Analysis of Single and Multi Phase Flows in Porous Media

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Introduction: Darcy's and Forchheimer's flows

Fluid flows in porous media with velocity u and pressure p:

- Darcy's Law:

$$\alpha u = -\nabla p,$$

- Forchheimer's "two term" law

$$\alpha u + \beta |u|\, u = -\nabla p,$$

- Forchheimer's "three term" law

$$\mathcal{A} u + \mathcal{B}\, |u|\, u + \mathcal{C} |u|^2\, u = -\nabla p.$$

- Forchheimer's "power" law

$$a u + c^n |u|^{n-1}\, u = -\nabla p,$$

Here $\alpha, \beta, a, c, n, \mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

Generalized Forchheimer equations

[Aulisa-Bloshanskaya-H.-Ibragimov 2009]
Generalizing the above equations as follows

$$g(|u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} \Rightarrow u = -K(|\nabla p|)\nabla p,$$

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

Class $FP(N, \vec{\alpha})$. Let $N > 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_N$,

$$FP(N, \vec{\alpha}) = \Big\{ g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_N s^{\alpha_N} \Big\},$$

where $a_0, a_N > 0$, $a_1, \ldots, a_{N-1} \geq 0$. Notation: $\alpha_N = \deg(g)$,
$\vec{a} = (a_0, a_1, \ldots, a_N)$, $a = \frac{\alpha_N}{\alpha_N + 1} \in (0,1)$, $b = \frac{\alpha_N}{\alpha_N + 2} \in (0,1)$.

Historical remarks

- Darcy-Dupuit: 1865
- Forchheimer: 1901
- Other nonlinear models: 1940s–1960s
- Incompressible fluids: Payne, Straughan and collaborators since 1990's, Celebi-Kalantarov-Ugurlu since 2005 (Brinkman-Forchheimer)
- Derivation of non-Darcy, non-Forchheimer flows: Marusic-Paloka and Mikelic 2009 (homogenization for Navier–Stokes equations), Balhoff et. al. 2009 (computational)

Works on generalized Forchheimer flows

A. Single-phase flows.

- 1990's Numerical study
- L^2-theory (for slightly compressible flows):
  Aulisa-Bloshanskaya-H.-Ibragimov (2009), H.-Ibragimov: Dirichlet B.C. (2011), H.-Ibragimov Flux B.C. (2012), Aulisa-Bloshanskaya-Ibragimov total flux, productivity index (2011, 2012), Inhomogeneous media Celik-H.(in preparation).
- L^α-theory: H.-Ibragimov-Kieu-Sobol (2012-preprint)
- L^∞, $W^{1,p}$-theory: H.-Kieu-Phan (2014), Celik-H.(in preparation).
- $W^{1,\infty}$-theory: interior H.-Kieu (2014-preprint), global Celik-H.-Kieu (in preparation).

B. Multi-phase flows.

- One-dimensional case: H.-Ibragimov-Kieu (2013).
- Multi-dimensional case: H.-Ibragimov-Kieu (preprint).

Note: there are more works on Forchheimer flows (2-terms or 3 terms).

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Single-phase Forchheimer flows

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho u) = 0.$$

For slightly compressible fluid:

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Then

$$\frac{dp}{dt} = \kappa \nabla \cdot \Big(K(|\nabla p|)\nabla p \Big) + K(|\nabla p|)|\nabla p|^2.$$

Since $\kappa \gg 1$, we neglect the last terms, after scaling the time variable:

$$\frac{dp}{dt} = \nabla \cdot \Big(K(|\nabla p|)\nabla p \Big).$$

Degeneracy - The Degree Condition

Lemma

Let $g(s,\vec{a})$ be in class $FP(N,\vec{\alpha})$. One has for any $\xi \geq 0$ that

$$\frac{C_1(\vec{a})}{(1+\xi)^a} \leq K(\xi,\vec{a}) \leq \frac{C_2(\vec{a})}{(1+\xi)^a},$$

$$C_3(\vec{a})(\xi^{2-a}-1) \leq K(\xi,\vec{a})\xi^2 \leq C_2(\vec{a})\xi^{2-a}.$$

Degree Condition (DC)

$$\deg(g) \leq \frac{4}{n-2} \iff 2 \leq (2-a)^* = \frac{n(2-a)}{n-(2-a)}.$$

Under the (DC), $W^{1,2-a}(U) \subset L^2(U)$.
This talk: No conditions on the degree.

IBVP

We study the resulting parabolic equation for the pressure $p = p(x,t)$:

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p), \quad x \in U, \ t > 0.$$

Dirichlet boundary data:

$$p = \psi(x,t), \quad x \in \Gamma, \ t > 0,$$

where $\psi(x,t)$ is known.
The initial data

$$p(x,0) = p_0(x) \text{ is given.}$$

Extension $\Psi(x,t)$ of $\psi(x,t)$ to $x \in \bar{U}, \ t \geq 0$.

For $\alpha \geq 1$, we define

$$A(\alpha, t) = \Big[\int_U |\nabla \Psi(x,t)|^{\frac{\alpha(2-a)}{2}} dx \Big]^{\frac{2(\alpha-a)}{\alpha(2-a)}} + \Big[\int_U |\Psi_t(x,t)|^{\alpha} dx \Big]^{\frac{\alpha-a}{\alpha(1-a)}}$$

for $t \geq 0$, and

$$A(\alpha) = \limsup_{t\to\infty} A(\alpha, t) \quad \text{and} \quad \beta(\alpha) = \limsup_{t\to\infty} [A'(\alpha, t)]^{-}.$$

Also, define $\alpha_* = \frac{an}{2-a}$, and for $\alpha > 0$ define

$$\widehat{\alpha} = \max \{\alpha, 2, \alpha_*\}.$$

Whenever $\beta(\alpha)$ is in use, it is understood that the function $t \to A(\alpha, t)$ belongs to $C^1((0,\infty))$.
For a function $f : [0,\infty) \to \mathbb{R}$, we denote by *Envf* a continuous and increasing function $F : [0,\infty) \to \mathbb{R}$ such that $F(t) \geq f(t)$ for all $t \geq 0$.
Denote $\overline{p} = p - \Psi$.

Theorem (H.-Ibragimov-Kieu-Sobol)

Let $\alpha > 0$.

(i) *For all* $t \geq 0$,

$$\int_U |\bar{p}(x,t)|^\alpha dx \leq C\Big(1 + \int_U |\bar{p}(x,0)|^{\widehat{\alpha}} dx + [EnvA(\widehat{\alpha},t)]^{\frac{\widehat{\alpha}}{\widehat{\alpha}-a}}\Big).$$

(ii) *If* $A(\widehat{\alpha}) < \infty$ *then*

$$\limsup_{t\to\infty} \int_U |\bar{p}(x,t)|^\alpha dx \leq C\big(1 + A(\widehat{\alpha})^{\frac{\widehat{\alpha}}{\widehat{\alpha}-a}}\big).$$

(iii) *If* $\beta(\widehat{\alpha}) < \infty$ *then there is* $T > 0$ *such that*

$$\int_U |\bar{p}(x,t)|^\alpha dx \leq C\big(1 + \beta(\widehat{\alpha})^{\frac{\widehat{\alpha}}{\widehat{\alpha}-2a}} + A(\widehat{\alpha},t)^{\frac{\widehat{\alpha}}{\widehat{\alpha}-a}}\big) \quad \textit{for all } t \geq T.$$

