On the normal form of Navier-Stokes equations in Gevrey spaces

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Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p - \nabla \phi, \\ \operatorname{div} u = 0, \\ \mathbf{u}(x,0) = u^0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
ϕ is the potential of the body force,
u^0 is the initial velocity.

Let $L > 0$ and $\Omega = (0, L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Functional setting

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued L-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.

$$H = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3,$$
$$V = \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \quad \mathcal{D}(A) = \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3.$$

Norm on H: $|u| = \|u\|_{L^2(\Omega)}$. Norm on V: $\|u\| = |\nabla u|$.
The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u,v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}(A).$$

P_L is the Leray projection from $L^2(\Omega)$ onto H.
Spectrum of A:

$$\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}.$$

Denote by $R_N H$ the eigenspace of A corresponding to N.

Functional form of NSE

Denote by $\mathcal{R}$ the set of all initial data $u^0 \in V$ such that the solution $u(t)$ is regular for all $t > 0$. The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \ t > 0,$$

$$u(0) = u^0 \in \mathcal{R},$$

where the equation holds in $\mathcal{D}(A)$ for all $t > 0$ and $u(t)$ is continuous from $[0, \infty)$ into V.

Asymptotic expansion - Normalization map

For $u_0 \in \mathcal{R}$, the solution $u(t)$ has an asymptotic expansion: [Foias-Saut]

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \ldots,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in t with trigonometric polynomial values. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$,

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$.
Let $W(u^0) = \xi_1 \oplus \xi_2 \oplus \cdots$, where $\xi_j = R_j q_j(0)$, for $j = 1, 2, 3\ldots$ Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Fréchet space

$$S_A = R_1 H \oplus R_2 H \oplus \cdots .$$

Constructions of polynomials $q_j(t)$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, ...)$, then q_j's are the unique polynomial solutions to the following equations

$$q_j' + (A - j)q_j + \beta_j = 0,$$

with $R_j q_j(0) = \xi_j$, where β_j's are defined by

$$\beta_1 = 0 \text{ and for } j > 1,\ \beta_j = \sum_{k+l=j} B(q_k, q_l).$$

Explicitly, these polynomials $q_j(t)$'s are recurrently given by

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau + \sum_{n\geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} (\frac{d}{dt})^n (I - R_j)\beta_j,$$

where $[(A - j)(I - R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik\cdot x}$, for $u(x) = \sum_{|k|^2 \neq j} a_k e^{ik\cdot x} \in \mathcal{V}$.

Normal form in S_A

The S_A-valued function $\xi(t) = (\xi_n(t))_{n=1}^{\infty} = (W_n(u(t)))_{n=1}^{\infty} = W(u(t))$ satisfies the following system of differential equations

$$\frac{d\xi_1(t)}{dt} + A\xi_1(t) = 0,$$
$$\frac{d\xi_j(t)}{dt} + A\xi_j(t) + \sum_{k+l=j} R_j B(\mathcal{P}_k(\xi(t)), \mathcal{P}_l(\xi(t))) = 0, \ \ n > 1,$$

where $P_j(\xi) = q_j(0, \xi)$ for $\xi \in S_A$.
This system is the normal form (in S_A) of the Navier–Stokes equations associated with the asymptotic expansions of regular solutions.

The normal form (with power series)

For $d \geq 1$, let

$$\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^{\infty} \mathcal{P}_j^{[d]}(\xi) = \sum_{j=d}^{\infty} q_j^{[d]}(0, \xi).$$

For $d \geq 2$, let

$$\mathcal{B}^{[d]}(\xi) = \sum_{j=1}^{\infty} \mathcal{B}_j^{[d]}(\xi) = \sum_{j=1}^{\infty} \sum_{k+l=j} \sum_{m+n=d} R_j B\big(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)\big).$$

Rewrite the normal form:

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0.$$

Normal form in C^∞

Let E^∞ be the Fréchet space $C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V$.

Theorem (Foias-H.-Saut 2011)

The formal power series change of variable

$$u = \xi + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}(\xi),$$

where $\xi \in E^\infty$, reduces the Navier–Stokes equations to a Poincaré–Dulac normal form

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0.$$

Key Sobolev-norm estimates of homogeneous polynomials

Theorem (Foias-H.-Saut 2011)

Let $\alpha \geq 1/2$, $d \geq 1$. Then $\mathcal{P}^{[d]}(\xi)$ is a continuous homogeneous polynomial of degree d from $\mathcal{D}(A^{\alpha+3d/2})$ to $\mathcal{D}(A^{\alpha})$:

$$|A^{\alpha}\mathcal{P}^{[d]}(\xi)| \leq M(\alpha, d)|A^{\alpha+3d/2}\xi|^{d}.$$

