Asymptotic expansions in large time for solutions of non-autonomous differential equations

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1. Introduction

Foias-Saut result for Navier-Stokes equations

Functional form of NSE

$$\frac{du}{dt} + Au + B(u,u) = f(t),$$

where A is the (unbounded) Stokes operator with, after scaling, $\sigma(A) \subset \mathbb{N}$.
Note: quadratic nonlinearity.
When $f = 0$, solution $u(t)$ admits an expansion

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt}, \quad \text{with polynomials } q_n(t),$$

meaning

$$\|u(t) - \sum_{n=1}^{N} q_n(t) e^{-nt}\| = \mathcal{O}(e^{-(N+\varepsilon)t}) \quad ast \to \infty.$$

Extension to time-dependent forces

NSE with periodic boundary conditions.

• H.-Martinez (2017):

$$f(t) \sim \sum_{n=1}^{\infty} f_n(t) e^{-nt}.$$

Same expansion for $u(t)$:

$$u(t) \sim \sum_{n=1}^{\infty} q_n(t) e^{-nt}.$$

Note: exponential rates are in the additive semigroup generated by $\sigma(A)$.

• Cao-H. (2017)

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-n}.$$

Then

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-n}.$$

Our problems

Focus on ordinary differential equations (ODE) in $\mathbb{R}^n$:

$$\frac{dy}{dt} = -Ay + G(y) + f(t), \quad y(0) = y_0,$$

where

- unknown $y(t) \in \mathbb{R}^n$, given initial condition $y_0 \in \mathbb{R}^n$,
- A is an $n \times n$ matrix,
- $G(y)$ locally is Lipschitz, and has expansion

$$G(y) \sim \sum_{m=2}^{\infty} \mathcal{L}_m(y) \text{ as } y \to 0,$$

- each $\mathcal{L}_m : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous polynomial of degree m,
- $f(t)$ decays exponentially or algebraically at *any* rates.

Goal: Obtain asymptotic expansions for solutions $y(t)$ as $t \to \infty$.

Assumption 1

• Matrix A has positive eigenvalues

$$\Lambda_1 \le \Lambda_2 \le \ldots \le \Lambda_n,$$

and the corresponding eigenvectors form a basis of $\mathbb{R}^n$.

• Rewrite the spectrum

$$\sigma(A) = \{\Lambda_k : k = 1, 2, \ldots, n\} = \{\lambda_1 < \lambda_2 < \ldots\}.$$

Assumption 2

• Rewrite the homogeneous polynomials as

$$\mathcal{L}_m(y) = L_m(y, y, \dots, y) \quad (m \text{ times}),$$

where $L_m : \mathbb{R}^{n\times m} \to \mathbb{R}^n$ is an m-linear mapping.
• For each $N \geq 2$,

$$|G(y) - \sum_{m=2}^{N} \mathcal{L}_m(y)| = \mathcal{O}(|y|^{N+\varepsilon}) \text{ as } y \to 0,$$

for some $\varepsilon > 0$.

Global existence

Theorem

There exists $\varepsilon_0 > 0$ *such that if*

$$|y_0| < \varepsilon_0, \quad \|f\|_\infty = \sup_{t\geq 0} |f(t)| < \varepsilon_0,$$

then there exists a solution $y(t)$ *on* $[0,\infty)$.
In addition, if

$$\lim_{t\to\infty} f(t) = 0,$$

then

$$\lim_{t\to\infty} y(t) = 0.$$

Note: for small y: $|G(y)| \leq C|y|^2$.
Throughout, we consider global solution $y(t)$ on $[0,\infty)$ that converges to zero as $t \to \infty$.