For gradient and time derivative estimates, we denote

$$G_1(t) = \int_U |\nabla\Psi(x,t)|^2 dx + \Big[\int_U |\Psi_t(x,t)|^{r_0} dx\Big]^{\frac{2-a}{r_0(1-a)}} + \Big[\int_U |\Psi_t(x,t)|^{r_0} dx\Big]^{\frac{1}{r_0}},$$

$$G_2(t) = \int_U |\nabla\Psi_t(x,t)|^2 dx + \int_U |\Psi_t(x,t)|^2 dx,$$

$$G_3(t) = G_1(t) + G_2(t), \quad G_4(t) = G_3(t) + \int_U |\Psi_{tt}|^2 dx.$$

with $r_0 = \frac{n(2-a)}{(2-a)(n+1)-n}$.
Define

$$H(\xi) = \int_0^{\xi^2} K(\sqrt{s})ds \quad \text{for } \xi \geq 0.$$

Then

$$H(\xi) \sim K(\xi)\xi^2 \sim \xi^{2-a}.$$

For $t \geq 0$, recall from H.-Ibragimov (2011) that

$$\int_0^t \int_U H(|\nabla p|) dx d\tau \leq C \int_U \bar{p}^2(x,0) dx + C \int_0^t G_1(\tau) d\tau,$$

and

$$\int_U H(|\nabla p|)(x,t) dx + \int_0^t \int_U |\bar{p}_t(x,\tau)|^2 dx d\tau$$
$$\leq \int_U \left[H(|\nabla p(x,0)|) + \bar{p}^2(x,0) \right] dx + C \int_0^t G_3(\tau) d\tau.$$

Let $\alpha \geq \widehat{2}$. For $t > 0$, recall from H.-Ibragimov-Kieu-Sobol that

$$\int_U |\bar{p}_t(x,t)|^2 dx \leq C(1+t^{-1}) \Big(1 + \int_U |\bar{p}(x,0)|^\alpha dx + \int_U H(|\nabla p(x,0)|) dx$$
$$+ [EnvA(\alpha,t)]^{\frac{\alpha}{\alpha-a}} + \int_0^t G_4(\tau) d\tau \Big).$$

Lemma

For $t \geq 1$*,*

$$\int_{t-1}^{t}\int_U H(|\nabla p|)dxd\tau \leq C\int_U \bar{p}^2(x,t-1)dx + C\int_{t-1}^{t} G_1(\tau)d\tau,$$

$$\int_U H(|\nabla p|)(x,t)dx + \frac{1}{2}\int_{t-1/2}^{t}\int_U \bar{p}_t^2(x,\tau)dxd\tau$$

$$\leq C\int_U \bar{p}^2(x,t-1)dx + C\int_{t-1}^{t} G_3(\tau)d\tau,$$

$$\int_U \bar{p}_t^2(x,t)dx \leq C\int_U \bar{p}^2(x,t-1)dx + C\int_{t-1}^{t} G_4(\tau)d\tau.$$

Interior L^∞-estimates for pressure

Theorem

Let $U' \Subset U$. *If* $T_0 \geq 0$, $T > 0$ *and* $\theta \in (0,1)$ *then*

$$\sup_{[T_0+\theta T, T_0+T]} \|p(t)\|_{L^\infty(U')} \leq C(1+T)^{\frac{\kappa_1}{\kappa_0}} \Big(1 + (\theta T)^{-1}\Big)^{\frac{\kappa_1}{\alpha - a}} \cdot \Big(1 + \|p\|_{L^\alpha(U\times(T_0,T_0+T))}\Big)^{\kappa_2}.$$

Proof by De Giorgi's technique with cut-off functions in both spatial and time variables.

Parabolic Poincaré-Sobolev inequality

For each $T > 0$, denote $Q_T = U \times (0, T)$. Recall $\alpha_* = \frac{an}{2-a}$.

Lemma

Assume $\alpha \geq 2$ *and* $\alpha > \alpha_*$. *Let* $p = \alpha\Big(1 + \frac{2-a}{n}\Big) - a$. *Then*

$$\|u\|_{L^p(Q_T)} \leq C(1 + \delta T)^{1/p}[[u]],$$

where $\delta = 1$ *in general,* $\delta = 0$ *in case* u *vanishes on the boundary* ∂U, *and*

$$[[u]] = \operatorname*{ess\,sup}_{[0,T]} \|u(t)\|_{L^\alpha(U)} + \Big(\int_0^T \int_U |u|^{\alpha-2}|\nabla u|^{2-a} dxdt\Big)^{\frac{1}{\alpha-a}}.$$

In case $U = B_R$, *the inequality holds with* $[[u]]$ *defined by*

$$R^{n(\frac{1}{\alpha}-\frac{1}{p})} \operatorname*{ess\,sup}_{[0,T]} \|u(t)\|_{L^\alpha(B_R)} + R^{\frac{n+2-a}{\alpha-a}-\frac{n}{p}}\Big(\int_0^T \int_{B_R} |u|^{\alpha-2}|\nabla u|^{2-a} dxdt\Big)^{\frac{1}{\alpha-a}}.$$

Fast decaying geometry sequences with multiple rates

Lemma

Let $\{Y_i\}_{i=0}^{\infty}$ *be a sequence of non-negative numbers satisfying*

$$Y_{i+1} \le \sum_{k=1}^{m} A_k B_k^i Y_i^{1+\mu_k}, \quad i = 0, 1, 2, \cdots ,$$

where $A_k > 0$, $B_k > 1$ *and* $\mu_k > 0$ *for* $k = 1, 2, \ldots, m$. *Let* $B = \max\{B_k : 1 \le k \le m\}$ *and* $\mu = \min\{\mu_k : 1 \le k \le m\}$.

$$\textit{If} \quad \sum_{k=1}^{m} A_k Y_0^{\mu_k} \le B^{-1/\mu} \quad \textit{then} \ \lim_{i\to\infty} Y_i = 0.$$

In particular, if $Y_0 \le \min\{(m^{-1} A_k^{-1} B^{-\frac{1}{\mu}})^{1/\mu_k} : 1 \le k \le m\}$ *then* $\lim_{i\to\infty} Y_i = 0$.

Theorem

Let $U' \Subset U$.

(i) *If $t \in (0,1)$ then*

$$\|p(t)\|_{L^\infty(U')} \le Ct^{-\frac{\kappa_1}{\alpha-a}}\Big(1 + \|\bar{p}_0\|_{L^\alpha} + [EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_2}.$$

If $t \ge 1$ then

$$\|p(t)\|_{L^\infty(U')} \le C\Big(1 + \|\bar{p}_0\|_{L^\alpha} + [EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_2}.$$

(ii) *If $A(\alpha) < \infty$ then*

$$\limsup_{t\to\infty} \|p(t)\|_{L^\infty(U')} \le C\big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\big)^{\kappa_2}.$$

(iii) *If $\beta(\alpha) < \infty$ then there is $T > 0$ such that for all $t \ge T$,*

$$\|p(t)\|_{L^\infty(U')} \le C\big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + A(\alpha,t)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-1,t))}\big)^{\kappa_2}.$$

Interior L^s-estimates for pressure gradient

Ladyzhenskaya-Uraltseva type embedding:

Lemma

For each $s \geq 1$, there exists a constant $C > 0$ depending on s such that for each smooth cut-off function $\zeta(x) \in C_c^\infty(U)$, the following inequality holds

$$\int_U K(|\nabla p|)|\nabla p|^{2s+2}\zeta^2 dx \leq C \max_{\mathrm{supp}\zeta} |p|^2 \Big[\int_U K(|\nabla p|)|\nabla p|^{2s-2}|\nabla^2 p|^2 \zeta^2 dx + \int_U K(|\nabla p|)|\nabla p|^{2s}|\nabla\zeta|^2 dx \Big],$$

for every sufficiently regular function $p(x)$ such that the right hand side is well-defined.

Key property.

$$-aK(\xi) \leq \xi K'(\xi) \leq 0.$$

We establish the basic step for the Ladyzhenskaya-Uraltseva iteration.