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General Gevrey classes

For $\alpha \geq 0$, $\sigma \geq 0$, define

$$A^{\alpha}e^{\sigma A^{1/2}}u = \sum_{\mathbf{k}\neq 0} |\mathbf{k}|^{2\alpha}\hat{u}(\mathbf{k})e^{\sigma|\mathbf{k}|}e^{i\mathbf{k}\cdot\mathbf{x}}, \text{ for } u = \sum_{\mathbf{k}\neq 0} \hat{u}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \in H.$$

The domain of $A^{\alpha}e^{\sigma A^{1/2}}$ is

$$\mathcal{D}(A^{\alpha}e^{\sigma A^{1/2}}) = \{u \in H : |A^{\alpha}e^{\sigma A^{1/2}}u| < \infty\}.$$

Main result: Gevrey-norm estimates

Theorem

Given $\alpha \geq 1/2$*,* $\bar{\sigma} > \sigma > 0$*. Then*

$$|A^\alpha e^{\sigma A^{1/2}} \mathcal{P}^{[d]} \xi| \leq C(\alpha, d, \sigma, \bar{\sigma}) |A^\alpha e^{\bar{\sigma} A^{1/2}} \xi|^d.$$

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Indices

Set of (general) indices: $GI = \bigcup_{n=1}^{\infty} GI(n)$, where for $n \geq 1$,

$$GI(n) = \{\bar{\alpha} = (\alpha_k)_{k=1}^{\infty},\ \alpha_k \in \{0,1,2,\ldots\},\ \alpha_k = 0 \text{ for } k > n\}.$$

For $\bar{\alpha} \in GI$, define

$$|\bar{\alpha}| = \sum_{k=1}^{\infty} \alpha_k \text{ and } \|\bar{\alpha}\| = \sum_{k=1}^{\infty} k\alpha_k.$$

For $d, n \geq 1$, define the set of special multi-indices:

$$SI(d,n) = \Big\{\bar{\alpha} = (\alpha_k)_{=1}^{\infty} \in GI, |\bar{\alpha}| = d,\ \|\bar{\alpha}\| = n\Big\};$$

note $1 \leq d \leq n$ hence $SI(d,n) \subset GI(n)$. Also, for $n \geq d \geq 1$ and $n' \geq d' \geq 1$ we have

$$SI(d,n) + SI(d',n') \subset SI(d+d', n+n').$$

Degrees and Resonance

$$q_j(t,\xi) = \sum_{m=0}^{j-1} q_{j,m}(\xi) t^m = \sum_{m=0}^{j-1} \sum_{d=1}^{j} q_{j,m}^{[d]}(\xi) t^m = \sum_{d=1}^{j} q_j^{[d]}(t,\xi),$$

$$q_j^{[d]}(t,\xi) = \sum_{|\bar\alpha|=d} q_j^{[d],(\bar\alpha)}(t,\xi),$$

Lemma

(i) $\deg_t q_j(t,\xi) \le j-1$, $\deg_t q_j^{[d]}(t,\xi) \le d-1$.
(ii) If $q_j^{[d],(\bar\alpha)} \ne 0$ *then* $\bar\alpha \in SI(d,j)$.
(iii) Consequently, for each (non-zero) monomial of $\mathcal{P}_j(\xi)$, $j \ge 1$ *having degree* α_k *in* ξ_k, $k \ge 1$, *one has*

$$\sum \alpha_k = d, \quad \sum k\alpha_k = j.$$

Homogeneous gauge

Let $\xi = (\xi_k)_{k=1}^\infty \in S_A$ and $\bar\alpha = (\alpha_k)_{k=1}^\infty \in GI(n)$, define

$$\big[\,\xi\,\big]^{\bar\alpha} = |\xi_1|^{\alpha_1}|\xi_2|^{\alpha_2}\dots|\xi_n|^{\alpha_n}.$$

For $n \ge d \ge 1$, define

$$[[\,\xi\,]]_{d,n} = \left(\sum_{\bar\alpha\in SI(d,n)} \big[\,\xi\,\big]^{2\bar\alpha}\right)^{1/2} = \left(\sum_{|\bar\alpha|=d,\|\bar\alpha\|=n} \big[\,\xi\,\big]^{2\bar\alpha}\right)^{1/2}.$$

We have the following properties

$$\big[\,\xi\,\big]^{\bar\alpha}\big[\,\xi\,\big]^{\bar\alpha'} = \big[\,\xi\,\big]^{\bar\alpha+\bar\alpha'},$$

$$\big[\,\xi\,\big]^{r\bar\alpha} = (\big[\,\xi\,\big]^{\bar\alpha})^r \text{ for } r = 0,1,2,\dots,$$

$$\sum_{|\bar\alpha|=d} \big[\,\xi\,\big]^{2\bar\alpha} = |\xi|^{2d}.$$

$$[[\,\xi\,]]_{d,n} \le \Big(\sum_{\bar\alpha\in GI(n),|\bar\alpha|=d} \big[\,\xi\,\big]^{2\bar\alpha}\Big)^{1/2} \le |P_n\xi|^d.$$

Multiplicative and Poincaré inequalities

Lemma (Foias-H.-Saut 2011)

Let $\xi \in S_A$, $n \ge d \ge 1$ *and* $n' \ge d' \ge 1$. *Then*

$$[[\,\xi\,]]_{d,n} \cdot [[\,\xi\,]]_{d',n'} \le e^{d+d'}\, [[\,\xi\,]]_{d+d',n+n'}\,.$$

Note: The constant on the RHS is independent of n, n'.