2. Main results

- I. Exponentially decaying forces
- II. Power-decaying forces

I. Exponentially decaying forces

Notation. Exponential expansion (in time):

$$y(t) \overset{\text{exp.}}{\sim} \sum_{k=1}^{\infty} p_k(t) e^{-\alpha_k t},$$

where $\alpha_k > 0$ are strictly increasing constants, and p_k are polynomials, if for any $N \geq 1$, there exists $\varepsilon > 0$, such that

$$|y(t) - \sum_{k=1}^{N} p_k(t) e^{-\alpha_k t}| = \mathcal{O}(e^{-(\alpha_N + \varepsilon)t}) \quad \text{as } t \to \infty.$$

Assumption

Force

$$f(t) \overset{\text{exp.}}{\sim} \sum_{k=1}^{\infty} \tilde{p}_k(t) e^{-\alpha_k t}.$$

Let S be the additive semigroup generated by λ_k and α_k.
Re-order:

$$S = \{\mu_1 < \mu_2 < \mu_3 < \dots\}.$$

Re-write

$$f(t) \overset{\text{exp.}}{\sim} \sum_{k=1}^{\infty} p_k(t) e^{-\mu_k t} = \sum_{k=1}^{\infty} f_k(t).$$

For $\mu \in S$, denote $R_\mu = R_\lambda$ the projection if $\lambda \in \sigma(A)$, otherwise, $R_\mu = 0$.
Still have

$$AR_\mu y = \mu R_\mu y, \quad \mathbb{R}^n = \oplus_{k=1}^{\infty} R_{\mu_k}.$$

Theorem (Cao-H.)

For solution $y(t)$, there exist vector-valued polynomials $q_n(t)$ such that

$$y(t) \sim \sum_{k=1}^{\infty} q_k(t) e^{-\mu_k t} \quad \text{as } t \to \infty.$$

In fact, the polynomials $q_k(t)$'s solve the linear systems

$$q_k' = -(A - \mu_k) q_k + \sum_{m=2}^{N_k} \sum_{\mu_{j_{m,1}} + \mu_{j_{m,2}} + \dots \mu_{j_{m,m}} = \mu_k} L_m(q_{j_{m,1}}, q_{j_{m,2}}, \dots, q_{j_{m,m}}) + p_k(t).$$

Equivalently, $y_k(t) = q_k(t) e^{-\mu_k t}$ solve

$$y_k' = -A y_k + \sum_{m=2}^{N_k} \sum_{\mu_{j_{m,1}} + \mu_{j_{m,2}} + \dots \mu_{j_{m,m}} = \mu_k} L_m(y_{j_{m,1}}, y_{j_{m,2}}, \dots, y_{j_{m,m}}) + f_k(t).$$

Remarks.

- In the sums above, $1 \le j_{m,\ell} \le k-1$, and N_k is finite depending on k, and sufficiently large, for e.g., $N_k\mu_1 \ge \mu_k$.
- Each ODE is a linear system, with the forcing term defined by previous steps.
- The q_k's are unique polynomial solutions provided $R_{\mu_k} q_k(0)$ is given.
- In autonomous case ($f = 0$),

$$q_k' = -(A-\mu_k)q_k + \sum_{m=2}^{N_k} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_k} L_m(q_{j_{m,1}}, q_{j_{m,2}}, \ldots, q_{j_{m,m}}).$$

  Compare this with non-autonomous case.
- The q_k's depend on the initial data y_0.

Notation. Power expansion (in time):

$$y(t) \overset{\text{pow.}}{\sim} \sum_{j=1}^{\infty} \xi_j t^{-\alpha_j},$$

where $\alpha_j > 0$ are strictly increasing, and $\xi_j \in \mathbb{R}^n$ are constant vectors, if for any $N \geq 1$, there exists $\varepsilon > 0$, such that

$$|y(t) - \sum_{j=1}^{N} \xi_j t^{-\alpha_j}| = \mathcal{O}(t^{-(\alpha_N + \varepsilon)}) \quad \text{as } t \to \infty.$$

Assumption

The force has the expansion

$$f(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \tilde{\eta}_k t^{-\alpha_k},$$

where $\tilde{\eta}_k \in \mathbb{R}^n$, and

$$0 < \alpha_1 < \alpha_2 < \dots$$

Let $\mathcal{S} =$ (additive semigroup generated α_k's)$+(\mathbb{N} \cup \{0\})$.
Denote

$$\mathcal{S} = \{0 < \mu_1 = \alpha_1 < \mu_2 < \mu_3 < \dots\}.$$

Rewrite

$$f(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \eta_k t^{-\mu_k} = \sum_{k=1}^{\infty} f_k(t).$$

Theorem (Cao-H.)