Lemma

For each $s \geq 0$*, if* $T_0 \geq 0$*,* $T > 0$*, and* $\zeta(x,t)$ *is a smooth cut-off function then*

$$\sup_{[T_0, T_0+T]} \int_U |\nabla p(x,t)|^{2s+2} \zeta^2 dx + \int_{T_0}^{T_0+T} \int_U K(|\nabla p|) |\nabla^2 p|^2 |\nabla p|^{2s} \zeta^2 dxdt$$
$$\leq C \int_{T_0}^{T_0+T} \int_U K(|\nabla p|) |\nabla p|^{2s+2} |\nabla \zeta|^2 dxdt + C \int_{T_0}^{T_0+T} \int_U |\nabla p|^{2s+2} \zeta |\zeta_t| dx.$$

As a consequence, for $0 < \theta' < \theta < 1$,

$$\int_{T_0}^{T_0+T} \int_U K(|\nabla p|) |\nabla p|^{2s+4} \zeta^2 dxdt \leq C \sup_{[T_0+\theta' T, T_0+T]} \|p\|^2_{L^\infty(V)}$$
$$\cdot \int_{T_0}^{T_0+T} \int_U (1 + K(|\nabla p|) |\nabla p|^{2s+2+a}) (|\nabla \zeta|^2 + \zeta |\zeta_t|) dxdt.$$

Proposition

Let $U' \Subset V \Subset U$, $T_0 \geq 0$, $T > 0$ and $\theta \in (0,1)$. If $s \geq 2$ then

$$\int_{T_0+\theta T}^{T_0+T} \int_{U'} K(|\nabla p|)|\nabla p|^s dxdt \leq C\Big(1 + (\theta T)^{-1}\Big)^{s-2} \cdot \big(1 + \sup_{[T_0+\theta T/2, T_0+T]} \|p\|^2_{L^\infty(V)}\big)^{s-2} \cdot \int_{T_0}^{T_0+T} \int_U (1 + K(|\nabla p|)|\nabla p|^2) dxdt,$$

$$\sup_{t\in[T_0+\theta T, T_0+T]} \int_{U'} |\nabla p(x,t)|^s dxdt \leq C\Big(1 + (\theta T)^{-1}\Big)^{s+a-1} \cdot \big(1 + \sup_{[T_0+\theta T/2, T_0+T]} \|p\|^2_{L^\infty(V)}\big)^{s-2+a} \cdot \int_{T_0}^{T_0+T} \int_U (1 + K(|\nabla p|)|\nabla p|^2) dxdt.$$

Theorem

Let $U' \Subset U$ and $s \geq 2$. If $t \in (0,2)$ then

$$\int_{U'} |\nabla p(x,t)|^s dx \leq C t^{-\mu_1} (1 + \|\bar{p}_0\|_{L^\alpha})^{\mu_2+2} \cdot \Big(1 + [EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\mu_2} \cdot \Big(1 + \int_0^t G_1(\tau) d\tau\Big),$$

If $t \geq 2$ then

$$\int_{U'} |\nabla p(x,t)|^s dx \leq C (1 + \|\bar{p}_0\|_{L^\alpha})^{\mu_2+\alpha} \cdot \Big(1 + [EnvA(\alpha,t)]^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\mu_2+\alpha} \cdot \Big(1 + \int_{t-1}^t G_1(\tau) d\tau\Big).$$

Theorem

Let $U' \Subset U$ *and* $s \geq 2$.
(i) *If* $A(\alpha) < \infty$ *then*

$$\limsup_{t\to\infty} \int_{U'} |\nabla p(x,t)|^s dx \leq C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\mu_2+\alpha} \cdot \Big(1 + \limsup_{t\to\infty} \int_{t-1}^{t} G_1(\tau)d\tau\Big).$$

(ii) *If* $\beta(\alpha) < \infty$ *then there is* $T > 0$ *such that for all* $t > T$,

$$\int_{U'} |\nabla p(x,t)|^s dx \leq C\Big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\mu_2+\alpha} \cdot \Big(1 + \int_{t-1}^{t} G_1(\tau)d\tau\Big).$$

Interior L^∞-estimates for pressure gradient

For each $m = 1, 2, \ldots, n$, denote $u_m = p_{x_m}$ and $u = (u_1, u_2, \ldots, u_n) = \nabla p$. We have

$$\frac{\partial u_m}{\partial t} = \partial_m(\nabla \cdot (K(|u|)u)) = \nabla \cdot (K(|u|)\partial_m u) + \nabla \cdot \Big[K'(|u|)\frac{\sum_i u_i \partial_m u_i}{|u|} u\Big].$$

Since $\partial_i u_m = \partial_m u_i$, we have
$\partial_m u = (\partial_m u_1, \cdots, \partial_m u_n) = (\partial_1 u_m, \cdots, \partial_n u_m) = \nabla u_m$, and
$\sum_i u_i \partial_m u_i = \sum_i u_i \partial_i u_m = u \cdot \nabla u_m$. Therefore, we rewrite (1) as

$$\frac{\partial u_m}{\partial t} = \nabla \cdot (K(|u|)\nabla u_m) + \nabla \cdot \Big[K'(|u|)\frac{u \cdot \nabla u_m}{|u|} u\Big].$$

Then use De Giorgi's iteration.

Weighted parabolic Sobolev-Poincaré inequality

Lemma

Given $W(x,t) > 0$ on Q_T. Let r be a number that satisfies $\frac{2n}{n+2} < r < 2$. Set

$$\varrho = \varrho(r) \stackrel{\text{def}}{=\!=} 4(1 - 1/r^*).$$

Then

$$\|u\|_{L^{\varrho}(Q_T)} \le C[[u]]_{2,W;T}\Big\{\delta T^{\frac{1}{\varrho}} + \operatorname*{ess\,sup}_{t\in[0,T]}\Big(\int_U W(x,t)^{-\frac{r}{2-r}}\chi_{\operatorname{supp} u}(x,t)dx\Big)^{\frac{2-r}{\varrho r}}$$

where $\delta = 1$ in general, $\delta = 0$ in case u vanishes on the boundary ∂U, and

$$[[u]]_{2,W;T} = \operatorname*{ess\,sup}_{[0,T]}\|u(t)\|_{L^2(U)} + \Big(\int_0^T\int_U W(x,t)|\nabla u|^2 dxdt\Big)^{\frac{1}{2}}.$$

Theorem

Let $U' \Subset V \Subset U$. *For any* $T_0 \geq 0$, $T > 0$, *and* $\theta \in (0,1)$, *if* $t \in [T_0 + \theta T, T_0 + T]$ *then*

$$\|\nabla p(t)\|_{L^\infty(U')} \leq C(1 + (\theta T)^{-1})^{\frac{s_1+1}{2}} \lambda^{\frac{s_1}{2}} \|\nabla p\|_{L^2(V \times (T_0+\theta T/2, T_0+T))},$$

where

$$\lambda = \lambda(T_0, T, \theta; V) = \Big(\int_{T_0+\theta T/2}^{T_0+T} \int_V (1 + |\nabla p|)^{\frac{a s_0}{2-s_0}} dxdt \Big)^{\frac{2-s_0}{s_0}}.$$

Theorem

If $t \in (0,2)$ *then*

$$\|\nabla p(t)\|_{L^\infty(U')} \le Ct^{-\kappa_4/2}(1+\|\bar{p}_0\|_{L^\alpha})^{s_3(\kappa_5+1)} \cdot \Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{s_3\kappa_5} \cdot \Big(1+\int_0^t G_1(\tau)d\tau\Big)^{s_3/2}.$$

If $t \ge 2$ *then*

$$\|\nabla p(t)\|_{L^\infty(U')} \le C(1+\|\bar{p}_0\|_{L^\alpha})^{s_3(\kappa_5+\alpha/2)} \cdot \Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{s_3(\kappa_5+\alpha/2)} \cdot \Big(1+\int_{t-1}^t G_1(\tau)d\tau\Big)^{s_3/2}.$$

Theorem

(i) *If* $A(\alpha) < \infty$ *then*

$$\limsup_{t\to\infty} \|\nabla p(t)\|_{L^\infty(U')} \le C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{s_3(\kappa_5+\alpha/2)} \cdot \Big(1 + \limsup_{t\to\infty} \int_{t-1}^{t} G_1(\tau)d\tau\Big)^{s_3/2}.$$