Lemma (Foias-H.-Saut 2011)

For any $\xi \in S_A$, *any numbers* $\alpha, s \ge 0$ *and* $n \ge d \ge 1$, *one has*

$$[[\,A^\alpha \xi\,]]_{d,n} \le \left(\frac{d}{n}\right)^s \left[\left[\,A^{\alpha+s}\xi\,\right]\right]_{d,n} \le \left(\frac{d}{n}\right)^s |P_n A^{\alpha+s}\xi|^d.$$

Estimates for the bilinear form

Lemma

If $\alpha \geq 1/2$ then

$$|A^\alpha e^{\sigma A^{1/2}} B(u,v)| \leq K^\alpha |A^{\alpha+1/2} e^{\sigma A^{1/2}} u|\, |A^{\alpha+1/2} e^{\sigma A^{1/2}} v|,$$

for all $u, v \in \mathcal{D}(e^{\sigma A^{1/2}} A^{\alpha+1/2})$.

Using the fact that $\sup_{x \geq 0} x e^{-cx} = (ce)^{-1}$ for any $c > 0$, we have:

Corollary

If $\alpha \geq 1/2$ and $\sigma_1, \sigma_2 > \sigma > 0$ then

$$|A^\alpha e^{\sigma A^{1/2}} B(u,v)| \leq K^\alpha (\sigma_1 - \sigma)^{-1} (\sigma_2 - \sigma)^{-1} |A^\alpha e^{\sigma_1 A^{1/2}} u|\, |A^\alpha e^{\sigma_2 A^{1/2}} v|,$$

for all $u \in \mathcal{D}(e^{\sigma_1 A^{1/2}} A^\alpha)$, $v \in \mathcal{D}(e^{\sigma_2 A^{1/2}} A^\alpha)$.

Lemma

For $j \geq 2$ and $\alpha \geq 0$,

$$|A^\alpha e^{\sigma A^{1/2}} q_{j,0}^{[d]}(\xi)|^2 \leq 2(d!)(d-1)!\Big(|A^\alpha e^{\sigma A^{1/2}} \xi_j^{[d]}|^2 + \sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha e^{\sigma A^{1/2}} (I-R_j)\beta_{j,n}^{[d]}(\xi)|^2\Big);$$

$$|A^\alpha e^{\sigma A^{1/2}} q_{j,m}^{[d]}(\xi)|^2 \leq 2(d!)(d-1)!\Big(\frac{|A^\alpha e^{\sigma A^{1/2}} R_j \beta_{j,m-1}^{[d]}(\xi)|^2}{m^2} + \frac{1}{m!^2} \sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha e^{\sigma A^{1/2}} (I-R_j)\beta_{j,n}^{[d]}(\xi)|^2\Big)$$

for $m = 1, \ldots, (j-2)\big|_d$; and

$$|A^\alpha e^{\sigma A^{1/2}} q_{j,j-1}^{[d]}(\xi)|^2 = \frac{|A^\alpha e^{\sigma A^{1/2}} R_j \beta_{j,j-2}^{[d]}(\xi)|^2}{(j-1)^2}.$$

Estimates of homogeneous polynomials

Let $\Delta\sigma = \bar{\sigma} - \sigma$, $\sigma' = \sigma + \frac{3}{4}\Delta\sigma$, $\sigma_n = \sigma + \frac{\Delta\sigma}{2n}$ for $n \in \mathbb{N}$. Then

$$\bar{\sigma} > \sigma' > \sigma_1 > \sigma_2 > \sigma_3 > \ldots > \sigma.$$

Proposition

For $j \geq d \geq 1$ and $0 \leq m \leq (j-1)\big|_d$, one has

$$|A^\alpha e^{\sigma_d A^{1/2}} q_{j,m}^{[d]}(\xi)| \leq c(\alpha, d) j^{d-1} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j},$$

for all $\xi \in S_A$ and $\alpha \geq 1/2$, where

$$c(\alpha, d) = E(\alpha, d)^{d-1} = [8K^\alpha d^9 (d!) e^d (\Delta\sigma)^{-2}]^{d-1}.$$

