)|^{2-\beta_2} dx + \big[Env(f(t))\big]^{\frac{2}{2-\beta_2}} + \int_0^t e^{-\frac{1}{2}(t-\tau)} \|\nabla \Psi_t(\tau)\|^2 d\tau\Big).$$

Furthermore,

$$\limsup_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx \leq C(1 + \limsup_{t\to\infty} G_1(t)),$$

where

$$G_1(t) = G_1[\Psi](t) \stackrel{\text{def}}{=\!=} f(t)^{\frac{2}{2-\beta_2}} + \|\nabla \Psi_t(t)\|^2.$$

Theorem (Celik-H.-Ibragimov-Kieu 2016)

If

$$\lim_{t\to\infty} \|\nabla\Psi(t)\| = \lim_{t\to\infty} \|\Psi_t(t)\| = \lim_{t\to\infty} \|\nabla\Psi_t(t)\| = 0$$

then

$$\lim_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx = 0.$$

Improvements for large t

Theorem (Celik-H.-Ibragimov-Kieu 2016)

(i) If $t \ge 1$ then

$$\int_U |\nabla p(x,t)|^{2-\beta_2} dx \le C\Big(1 + \|\bar{p}(0)\|^2 + [Env(f(t))]^{\frac{2}{2-\beta_2}} + \int_{t-1}^{t} \|\nabla \Psi_t(\tau)\|^2 d\tau\Big)$$

(ii) One has

$$\limsup_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx \le C\Big(1 + \limsup_{t\to\infty} f(t)^{\frac{2}{2-\beta_2}} + \limsup_{t\to\infty} \int_{t-1}^{t} \|\nabla \Psi_t(\tau)\|^2 d\tau\Big).$$

(iii) Moreover, $\lim_{t\to\infty} \int_U |\nabla p(x,t)|^{2-\beta_2} dx = 0$ *provided* $\lim_{t\to\infty} \|\nabla \Psi(t)\| = \lim_{t\to\infty} \|\Psi_t(t)\| = \lim_{t\to\infty} \int_{t-1}^{t} \|\nabla \Psi_t(\tau)\|^2 d\tau = 0.$

Continuous dependence on the boundary data

• We consider $K(\xi) = \bar{K}(\xi)$, $K_I(\xi)$, $\hat{K}(\xi)$, $K_M(\xi)$.
• For $i = 1, 2$, let $p_i(x,t)$ be a solution with boundary data $\psi_2(x,t)$, let $\Psi_i(x,t)$ be an extension of $\psi_i(x,t)$ to $\bar{U} \times [0,\infty)$, and $\bar{p}_i = p_i - \Psi_i$.
• Denote

$$\Phi = \Psi_1 - \Psi_2 \quad \text{and} \quad \bar{P} = \bar{p}_1 - \bar{p}_2 = p_1 - p_2 - \Phi.$$

• Then

$$\frac{\partial \bar{P}}{\partial t} = \nabla \cdot (K(|\nabla p_1|)\nabla p_1) - \nabla \cdot (K(|\nabla p_2|)\nabla p_2) - \Phi_t \quad \text{on } U \times (0,\infty),$$
$$\bar{P} = 0 \quad \text{on } \partial U \times (0,\infty).$$

- Set

$$\mathcal{Y}_0 = 1 + \sum_{i=1,2} \Big(\|\bar{p}_i(0)\|^2 + \|\nabla p_i(0)\|^{2-\beta_2}_{L^{2-\beta_2}} \Big),$$

$$\widetilde{\mathcal{Y}}(t) = \mathcal{Y}_0 + \sum_{i=1,2} [Env(f[\Psi_i](t))]^{\frac{2}{2-\beta_2}}$$

$$+ \begin{cases} \int_0^t e^{-\frac{1}{2}(t-\tau)} \sum_{i=1,2} \|\nabla \Psi_{i,t}(\tau)\|^2 d\tau & \text{if } 0 \le t < 1, \\ \int_{t-1}^t \sum_{i=1,2} \|\nabla \Psi_{i,t}(\tau)\|^2 d\tau & \text{if } t \ge 1. \end{cases}$$

- Let

$$D(t) = \|\Phi_t(t)\| + \|\nabla \Phi(t)\|_{L^{2-\beta_2}} + \|\nabla \Phi(t)\|^{2+\beta_1}_{L^{2+\beta_1}}.$$

- For asymptotic estimates, we use

$$\widetilde{\mathcal{A}} = \Big(\sum_{i=1,2} \limsup_{t\to\infty} f[\Psi_i](t) \Big)^{\frac{1}{2-\beta_2}},$$