For any solution $y(t)$, ones have

$$y(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \xi_k t^{-\mu_k} \quad \textit{as } t \to \infty,$$

where constant vectors $\xi_k \in \mathbb{R}^n$ satisfy

$$A\xi_k = \sum_{m=2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_k} L_m(\xi_{j_{m,1}}, \xi_{j_{m,2}}, \ldots, \xi_{j_{m,m}}) + \eta_k + \xi_p\mu_p$$

in case there exists $1 \le p \le k-1$ such that $\mu_p + 1 = \mu_k$; or

$$A\xi_k = \sum_{m=2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_k} L_m(\xi_{j_{m,1}}, \xi_{j_{m,2}}, \ldots, \xi_{j_{m,m}}) + \eta_k,$$

in case $\mu_p + 1 \ne \mu_k$ for all $1 \le p \le k-1$.

Remarks

- The ξ_k's and hence the expansion are *independent* on initial data y_0, contrasting with the exponential case.
- It means that all (decaying) solutions have the *same* power expansion.

Example

Assume:

- $\alpha_k = k$ for all $k \in \mathbb{N}$
- $G(y) = B(y, y)$.

Then $\mu_k = k$, and $\mathcal{S} = \mathbb{N}$. Expansion

$$y(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \xi_k t^{-k},$$

where

$$\xi_1 = A^{-1}\eta_1,$$

and for $k \geq 2$,

$$\xi_k = A^{-1}\Big\{(k-1)\xi_{k-1} + \sum_{j=1}^{k-1} B(\xi_j, \xi_{k-j}) + \eta_k\Big\}.$$

3. Sketch of proofs

- I. Case of exponential decay
- II. Case of power decay

I. Case of exponential decay

Recall

$$f(t) \overset{\text{exp.}}{\sim} \sum_{k=1}^{\infty} p_k(t) e^{-\mu_k t} = \sum_{k=1}^{\infty} f_k(t).$$

Need to prove

$$y(t) \overset{\text{exp.}}{\sim} \sum_{k=1}^{\infty} q_k(t) e^{-\mu_k t}.$$

Induction step.

Let $y_k(t) = q_k(t)e^{-\mu_k t}$, for $1 \le k \le N$, $\bar{y}_N = \sum_{k=1}^N y_k$ and $v_N = y - \bar{y}_N$.
Induction hypotheses: for $k = 1, 2, \dots, N$

$$v_k = \mathcal{O}(e^{-(\mu_k+\delta_k)t}),$$

and equations for y_k's hold true for $k = 1, 2, \dots, N$.
We will construct the polynomial $q_{N+1}(t)$ such that

$$|w_N(t) - q_{N+1}(t)| = \mathcal{O}(e^{-\delta_{N+1}t}),$$

where

$$w_N(t) = e^{\mu_{N+1}t}v_N(t).$$

Equation for $w_N(t)$:

$$w_N' = -(A - \mu_{N+1}) w_N + \sum_{m\ge 2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_{N+1}} L_m(q_{j_{m,1}}, q_{j_{m,2}}, \ldots, q_{j_{m,m}}) + \mathcal{O}(e^{-\delta t}).$$

For $\mu \in S$, taking R_μ of the equation gives

$$(R_\mu w_N)' = -(\mu - \mu_{N+1}) R_\mu w_N + \sum_{m\ge 2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_{N+1}} R_\mu L_m(q_{j_{m,1}}, q_{j_{m,2}}, \ldots, q_{j_{m,m}}) + \mathcal{O}(e^{-\delta t}).$$