(ii) *If* $\beta(\alpha) < \infty$ *then there is* $T > 0$ *such that when* $t > T$ *we have*

$$\|\nabla p(t)\|_{L^\infty(U')} \le C\Big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{s_3(\kappa_5+\alpha/2)} \cdot \Big(1 + \int_{t-1}^{t} G_1(\tau)d\tau\Big)^{s_3/2}.$$

Interior L^∞-estimates for time derivative of pressure

Let $q = p_t$. Then

$$\frac{\partial q}{\partial t} = \nabla \cdot \big(K(|\nabla p|)\nabla p\big)_t.$$

Using De Giorgi's iterations and weighted parabolic Sobolev-Poincaré inequality, we obtain

Proposition

Let $U' \Subset V \Subset U$. If $T_0 \geq 0$, $T > 0$ and $\theta \in (0,1)$, then

$$\sup_{[T_0+\theta T, T_0+T]} \|p_t\|_{L^\infty(U')} \leq C\lambda^{\frac{s_1}{2}} \big(1 + (\theta T)^{-1}\big)^{\frac{s_1+1}{2}} \|p_t\|_{L^2(U\times(T_0,T_0+T))},$$

where

$$\lambda = \lambda(T_0, T, \theta; V) = \Big(\int_{T_0+\theta T/2}^{T_0+T}\int_V (1+|\nabla p|)^{\frac{as_0}{2-s_0}}\,dxdt\Big)^{\frac{2-s_0}{s_0}}.$$

Theorem

Let $U' \Subset U$. For $t \in (0,2)$,

$$\|p_t(t)\|_{L^\infty(U')} \le Ct^{-\kappa_6/2}\Big(1+\|\bar p_0\|_{L^\alpha}\Big)^{\kappa_7}\Big(1+\int_U H(|\nabla p(x,0)|)dx\Big)^{1/2}.$$

$$\cdot\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_8}\Big(1+\int_0^t G_3(\tau)d\tau\Big)\Big)^{s_3/2}.$$

For $t \ge 2$,

$$\|p_t(t)\|_{L^\infty(U')} \le C(1+\|\bar p_0\|_{L^\alpha})^{\kappa_9}$$
$$\cdot\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_9}$$
$$\cdot\Big(1+\int_{t-1}^t G_3(\tau)d\tau\Big)^{s_3/2}.$$

Theorem

Let $U' \Subset U$.
(i) *If* $A(\alpha) < \infty$ *then*

$$\limsup_{t\to\infty} \|p_t(t)\|_{L^\infty(U')} \le C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_9} \cdot \Big(1 + \limsup_{t\to\infty} \int_{t-1}^{t} G_3(\tau)d\tau\Big)^{s_3/2}.$$

(ii) *If* $\beta(\alpha) < \infty$ *then there is* $T > 0$ *such that for all* $t > T$,

$$\|p_t(t)\|_{L^\infty(U')} \le C\Big(1 + \beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_9} \cdot \Big(1 + \int_{t-1}^{t} G_3(\tau)d\tau\Big)^{s_3/2}.$$

Interior L^2-estimates for pressure's Hessian

We estimate the L^2-norm of the Hessian $\nabla^2 p = (p_{x_i x_j})_{i,j=1,2,\ldots,n}$.

Lemma

Let $U' \Subset V \Subset U$. For $t > 0$,

$$\|\nabla^2 p(t)\|_{L^2(U')} \le C\Big(1 + \|\nabla p(t)\|_{L^\infty(V)}\Big)^a \Big(\int_U [|\nabla p|^{2-a} + |p_t|^2] dx\Big)^{1/2}.$$

Theorem

Let $U' \Subset U$.

(i) *For $t \in (0,2)$, one has*

$$\|\nabla^2 p(t)\|_{L^2(U')} \le C t^{-\kappa_{10}/2}(1+\|\bar p_0\|_{L^\alpha})^{\kappa_{11}}\Big(1+\int_U H(|\nabla p(x,0)|)dx\Big)^{1/2}$$
$$\cdot\Big(1+EnvA(\alpha,t)^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(0,t))}\Big)^{\kappa_{12}}\Big(1+\int_0^t G_4(\tau)d\tau\Big)^{s_4/2}.$$

(ii) *If $t \ge 2$ then one has*

$$\|\nabla^2 p(t)\|_{L^2(U')} \le C(1+\|\bar p_0\|_{L^\alpha})^{\kappa_{11}}$$
$$\cdot\Big(1+[EnvA(\alpha,t)]^{\frac{1}{\alpha-a}}+\|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_{11}}$$
$$\cdot\Big(1+\int_{t-1}^t G_4(\tau)d\tau\Big)^{s_4/2}.$$

Theorem (Continued)

(iii) *If* $A(\alpha) < \infty$ *then*

$$\limsup_{t\to\infty} \|\nabla^2 p\|_{L^2(U')} \le C\Big(1 + A(\alpha)^{\frac{1}{\alpha-a}} + \limsup_{t\to\infty} \|\Psi\|_{L^\alpha(U\times(t-1,t))}\Big)^{\kappa_{11}} \cdot \Big(1 + \limsup_{t\to\infty} \int_{t-1}^{t} G_4(\tau)d\tau\Big)^{s_4/2}.$$

(iv) *If* $\beta(\alpha) < \infty$ *then there is* $T > 0$ *such that for* $t > T$*,*

$$\|\nabla^2 p(t)\|_{L^2(U')} \le C\Big(1+\beta(\alpha)^{\frac{1}{\alpha-2a}} + \sup_{[t-1,t]} A(\alpha,\cdot)^{\frac{1}{\alpha-a}} + \|\Psi\|_{L^\alpha(U\times(t-2,t))}\Big)^{\kappa_{11}} \cdot \Big(1 + \int_{t-1}^{t} G_4(\tau)d\tau\Big)^{s_4/2}.$$

Table of Contents

Contents

Two-phase incompressible Forchheimer flows

For each ith-phase ($i = 1, 2$), saturation $S_i \in [0, 1]$, density $\rho_i \geq 0$, velocity $\mathbf{u_i} \in \mathbb{R}^\mathbf{n}$, and , and pressure $p_i \in \mathbb{R}$. The saturations satisfy

$$S_1 + S_2 = 1.$$

Each phase's velocity obeys the generalized Forchheimer equation. Conservation of mass holds for each of the phases:

$$\partial_t(\phi \rho_i S_i) + \mathrm{div}(\rho_i \mathbf{u}_i) = 0, \quad i = 1, 2.$$

Due to incompressibility of the phases, i.e. $\rho_i = const. > 0$, it is reduced to

$$\phi \partial_t S_i + \mathrm{div}\ \mathbf{u}_i = 0, \quad i = 1, 2.$$

Let p_c be the capillary pressure between two phases, more specifically,

$$p_1 - p_2 = p_c.$$

Darcy's flows. Kruzkov, Sukorjanski, Alt, DiBenedetto, Cances, Mikelic, Galusinski, Saad, Chemetov, Neves ...