In particular, when $m = 0$ one has

$$|A^\alpha e^{\sigma_d A^{1/2}} \mathcal{P}_j^{[d]}(\xi)| \leq c(\alpha, d) j^{d-1} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}.$$

Proof. By induction in j and the use of Multiplicative Inequality:

$$
\begin{aligned}
&|A^\alpha e^{\sigma_d A^{1/2}} \beta^{[d]}_{j,m}(\xi)| \\
&\le \sum_{l+l'=j} \sum_{r+r'=m} \sum_{s+s'=d} K^\alpha (\sigma_s - \sigma_d)^{-1} (\sigma_{s'} - \sigma_d)^{-1} |A^\alpha e^{\sigma_s A^{1/2}} q^{[s]}_{l,r}(\xi)| \\
&\cdot |A^\alpha e^{\sigma_{s'} A^{1/2}} q^{[s']}_{l',r'}(\xi))| \le \sum_{l+l'=j} \sum_{r+r'=m} \sum_{s+s'=d} K^\alpha (\sigma_s - \sigma_d)^{-1} (\sigma_{s'} - \sigma_d)^{-1} \\
&\cdot E(\alpha, s)^{s-1} l^{s-1} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{s,l} E(\alpha, s)^{s'-1} l'^{s'-1} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{s',l'} \\
&\le \sum_{l+l'=j} \sum_{r+r'=m} \sum_{s+s'=d} K^\alpha 4 d^4 (\Delta\sigma)^{-2} \\
&\quad \times E(\alpha, d)^{s+s'-2} e^{s+s'} j^{s+s'-2} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{s+s',l+l'} \\
&\le \sum_{l+l'=j} \sum_{r+r'=m} \sum_{s+s'=d} 4K^\alpha d^4 (\Delta\sigma)^{-2} E(\alpha, d)^{d-2} e^d j^{d-2} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j} \\
&\le d^2 \cdot 4K^\alpha d^4 (\Delta\sigma)^{-2} E(\alpha, d)^{d-2} e^d j^{d-2} \cdot j \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}.
\end{aligned}
$$

Proof. (cont) Letting $N = N(\alpha, d) = 4K^{\alpha} d^6 e^d (\Delta\sigma)^{-2} E(\alpha, d)^{d-2}$, we have

$$|A^{\alpha} e^{\sigma_d A^{1/2}} \beta_{j,m}^{[d]}(\xi)| \le N j^{d-1} \left[\left[A^{\alpha + h\delta(d-1)} \xi \right]\right]_{d,j}.$$

Then

$$\begin{aligned}
|A^{\alpha} e^{\sigma_d A^{1/2}} q_{j,m}^{[d]}|^2 &\le j^{2(d-1)} (d-1)! d! 2 \Big(N^2 \left[\left[A^{\alpha} e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}^2 \\
&\quad + \sum_{n=0}^{(j-2)|_{d-1}} N^2 \left[\left[A^{\alpha} e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}^2 \Big) \\
&\le j^{2(d-1)} d!^2 N^2 2 \left[\left[A^{\alpha} e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}^2 \\
&= j^{2(d-1)} 2N^2 (d!)^2 \left[\left[A^{\alpha} e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}^2.
\end{aligned}$$

We can verify

$$2N^2 (d!)^2 \le c(\alpha, d)^2.$$

Proof of the Main Result

Let $\bar\sigma > \sigma > 0$, $\alpha \geq 1/2$, $d \geq 1$, then

$$|A^\alpha e^{\sigma A^{1/2}} \mathcal{P}^{[d]}(\xi)| \leq C |A^\alpha e^{\bar\sigma A^{1/2}} \xi|^d.$$

Proof. Since $\bar\sigma > \sigma' > \sigma_n > \sigma > 0$ for all n, and $r > 0$, then

$$|A^\alpha e^{\underline{\sigma} A^{1/2}} \mathcal{P}^{[d]}(\xi)| \leq \sum_{j=d}^\infty |A^\alpha e^{\underline{\sigma} A^{1/2}} \mathcal{P}_j^{[d]}(\xi)| \leq \sum_{j=d}^\infty |A^\alpha e^{\sigma_d A^{1/2}} \mathcal{P}_j^{[d]}(\xi)|$$

$$\leq \sum_{j=d}^\infty c(\alpha, d) j^{d-1} \left[\left[A^\alpha e^{\sigma' A^{1/2}} \xi \right]\right]_{d,j}$$

$$\leq \sum_{j=d}^\infty c(\alpha, d) \frac{d^{d+r}}{j^{1+r}} |A^{\alpha+d+r} e^{\sigma' A^{1/2}} \xi|^d = C |A^{\alpha+d+r} e^{\sigma' A^{1/2}} \xi|^d$$

$$\leq C' |A^\alpha e^{\bar\sigma A^{1/2}} \xi|^d.$$

THANK YOU FOR YOUR ATTENTION!