Lemma

Let $(X, \|\cdot\|)$ be a Banach space. Suppose $y(t)$ is in $C([0,\infty),X)$ and $C^1((0,\infty),X)$ that solves the following ODE

$$\frac{dy}{dt} + \alpha y = p(t) + g(t) \quad \text{for } t > 0,$$

where constant $\alpha \in \mathbb{R}$, $p(t)$ is a X-valued polynomial in t, and $g(t) \in C([0,\infty),X)$ satisfies

$$\|g(t)\| \le Me^{-\delta t} \quad \forall t \ge 0, \quad \text{for some } M, \delta > 0.$$

Define $q(t)$ for $t \in \mathbb{R}$ by

$$q(t) = \begin{cases} e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha\tau} p(\tau) d\tau & \text{if } \alpha > 0, \\ y(0) + \int_0^\infty g(\tau)d\tau + \int_0^t p(\tau)d\tau & \text{if } \alpha = 0, \\ -e^{-\alpha t} \int_t^\infty e^{\alpha\tau} p(\tau)d\tau & \text{if } \alpha < 0. \end{cases}$$

Then $q(t)$ is an X-valued polynomial that satisfies

$$\frac{dq(t)}{dt} + \alpha q(t) = p(t) \quad \forall t \in \mathbb{R},$$

and the following estimates hold.

(i) If $\alpha > 0$ then

$$\|y(t)-q(t)\| \le \Big(\|y(0)-q(0)\| + \frac{M}{|\alpha-\delta|}\Big) e^{-\min\{\delta,\alpha\}t}, t \ge 0, \text{ for } \alpha \ne \delta,$$

and

$$\|y(t)-q(t)\| \le (\|y(0)-q(0)\| + Mt)e^{-\delta t}, t \ge 0, \text{ for } \alpha = \delta.$$

(ii) If $(\alpha = 0)$ or $(\alpha < 0$ and $\lim_{t\to\infty} e^{\alpha t} y(t) = 0)$ then

$$\|y(t)-q(t)\| \le \frac{Me^{-\delta t}}{|\alpha-\delta|} \quad \forall t \ge 0.$$

Applying the above ODE lemma, then there exists polynomial $q_{N+1,j} \in R_{\mu_j}(\mathbb{R}^n)$ such that

$$|R_{\mu_j} w_N(t) - q_{N+1,j}(t)| = \mathcal{O}(e^{-\delta_{N+1} t}),$$

Define $q_{N+1} = \sum_j q_{N+1,j}$ (finite sum). Then

$$|w_N(t) - q_{N+1}(t)| = \mathcal{O}(e^{-\delta_{N+1} t}),$$

which yields

$$|y(t) - \sum_{k=1}^{N} q_k(t) e^{-\mu_k t} - q_{N+1}(t) e^{-\mu_{N+1} t}| = \mathcal{O}(e^{-(\mu_{N+1} + \delta_{N+1}) t}).$$

II. Case of power decay

Recall

$$f(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \eta_k t^{-\mu_k}.$$

Need to prove

$$y(t) \overset{\text{pow.}}{\sim} \sum_{k=1}^{\infty} \xi_k t^{-\mu_k}.$$

Induction step. Let $y_k = \xi_k t^{-\mu_k}$, $\bar{y}_N = \sum_{k=1}^{N} y_k$ and $v_N = y - \bar{y}_N$.
Suppose

$$|v_N| = \mathcal{O}(t^{-(\mu_N+\delta_N)}).$$

Let

$$w_N = t^{\mu_{N+1}} v_N.$$

Induction step.