Denote $S = S_1$ and $p_c = p_c(S)$. Then

$$g_i(|\mathbf{u}_i|)\mathbf{u}_i = -f_i(S)\nabla p_i, \quad i = 1, 2,$$

$$\nabla p_1 - \nabla p_2 = p_c'(S)\nabla S.$$

Hence

$$F_2(S)g_2(|\mathbf{u}_2|)\mathbf{u}_2 - F_1(S)g_1(|\mathbf{u}_1|)\mathbf{u}_1 = \nabla S,$$

where

$$F_i(S) = \frac{1}{p_c'(S)f_i(S)}, \quad i = 1, 2.$$

In summary,

$$\begin{aligned}
&0 \le S = S(\mathbf{x}, t) \le 1,\\
&S_t = -\mathrm{div}\ \mathbf{u}_1,\\
&S_t = \mathrm{div}\ \mathbf{u}_2,\\
&\nabla S = F_2(S)\mathbf{G}_2(\mathbf{u}_2) - F_1(S)\mathbf{G}_1(\mathbf{u}_1).
\end{aligned}$$

One-dimensional problem

Assumption A.

$$f_1, f_2 \in C([0,1]) \cap C^1((0,1)),$$

$$f_1(0) = 0, \quad f_2(1) = 0,$$

$$f_1'(S) > 0, \quad f_2'(S) < 0 \text{ on } (0,1).$$

Assumption B.

$$p_c' \in C^1((0,1)), \quad p_c'(S) > 0 \text{ on } (0,1).$$

Theorem (H.-Kieu-Ibragimov 2013)

- *There are 16 types of non-constant steady states (based on their monotonicity and asymptotic behavior as $x \to \pm\infty$).*
- *The steady states which are never zero nor one are linearly stable.*

Multi-dimensional problem

In $\mathbb{R}^n$, steady states:

$$\mathrm{div}\ \mathbf{u}_1 = \mathrm{div}\ \mathbf{u}_2 = 0, \quad \nabla S = F_2(S)\mathbf{G}_2(\mathbf{u}_2) - F_1(S)\mathbf{G}_1(\mathbf{u}_1).$$

Steady states with geometric constraints:

$$\mathbf{u}_1^*(\mathbf{x}) = c_1|\mathbf{x}|^{-n}\mathbf{x}, \quad \mathbf{u}_2^*(\mathbf{x}) = c_2|\mathbf{x}|^{-n}\mathbf{x}, \quad S_*(\mathbf{x}) = S(|\mathbf{x}|),$$

where c_1, c_2 are constants and $S(r)$ is a solution of the following ODE:

$$S' = F(r, S(r)) \quad \text{for } r > r_0, \quad S(r_0) = s_0, \quad 0 < S(r) < 1.$$

where s_0 is always a number in $(0,1)$ and

$$F(r, S(r)) = G_2(c_2 r^{1-n})F_2(S) - G_1(c_1 r^{1-n})F_1(S).$$

Theorem

There exists a maximal interval of existence $[r_0, R_{\max})$, *where* $R_{\max} \in (r_0, \infty]$, *and a unique solution* $S \in C^1([r_0, R_{\max}); (0,1))$. *Moreover, if* $R_{\max}$ *is finite then either*

$$\lim_{r \to R_{\max}^-} S(r) = 0 \quad \text{or} \quad \lim_{r \to R_{\max}^-} S(r) = 1.$$

Theorem

If solution $S(r)$ *exists in* $[r_0, \infty)$, *then it eventually becomes monotone and, consequently,* $s_\infty = \lim_{r\to\infty} S(r)$ *exists.*

In case $n = 2$ *and* $c_1^2 + c_2^2 > 0$, *let* $s^* = (f_1/f_2)^{-1}\left(\frac{c_1 a_1^0}{c_2 a_2^0}\right)$.

(i) *If* $c_1 \le 0$ *and* $c_2 \ge 0$ *then* $s_\infty = 1$.

(ii) *If* $c_1 \ge 0$ *and* $c_2 \le 0$ *then* $s_\infty = 0$.

(iii) *If* $c_1, c_2 < 0$ *then* $s_\infty = s^*$.

(iv) *If* $c_1, c_2 > 0$ *then* $s_\infty \in \{0, 1, s^*\}$.

Linearized problem

The formal linearized system at the steady state $(\mathbf{u}_1^*(\mathbf{x}), \mathbf{u}_2^*(\mathbf{x}), S_*(\mathbf{x}))$ is

$$\sigma_t = -\mathrm{div}\ \mathbf{v}_1, \quad \sigma_t = \mathrm{div}\ \mathbf{v}_2,$$
$$\nabla\sigma = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*)\mathbf{v}_2 + F_2'(S_*)\sigma\mathbf{G}_2(\mathbf{u}_2^*) - \Big(F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{v}_1 + F_1'(S_*)\sigma\mathbf{G}_1(\mathbf{u}_1^*)\Big).$$

Let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Then $\mathrm{div}\ \mathbf{v} = 0$. Assume $\mathbf{v} = \mathbf{V}(\mathbf{x}, t)$ is given . Let

$$\underline{\mathbf{B}} = \underline{\mathbf{B}}(\mathbf{x}) = F_2(S_*)\mathbf{G}_2'(\mathbf{u}_2^*) + F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*),$$
$$\underline{\mathbf{A}} = \underline{\mathbf{A}}(\mathbf{x}) = \underline{\mathbf{B}}(\mathbf{x})^{-1}$$
$$\mathbf{b} = \mathbf{b}(\mathbf{x}) = F_2'(S_*)\mathbf{G}_2(\mathbf{u}_2^*) - F_1'(S_*)\mathbf{G}_1(\mathbf{u}_1^*),$$
$$\mathbf{c} = \mathbf{c}(\mathbf{x}, t) = F_1(S_*)\mathbf{G}_1'(\mathbf{u}_1^*)\mathbf{V}(\mathbf{x}, t).$$

Decoupling the linearized system:

$$\sigma_t = \nabla\cdot\Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b})\Big] + \nabla\cdot(\underline{\mathbf{A}}\mathbf{c}),$$
$$\mathbf{v}_2 = \underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b}) + \underline{\mathbf{A}}\mathbf{c}, \quad \mathbf{v}_1 = \mathbf{V} - \mathbf{v}_2.$$

Lemma

For any $c_1^2 + c_2^2 > 0$ and $\mathbf{x} \neq 0$, matrices $\underline{\mathbf{B}}(\mathbf{x})$ and $\underline{\mathbf{A}}(\mathbf{x})$ are symmetric, invertible and positive definite.

Also, matrix $\underline{\mathbf{B}}$ has the following special property:

$$\begin{aligned}\underline{\mathbf{B}}(\mathbf{x})\mathbf{x} &= \sum_{i=1}^{2} \left\{ F_i(\hat{S}(|\mathbf{x}|)) \left[g_i(|c_i||\mathbf{x}|^{1-n}) + g_i'(|c_i||\mathbf{x}|^{1-n})|c_i||\mathbf{x}|^{1-n} \right] \right\} \mathbf{x} \\ &= \phi(|\mathbf{x}|)\mathbf{x},\end{aligned}$$

where

$$\phi(r) = \sum_{i=1}^{2} F_i(\hat{S}(r)) \left[g_i(|c_i|r^{1-n}) + g_i'(|c_i|r^{1-n})|c_i|r^{1-n} \right].$$

Now consider "good" steady states.

In Bounded domains

Let $R > r_0 > 0$, $U \subset \mathcal{U} \stackrel{\text{def}}{=\!=} B_R \setminus \bar{B}_{r_0}$. Denote $\Gamma = \partial U$, $D = U \times (0,\infty)$ and $\mathcal{D} = \mathcal{U} \times (0,\infty)$.

Initial-boundary value problem (IBVP):

$$\begin{cases} \sigma_t = \nabla \cdot \Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b}) \Big] + \nabla \cdot (\underline{\mathbf{A}}\mathbf{c}) & \text{on } U \times (0,\infty), \\ \sigma = g(\mathbf{x}, t) & \text{on } \Gamma \times (0,\infty), \\ \sigma = \sigma_0(\mathbf{x}) & \text{on } U \times \{t = 0\}. \end{cases}$$

Condition (E1). $F_1, F_2 \in C^7((0,1))$ and $V \in C^6_{\mathbf{x}}(\bar{D})$; $V_t \in C^3_{\mathbf{x}}(\bar{D})$.