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{A}}^2 + \sum_{i=1,2} \limsup_{t\to\infty} \int_{t-1}^t \|\nabla \Psi_{i,t}(\tau)\|^2 d\tau,$$

$$\mathcal{D} = \limsup_{t\to\infty} D(t).$$

Theorem (Celik-H.-Ibragimov-Kieu 2016)

For $t \geq 0$,

$$\|\bar{P}(t)\|^2 \leq \|\bar{P}(0)\|^2 + C\Big\{Env\Big[\widetilde{\mathcal{Y}}(t)^{\frac{\beta_1+\beta_2}{2-\beta_2}+\frac{1}{2}} D(t)\Big]\Big\}^{\frac{2}{2+\beta_1}}.$$

If $\widetilde{\mathcal{K}} < \infty$ then

$$\limsup_{t\to\infty} \|\bar{P}(t)\|^2 \leq C\Big\{(1+\widetilde{\mathcal{K}})^{\frac{\beta_1+\beta_2}{2-\beta_2}+\frac{1}{2}} \mathcal{D}\Big\}^{\frac{2}{2+\beta_1}}.$$

Structural stability

- Consider $K(\xi) = K_I(\xi, \vec{a})$ and study the dependence of the solutions on the coefficient vector $\vec{a}$.
- Let $N \geq 1$ and the exponent vector $\vec{\alpha} = (-\alpha, 0, \alpha_1, \ldots, \alpha_N)$ be fixed.
- Denote the set of admissible $\vec{a}$

$$S = \{\vec{a} = (a_{-1}, a_0, \ldots, a_N) : a_{-1}, a_N > 0, a_0, a_1, \ldots, a_{N-1} \geq 0\}.$$

Lemma (Perturbed Monotonicity)

For any coefficient vectors $\vec{a}^{(1)}, \vec{a}^{(2)} \in S$, *and any* $y, y' \in \mathbb{R}^n$, *one has*

$$\big(K_I(|y'|, \vec{a}^{(1)})y' - K_I(|y|, \vec{a}^{(2)})y\big) \cdot (y' - y) \geq \frac{d_6 |y - y'|^{2+\beta_1}}{(1 + |y| + |y'|)^{\beta_1+\beta_2}}$$
$$- d_7 K\left(|y| \vee |y'|, \vec{a}^{(1)} \wedge \vec{a}^{(2)}\right) (|y| \vee |y'|) \, |\vec{a}^{(1)} - \vec{a}^{(2)}| \, |y - y'|,$$

where $d_6 = d_6(\vec{a}^{(1)}, \vec{a}^{(2)})$ *and* $d_7 = d_7(\vec{a}^{(1)}, \vec{a}^{(2)})$ *are positive constants defined by*

$$d_6 = \frac{1 - \beta_2}{(\beta_1 + 1)\Big[2(N+2)\max\left\{1, a_i^{(j)} : i = -1, 0, \dots, N,\ j = 1, 2\right\}\Big]^{\beta_1+1}},$$
$$d_7 = \frac{N+1}{(1-\alpha)\min\left\{a_{-1}^{(1)}, a_{-1}^{(2)}, a_N^{(1)}, a_N^{(2)}\right\}}.$$

$$\mathcal{Y}(t) = \mathcal{Y}_0 + [Env(f(t))]^{\frac{2}{2-\beta_2}} + \begin{cases} \int_0^t \|\nabla \Psi_t(\tau)\|^2 d\tau & \text{if } 0 \le t < 1, \\ \int_{t-1}^t \|\nabla \Psi_t(\tau)\|^2 d\tau & \text{if } t \ge 1, \end{cases}$$

$$\mathcal{A} = \limsup_{t\to\infty} f(t)^{\frac{1}{2-\beta_2}} \quad \text{and} \quad \mathcal{K} = \mathcal{A}^2 + \limsup_{t\to\infty} \int_{t-1}^t \|\nabla \Psi_t(\tau)\|^2 d\tau.$$

Theorem (Celik-H.-Ibragimov-Kieu 2016)

(i) *For $t \ge 0$, one has*

$$\int_U |P(x,t)|^2 dx \le \int_U |P(x,0)|^2 dx + C[Env(\mathcal{Y}(t))]^{\frac{2}{2-\beta_2}} |\vec{a}^{(1)} - \vec{a}^{(2)}|^{\frac{2}{2+\beta_1}}.$$

(ii) *If $\mathcal{K} < \infty$ then*

$$\limsup_{t\to\infty} \int_U |P(x,t)|^2 dx \le C(1+\mathcal{K})^{\frac{2}{2-\beta_2}} |\vec{a}^{(1)} - \vec{a}^{(2)}|^{\frac{2}{2+\beta_1}}.$$

THANK YOU!