Equation for $w_N(t)$:

$$
\begin{aligned}
w_N' &= -Aw_N \\
&+ t^{\mu_{N+1}}\Big\{ t^{-\mu_{N+1}} \sum_{m\geq 2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_{N+1}} L_m(\xi_{j_{m,1}},\xi_{j_{m,2}},\ldots,\xi_{j_{m,m}})\Big) \\
&+ \eta_{N+1} t^{-\mu_{N+1}} \\
&+ \sum_{k=1}^{N} \Big(t^{-\mu_k} \sum_{m\geq 2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_k} L_m(\xi_{j_{m,1}},\xi_{j_{m,2}},\ldots,\xi_{j_{m,m}}) \\
&- A\xi_k t^{-\mu_k} + \eta_k t^{-\mu_k}\Big) + \sum_{p=1}^{N} \mu_p \xi_p t^{-(\mu_p+1)} \Big\} + \mathcal{O}(t^{-\delta}).
\end{aligned}
$$

Note $\mu_N + 1 \geq \mu_{N+1}$. Moreover

$$\{\mu_p + 1 : 1 \leq p \leq N-1\} \cap [\mu_1, \mu_{N+1}) \subset \{\mu_k : 1 \leq k \leq N\}.$$

Then distribute the red sum into the others including possible $\mathcal{O}(t^{-\delta})$ gives

$$\begin{aligned}
w_N' &= -Aw_N \\
&+ t^{\mu_{N+1}}\Big\{t^{-\mu_{N+1}} \sum_{m\geq 2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_{N+1}} L_m(\xi_{j_{m,1}}, \xi_{j_{m,2}}, \ldots, \xi_{j_{m,m}})\Big) \\
&+ \eta_{N+1} t^{-\mu_{N+1}} + \mu_p \xi_p t^{-(\mu_p+1)}\Big|_{\mu_p+1=\mu_{N+1}} \\
&+ \sum_{k=1}^{N} \Big(-A\xi_k t^{-\mu_k} + t^{-\mu_k} \sum_{m\geq 2} \sum_{\mu_{j_{m,1}}+\mu_{j_{m,2}}+\ldots\mu_{j_{m,m}}=\mu_k} L_m(\xi_{j_{m,1}}, \xi_{j_{m,2}}, \ldots, \xi_{j_{m,m}}) \\
&+ \eta_k t^{-\mu_k} + \mu_p \xi_p t^{-(\mu_p+1)}\Big|_{\mu_p+1=\mu_k} \Big) \Big\} \\
&+ \mathcal{O}(t^{-\delta}).
\end{aligned}$$

Thus,

$$w_N' = -Aw_N + A\xi_{N+1} + \mathcal{O}(t^{-\delta}).$$

Lemma

If for some $\alpha > 0$,

$$y' = -Ay + \xi + \mathcal{O}(t^{-\alpha}),$$

then

$$y(t) = A^{-1}\xi + \mathcal{O}(t^{-\alpha}).$$

Proof.

$$\begin{aligned} y(t) &= e^{-tA}y_0 + e^{-tA}\int_0^t e^{\tau A}\xi d\tau + \int_0^t e^{-(t-\tau)A}\mathcal{O}(\tau^{-\alpha})d\tau \\ &= e^{-tA}y_0 + e^{-tA}A^{-1}(e^{tA}\xi - \xi) + \mathcal{O}(t^{-\alpha}) \\ &= A^{-1}\xi + \mathcal{O}(t^{-\alpha}). \quad \square \end{aligned}$$

Then $\quad w_N(t) = A^{-1}(A\xi_{N+1}) + \mathcal{O}(t^{-\delta}) = \xi_{N+1} + \mathcal{O}(t^{-\delta}).$

Thus,

$$v_N(t) = \xi_{N+1}t^{-\mu_{N+1}} + \mathcal{O}(t^{-(\mu_{N+1}+\delta)}).$$

4. Application to solutions near special periodic orbits

Application (demonstration)

On the plane $n = 2$, $y = (y_1, y_2)$, $r = |y| = \sqrt{y_1^2 + y_2^2}$.
In polar coordinates, i.e., $y(t) = r(t)(\cos(\theta(t), \sin(\theta(t))$, assume

$$\begin{cases} r' = (r-1)(r-2), \\ \theta' = 1. \end{cases}$$

Then $r = 1, 2$ and $\theta = \theta_0 + t$ are periodic solutions.
The first ($r = 1$) is asymptotically stable, and the second ($r = 2$) is unstable.
Denote the first periodic orbit by $y^*(t) = (\cos(\theta_0 + t), \sin(\theta_0 + t))$.