Theorem

Assume **(E1)** *and* $\Delta_4 \stackrel{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)|) + \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|$ *is finite. Then the solution* $\sigma(\mathbf{x},t)$ *of the linearized equation satisfies*

$$\sup_{\mathbf{x}\in U} |\sigma(\mathbf{x},t)| \le C\Big[e^{-\eta_1 t}\sup_U |\sigma_0(\mathbf{x})| + \Delta_4\Big] \quad \textit{for all } t > 0.$$

Moreover,

$$\limsup_{t\to\infty}\Big[\sup_{\mathbf{x}\in U}|\sigma(\mathbf{x},t)|\Big] \le C\Delta_5,$$

where
$\Delta_5 = \limsup_{t\to\infty}\Big[\sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma}|g(\mathbf{x},t)|\Big].$

Theorem

Assume **(E1)**, *and* $\Delta_6 \stackrel{\text{def}}{=\!=} \sup_D(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)| + |\nabla^2 \mathbf{V}(\mathbf{x},t)|)$ *and* $\Delta_7 \stackrel{\text{def}}{=\!=} \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|$ *are finite. Then for any* $U' \Subset U$, *there is* $\tilde{M} > 0$ *such that for* $i = 1, 2$, $\mathbf{x} \in U'$ *and* $t > 0$,

$$\sup_{\mathbf{x}\in U'} |\mathbf{v}_i(\mathbf{x},t)| \le \tilde{M}\Big(1 + \frac{1}{\sqrt{t}}\Big)\Big[e^{-\eta_1 t} \sup_U |\sigma_0(\mathbf{x})| + \Delta_6 + \sqrt{\Delta_6} + \Delta_7\Big].$$

Consequently, if

$$\lim_{t\to\infty} \Big\{ \sup_{\mathbf{x}\in U}(|\mathbf{V}(\mathbf{x},t)| + |\nabla \mathbf{V}(\mathbf{x},t)| + |\nabla^2 \mathbf{V}(\mathbf{x},t)|) + \sup_{\mathbf{x}\in\Gamma} |g(\mathbf{x},t)| \Big\} = 0,$$

then for any $\mathbf{x} \in U$,

$$\lim_{t\to\infty} \mathbf{v}_1(\mathbf{x},t) = \lim_{t\to\infty} \mathbf{v}_2(\mathbf{x},t) = 0.$$

Structure and Transformation

Rewrite vector function $\mathbf{b}(\mathbf{x})$ explicitly as

$$\mathbf{b}(\mathbf{x}) = \Big(F_2'(S_*(\mathbf{x}))g_2(\frac{|c_2|}{|\mathbf{x}|^{n-1}})\frac{c_2}{|\mathbf{x}|^n} - F_1'(S_*(\mathbf{x}))g_1(\frac{|c_1|}{|\mathbf{x}|^{n-1}})\frac{c_1}{|\mathbf{x}|^n}\Big)\mathbf{x} = \lambda(|\mathbf{x}|)\mathbf{x},$$

where

$$\lambda(r) = F_2'(\hat{S}(r))g_2(\frac{|c_2|}{r^{n-1}})\frac{c_2}{r^n} - F_1'(\hat{S}(r))g_1(\frac{|c_1|}{r^{n-1}})\frac{c_1}{r^n}.$$

By defining

$$\Lambda(\mathbf{x}) = \frac{1}{2}\int_{r_0^2}^{|\mathbf{x}|^2} \lambda(\sqrt{\xi})d\xi = \int_{r_0}^{|\mathbf{x}|} r\lambda(r)dr,$$

we have for $\mathbf{x} \neq 0$ that

$$\mathbf{b}(\mathbf{x}) = \nabla\Lambda(\mathbf{x}).$$

Let

$$w(\mathbf{x}, t) = e^{-\Lambda(\mathbf{x})}\sigma(\mathbf{x}, t).$$

Then w satisfies

$$w_t - \nabla \cdot \Big(\underline{\mathbf{A}}\nabla w\Big) - \nabla\Lambda \cdot \underline{\mathbf{A}}\nabla w = e^{-\Lambda}\nabla \cdot (\underline{\mathbf{A}}\mathbf{c}).$$

New system

Define the differential operator

$$\mathcal{L}w = \partial_t w - \nabla \cdot (\underline{\mathbf{A}}\nabla w) - \mathbf{b} \cdot \underline{\mathbf{A}}\nabla w.$$

Corresponding IBVP for $w(\mathbf{x}, t)$ is

$$\begin{cases} \mathcal{L}w = f_0 & \text{in } U \times (0,\infty), \\ w(\mathbf{x},0) = w_0(\mathbf{x}) & \text{in } U, \\ w(\mathbf{x},t) = G(\mathbf{x},t) & \text{on } \Gamma \times (0,\infty), \end{cases}$$

where $w_0(\mathbf{x})$ and $G(\mathbf{x}, t)$ are given initial data and boundary data, respectively, and $f_0(\mathbf{x}, t)$ is a known function.

- For the velocities, we have

$$\mathbf{v}_2 = \underline{\mathbf{A}}\big[\nabla(e^\Lambda w) - e^\Lambda w\mathbf{b}\big] + \underline{\mathbf{A}}\mathbf{c} = \underline{\mathbf{A}}\big[e^\Lambda \nabla w + we^\Lambda \nabla\Lambda - e^\Lambda w\mathbf{b}\big] + \underline{\mathbf{A}}\mathbf{c}.$$

Thus,

$$\mathbf{v}_2 = e^\Lambda \underline{\mathbf{A}}\nabla w + \underline{\mathbf{A}}\mathbf{c}, \quad \mathbf{v}_1 = \mathbf{V} - \mathbf{v}_2.$$

Barrier function. Define

$$W(\mathbf{x},t)=\begin{cases} t^{-s}e^{-\frac{\varphi(\mathbf{x})}{t}} & \text{if } t>0,\\ 0 & \text{if } t\le 0,\end{cases}$$

where the number $s>0$ and the function $\varphi(\mathbf{x})>0$ will be decided later. Then

$$\mathcal{L}W=t^{-s-2}e^{-\frac{\varphi}{t}}\Big\{t\big(-s+\nabla\cdot(\underline{\mathbf{A}}\nabla\varphi)+\mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi\big)+\varphi-(\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi\Big\}.$$

Thus, $\mathcal{L}W\le 0$ if

$$s\ge\nabla\cdot(\underline{\mathbf{A}}\nabla\varphi)+\mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi \quad\text{and}\quad \varphi\le(\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi.$$

We will choose φ to satisfy

$$\underline{\mathbf{A}}\nabla\varphi = \kappa_0 \mathbf{x},$$

where κ_0 is a positive constant selected later. Equivalently,

$$\nabla\varphi = \kappa_0 \underline{\mathbf{A}}^{-1}\mathbf{x} = \kappa_0 \underline{\mathbf{B}}\mathbf{x} = \kappa_0 \phi(|\mathbf{x}|)\mathbf{x}.$$

Define for $\mathbf{x} \in \bar{\mathcal{U}}$ the function

$$\varphi(\mathbf{x}) = \kappa_0 \Big(\varphi_0 + \int_{r_0}^{|\mathbf{x}|} r\phi(r)dr \Big), \quad \text{where } \varphi_0 = \frac{C_0 r_0^2}{2} \text{ and } \kappa_0 = \frac{C_0}{2C_1}.$$

Select

$$s = s_R \stackrel{\text{def}}{=\!=} \kappa_0(n + C_2 R).$$

Lemma

The function $W(\mathbf{x},t)$ belongs to $C^{2,1}_{\mathbf{x},t}(\mathcal{D}) \cap C(\bar{\mathcal{D}})$ and satisfies $\mathcal{L}W \le 0$ in $\mathcal{D}$.

Lemma of growth in time

We fix $s = s_R$ and also the following two parameters

$$q = \frac{\kappa_0 C_0}{2s} \quad \text{and} \quad \eta_0 = \left(\frac{r_0}{R}\right)^{2s},$$

and denote $D_1 = U \times (0, qR^2]$.

Lemma (Lemma of growth in time)

Assume $w(\mathbf{x}, t) \in C^{2,1}_{\mathbf{x},t}(D_1) \cap C(\bar{D}_1)$. *If*

$$\mathcal{L}w \le 0 \text{ on } D_1 \quad \text{and} \quad w \le 0 \text{ on } \Gamma \times (0, qR^2),$$

then

$$\max\{0, \sup_U w(\mathbf{x}, qR^2)\} \le \frac{1}{1+\eta_0} \max\{0, \sup_U w(\mathbf{x}, 0)\}.$$

Proof.

Let $M = \max\{0, \sup_{\bar{U}} w(x,0)\}$, $\tilde{W} = M[1 - \eta W]$, $\eta > 0$ selected later, $t_1 = qR^2$.
Applying maximum principle for $\tilde{W}$ gives

$$w(x, t_1) \le \tilde{W}(x, t_1) \le M(1 - \eta C(s, R)) = M(1 - \eta_0) \le M/(1 + \eta_0).$$

Proposition (Homogeneous problem)

Assume $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *satisfies*

$$\mathcal{L}w = 0 \text{ in } D \quad \text{and} \quad w = 0 \text{ on } \Gamma \times (0,\infty).$$

Let $\eta_1 = \frac{\ln(1+\eta_0)}{qR^2}$. *Then*

$$-e^{-\eta_1 t} \inf_U |w(\mathbf{x},0)| \le w(\mathbf{x},t) \le (1+\eta_0) e^{-\eta_1 t} \sup_U |w(\mathbf{x},0)| \quad \forall (\mathbf{x},t) \in D.$$

Proposition (Non-homogeneous problem)

Assume $f_0 \in C(\bar{D})$ *and*
$\Delta_1 \stackrel{\text{def}}{=\!=} \sup_{U\times(0,\infty)} |f_0(\mathbf{x},t)| + \sup_{\Gamma\times(0,\infty)} |G(\mathbf{x},t)| < \infty$ *The solution* $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *satisfies*

$$|w(\mathbf{x},t)| \le C\left[e^{-\eta_1 t} \sup_U |w_0(\mathbf{x})| + \Delta_1\right] \quad \forall (\mathbf{x},t) \in D.$$

Bernstein's estimates

Proposition

Assume $f_0 \in C(\bar{D})$, $\nabla f_0 \in C(D)$, $\Delta_1 < \infty$ *and*

$$\Delta_3 \overset{\text{def}}{=\!=} \sup_D |\nabla f_0| < \infty.$$

For any $U' \Subset U$ *there is* $\tilde{M} > 0$ *such that if* $w(\mathbf{x}, t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *is a solution of (16) that also satisfies* $w \in C^3_{\mathbf{x}}(D)$ *and* $w_t \in C^1_{\mathbf{x}}(D)$, *then*

$$|\nabla w(\mathbf{x}, t)| \le \tilde{M}\left[1 + \frac{1}{\sqrt{t}}\right]\left[e^{-\eta_1 t} \sup_U |w(\mathbf{x}, 0)| + \Delta_1 + \sqrt{\Delta_3}\right] \quad \forall (\mathbf{x}, t) \in U' \times (0, \infty$$

In unbounded domains

Outer domain $U = \mathbb{R}^n \setminus \bar{B}_{r_0}$.

Notation. For $R > r > 0$, denote $\mathcal{O}_r = \mathbb{R}^n \setminus \bar{B}_r$, $\mathcal{O}_{r,R} = B_R \setminus \bar{B}_r$.

Let $\Gamma = \partial U = \{\mathbf{x} : |\mathbf{x}| = r_0\}$ and $D = U \times (0, \infty)$.

Similar IBVP for $\sigma(\mathbf{x}, t)$:

$$\begin{cases} \sigma_t = \nabla \cdot \Big[\underline{\mathbf{A}}(\nabla\sigma - \sigma\mathbf{b})\Big] + \nabla \cdot (\underline{\mathbf{A}}\mathbf{c}) & \text{on } U \times (0, \infty), \\ \sigma = g(\mathbf{x}, t) & \text{on } \Gamma \times (0, \infty), \\ \sigma = \sigma_0(\mathbf{x}) & \text{on } U \times \{t = 0\}. \end{cases}$$

Define the differential operator

$$\mathcal{L}w = \partial_t w - \nabla \cdot (\underline{\mathbf{A}}\nabla w) - \mathbf{b} \cdot \underline{\mathbf{A}}\nabla w.$$

Corresponding IBVP for $w(\mathbf{x}, t)$ is

$$\begin{cases} \mathcal{L}w = f_0 & \text{in } U \times (0, \infty), \\ w(\mathbf{x}, 0) = w_0(\mathbf{x}) & \text{in } U, \\ w(\mathbf{x}, t) = G(\mathbf{x}, t) & \text{on } \Gamma \times (0, \infty). \end{cases}$$

Maximum principle for unbounded domain

Theorem

Let $T > 0$ and $w(\mathbf{x}, t)$ be a bounded function in $C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ that solves $\mathcal{L}w = f_0$ in U_T, where $f_0 \in C(\bar{U}_T)$. Then

$$\sup_{\bar{U}_T} |w(\mathbf{x}, t)| \le \sup_{\partial_p U_T} |w(\mathbf{x}, t)| + (T + 1) \sup_{\bar{U}_T} |f_0|.$$

Barrier function:

$$W(\mathbf{x}, t) \overset{\text{def}}{=\!=} (T - t)^{-s} e^{\frac{\varphi(\mathbf{x})}{T-t}} \quad \text{for } (\mathbf{x}, t) \in \mathcal{O}_{r_0,R} \times (0, T),$$

where constant $s > 0$ and function $\varphi(\mathbf{x}) > 0$ will be decided later.
Elementary calculations give

$$\mathcal{L}W = (T-t)^{-s-2} e^{\frac{\varphi}{T-t}} \Big\{ (T-t)\big(s - \nabla\cdot(\underline{\mathbf{A}}\nabla\varphi) - \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi\big) + \varphi - (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi \Big\}.$$

Then $\mathcal{L}W \ge 0$ if

$$s \ge \nabla\cdot(A\nabla\varphi) + \mathbf{b}\cdot\underline{\mathbf{A}}\nabla\varphi \quad \text{and} \quad \varphi \ge (\underline{\mathbf{A}}\nabla\varphi)\cdot\nabla\varphi.$$

Choose

$$\varphi(\mathbf{x}) = \kappa_1\Big(\varphi_1 + \int_{r_0}^{|\mathbf{x}|} r\phi(r)dr\Big),$$

where $\varphi_1 = \frac{C_1 r_0^2}{2} > 0$ and $\kappa_1 = \frac{C_1}{2C_0}$, and

$$s = s_R \stackrel{\text{def}}{=\!=} C_3(1+R).$$

• With this barrier function, we can prove the maximum principle in the outter domain, and hence the uniqueness of bounded solution w.

Lemma of growth in spatial variables

Let $R > 0$ and $\ell \geq R + r_0$. Denote

$\mathcal{O}_R(\ell) = \mathcal{O}_{\ell-R,\ell+R} = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|-\ell| < R\}$ and $\mathcal{S}_\ell = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = \ell\}$

Define the barrier function of Landis type

$$\mathcal{W}(\mathbf{x}, t) = \frac{1}{(t+1)^s} e^{-\frac{\psi(\mathbf{x})}{t+1}} \quad \text{for } |\mathbf{x}| \geq r_0, \ t \geq 0, \qquad (*)$$

where parameter $s > 0$ and function $\psi > 0$. Then $\mathcal{L}\mathcal{W} \leq 0$ if

$$s \geq \nabla \cdot (\underline{\mathbf{A}} \nabla \psi) + \mathbf{b} \cdot \underline{\mathbf{A}} \nabla \psi \quad \text{and} \quad \psi \leq (\underline{\mathbf{A}} \nabla \psi) \cdot \nabla \psi.$$

We can choose $s = C_3(1 + R)$ and

$$\psi(x, t) = \kappa_2 \int_\ell^{|x|} (r - \ell)\phi(r) dr.$$

Lemma

Given any $R > 0$ and $\ell \geq R + r_0$. Then the function $\mathcal{W}(\mathbf{x}, t)$ in (*) belongs to $C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ and satisfies $\mathcal{L}\mathcal{W} \leq 0$ on $\mathcal{O}_R(\ell) \times (0, \infty)$.