Expansion

Let $z = r - 1$, then

$$z' = z(z-1) = -z + z^2, \quad z(0) = z_0 \in (-1, 1).$$

Then $z(t)$ admits an expansion:

$$z(t) = \sum_{k=1}^{\infty} q_k(t) e^{-kt},$$

where real-valued polynomials q_k's solve

$$\frac{dq_k}{dt} = (k-1)q_k + \sum_{j+\ell=k} q_j q_\ell.$$

Hence the solution $y(t)$ has expansion

$$y(t) \overset{\text{exp.}}{\sim} y^*(t)\Big(1 + \sum_{k=1}^{\infty} q_k(t) e^{-kt}\Big).$$

Calculations

- $q_1(t) = \xi_1 e^{-t}$,
- $q_2'(t) = q_2(t) + \xi_1^2$. Hence, $q_2(t) = -e^t \int_{-\infty}^t e^{-\tau}\xi_1^2 d\tau = \xi_1^2$.
- Claim: $q_k(t) = c_k \xi_1^k$. Indeed, prove by induction,

$$q_k' = (k-1)q_k + s_k\xi_1^k, \quad s_k = \sum_{j=1}^{k-1} c_j.$$

Then, $q_k(t) = -e^{(k-1)t}\int_{-\infty}^t e^{-(k-1)\tau} s_k \xi_1^k d\tau = \frac{s_k}{k-1}\xi_1^k$, where

$$c_1 = 1, \; c_k = \frac{1}{k-1}(\sum_{j=1}^{k-1} c_j) = 1.$$

Thus, $y(t) \overset{\text{exp.}}{\sim} y^*(t)\Big(1 + \sum_{k=1}^\infty \xi_1^k e^{-kt}\Big)$.

- Explicitly, $z(t) = \frac{1}{1-(z_0-1)/z_0 e^t} = \frac{\xi_1 e^{-t}}{1-\xi_1 e^{-t}} = \sum_{k=1}^\infty \xi_1^k e^{-kt}$, with $\xi_1 = z_0/(z_0-1)$.

Non-autonomous case I: Exponential perturbation

$$r' = (r-1)(r-2) + \sum_{k=1}^{\infty} p_k(t)e^{-kt}, \quad \theta' = 1.$$

Then

$$z' = -z + z^2 + \sum_{k=1}^{\infty} p_k(t)e^{-kt}.$$

Similarly,

$$y(t) \overset{\text{exp.}}{\sim} y^*(t)\Big(1 + \sum_{k=1}^{\infty} q_k(t)e^{-kt}\Big),$$

where

$$\frac{dq_k}{dt} = (k-1)q_k + \sum_{j+\ell=k} q_j q_\ell + p_k.$$

Non-autonomous case II: Power perturbation

Assume there are $d_k \in \mathbb{R}$:

$$r' = (r-1)(r-2) + \sum_{k=1}^{\infty} d_k t^{-k}, \quad \theta' = 1.$$

Then

$$z' = -z + z^2 + \sum_{k=1}^{\infty} d_k t^{-k}.$$

We obtain

$$y(t) \overset{\text{pow.}}{\sim} y^*(t)\Big(1 + \sum_{k=1}^{\infty} a_k t^{-k}\Big),$$

where

$$a_1 = d_1,\ a_k = (k-1)a_{k-1} + \sum_{j+\ell=k} a_j a_\ell + d_k \text{ for } k \geq 2.$$

THANK YOU FOR YOUR ATTENTION.