Lemma (Lemma of growth in spatial variables)

Given $T > 0$, *let*

$$R = R(T) = C_4(1+T),$$
$$\eta_0 = \eta_0(T) = \left(1 - \frac{1}{2^{C_5(T+1)}}\right)\frac{1}{(T+1)^{2C_5(T+1)}},$$

where $C_4 = \max\{1, \frac{8C_3}{\kappa_2 e C_0}\}$ *and* $C_5 = C_3C_4$. *Suppose* $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ *satisfies* $\mathcal{L}w \le 0$ *on* U_T *and* $w(\mathbf{x},0) \le 0$ *on* $\bar{U}$. *Let* ℓ *be any number such that* $\ell \ge R + r_0$, *then*

$$\max\left\{0, \sup_{\mathcal{S}_\ell \times [0,T]} w(\mathbf{x},t)\right\} \le \frac{1}{1+\eta_0} \max\left\{0, \sup_{\bar{\mathcal{O}}_R(\ell)\times[0,T]} w(\mathbf{x},t)\right\}.$$

Dichotomies

Lemma

Let $T > 0$ and R, η_0 and $w(\mathbf{x}, t)$ be as in Lemma 36. For $i \geq 1$, let

$$\bar{m}_i = \max\Big\{0, \sup_{\mathcal{S}_{r_0+iR}\times[0,T]} w(\mathbf{x}, t)\Big\}.$$

Part A (Dichotomy for one cylinder). Then for any $i \geq 1$, we have either of the following cases.

(a) *If $\bar{m}_{i+1} \geq \bar{m}_{i-1}$, then $\bar{m}_{i+1} \geq (1 + \eta_0)\bar{m}_i$.*

(b) *If $\bar{m}_{i-1} \geq \bar{m}_{i+1}$, then $\bar{m}_{i-1} \geq (1 + \eta_0)\bar{m}_i$.*

Part B (Dichotomy for many cylinders). For any $k \geq 0$, we have the following two possibilities:

(i) *There is $i_0 \geq k + 1$ such that $\bar{m}_{i_0+j} \geq (1 + \eta_0)^j \bar{m}_{i_0}$ for all $j \geq 0$.*

(ii) *For all $j \geq 0$, $\bar{m}_{k+j} \leq (1 + \eta_0)^{-j} \bar{m}_k$.*

Theorem

Let $w \in C^{2,1}_{\mathbf{x},t}(U_T) \cap C(\bar{U}_T)$ *be a bounded solution of the IBVP on* U_T *with* $f_0 \in C(\bar{U}_T)$. *If*

$$\lim_{|\mathbf{x}|\to\infty} w_0(\mathbf{x}) = 0,$$

$$\lim_{|\mathbf{x}|\to\infty} \sup_{0\le t\le T} |f_0(\mathbf{x},t))| = 0,$$

then

$$\lim_{r\to\infty} \Big(\sup_{\mathcal{S}_r\times[0,T]} |w(\mathbf{x},t)| \Big) = 0.$$

Corollary

Let $w(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$ *be a bounded solution of (16) on* D *with* $f_0 \in C(\bar{D})$*. Assume* $w_0 \in C(\bar{U})$*,* $G \in C(\Gamma \times [0,\infty))$ *are bounded, satisfy same conditions as above for each* $T > 0$*. Then there exists an increasing, continuous function* $r(t) > 0$ *satisfying* $\lim_{t\to\infty} r(t) = \infty$ *such that*

$$\lim_{t\to\infty} \left(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}} |w(\mathbf{x},t)| \right) = 0.$$

Dealing with weight $e^{\Lambda(x)}$

From $w(x,t)$, we return to $\sigma(x,t) = we^{\Lambda(x)}$.

- In the case $n \geq 3$,

$$0 < C_7^{-1} \leq e^{\Lambda(\mathbf{x})} \leq C_7 \quad \forall |\mathbf{x}| \geq r_0.$$

- In the case $n = 2$,

$$e^{\Lambda(\mathbf{x})} \leq C_8 \quad \forall |\mathbf{x}| \geq r_0.$$

Theorem

Let $n \geq 3$. *Assume* **(E1)** *and*

$$\Delta_{10} \overset{\text{def}}{=\!=} \max\{\sup_{U} |\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty,$$

$$\Delta_{11} \overset{\text{def}}{=\!=} \sup_{D} |\nabla \cdot (\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty.$$

Then,

(i) *There exists a solution* $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D) \cap C(\bar{D})$. *This solution is unique in class of solutions* $\sigma(\mathbf{x},t)$ *that satisfy*

$$\sup_{U\times[0,T]} |\sigma(\mathbf{x},t)| < \infty \quad \text{for any } T > 0.$$

(ii) *There is* $C > 0$ *such that for* $(\mathbf{x},t) \in D$,

$$|\sigma(\mathbf{x},t)| \leq C\big[\Delta_{10} + \Delta_{11}(t+1)\big].$$

Theorem (continued)

(iii) *In addition, if*

$$\lim_{|\mathbf{x}|\to\infty} \sigma_0(\mathbf{x}) = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{0\le t\le T} |\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| = 0 \textit{ for each } T>0,$$

then

$$\lim_{r\to\infty} \Big(\sup_{\mathcal{S}_r\times[0,T]} |\sigma(\mathbf{x},t)| \Big) = 0 \quad \textit{for any } T>0,$$

and furthermore, there is a continuous, increasing function $r(t)>0$ *with* $\lim_{t\to\infty} r(t) = \infty$ *such that*

$$\lim_{t\to\infty} \Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}} |\sigma(\mathbf{x},t)| \Big) = 0.$$

Theorem

Let $n = 2$ and $\hat{S}(r)$ be a solution (for the steady state) with $c_1, c_2 < 0$. Assume **(E1)** *and*

$$\Delta_{12} \stackrel{\text{def}}{=\!=} \max\{\sup_U e^{-\Lambda(\mathbf{x})}|\sigma_0(\mathbf{x})|, \sup_{\Gamma\times[0,\infty)} |g(\mathbf{x},t)|\} < \infty,$$

$$\Delta_{13} \stackrel{\text{def}}{=\!=} \sup_D e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| < \infty.$$

Then the following statements hold true.
(i) *There exists a solution $\sigma(\mathbf{x},t) \in C^{2,1}_{\mathbf{x},t}(D)\cap C(\bar{D})$. This solution is unique in class of solutions $\sigma(\mathbf{x},t)$ that satisfy*

$$\sup_{U\times[0,T]} e^{-\Lambda(\mathbf{x})}|\sigma(\mathbf{x},t)| < \infty \quad \text{for any } T > 0.$$

(ii) *There is $C > 0$ such that for $(\mathbf{x},t)\in D$,*

$$|\sigma(\mathbf{x},t)| \le C\big[\Delta_{12} + \Delta_{13}(t+1)\big].$$

Theorem (continued)

(iii) *In addition, if*

$$\lim_{|\mathbf{x}|\to\infty} e^{-\Lambda(\mathbf{x})}\sigma_0(\mathbf{x}) = 0 \quad \textit{and} \quad \lim_{|\mathbf{x}|\to\infty} \sup_{0\le t\le T} e^{-\Lambda(\mathbf{x})}|\nabla\cdot(\underline{\mathbf{A}}(\mathbf{x})\mathbf{c}(\mathbf{x},t))| = 0$$

for each $T > 0$*, then*

$$\lim_{r\to\infty}\Big(\sup_{\mathcal{S}_r\times[0,T]}|\sigma(\mathbf{x},t)|\Big) = 0 \quad \textit{for any } T > 0,$$

and furthermore, there is a continuous, increasing function $r(t) > 0$ *with* $\lim_{t\to\infty} r(t) = \infty$ *such that*

$$\lim_{t\to\infty}\Big(\sup_{\mathbf{x}\in\bar{\mathcal{O}}_{r(t)}}|\sigma(\mathbf{x},t)|\Big) = 0.$$

THANK YOU FOR YOUR ATTENTION!