An Interface Boundary Value Problem For Incompressible Fluids In Two-layer Domains

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Outline

Navier–Stokes Equations (NSE)

$x = (x_1, x_2, x_3)$: spatial variable, t: time,
$u = (u_1, u_2, u_3) \in \mathbb{R}^3$: velocity,
$\rho \in \mathbb{R}$: density,
$p \in \mathbb{R}$: pressure,
$f(t)$ is the body force/unit mass.

- Conservation of mass:

$$\partial_t \rho + \nabla \cdot (\rho u) = 0.$$

- Conservation of moments:

$$\frac{D(\rho u)}{Dt} = \nabla \cdot (-pI_3 + 2\mu D(u)) + \rho f,$$

where D/Dt is the material derivative, i.e., $\partial_t + (u \cdot \nabla)$, positive constant μ is the dynamic viscosity, and

$$D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$

Navier–Stokes Equations (NSE)

Incompressible fluid: $\rho = constant$. We derive from the above:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u = -\frac{1}{\rho}\nabla p + f, \\ \mathrm{div}\ u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

$\nu = \mu/\rho > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
We assume $\rho = 1$ through out.
Domain Ω is open, bounded in $\mathbb{R}^3$.
No-slip boundary condition $u = 0$ on $\partial\Omega$.

Weak formulation

A pair of functions $(u, p) \in L^2_{loc}(\Omega \times (0,\infty))^3 \times L^1_{loc}(\Omega \times (0,\infty))$ is a weak solution to the NSE if for any $\varphi(x,t) \in C_c^\infty(\Omega \times (0,\infty))^3$

$$-\int_0^\infty \int_U u \cdot \varphi_t dxdt - \nu \int_0^\infty \int_U u \cdot \Delta\varphi dxdt - \sum_{i,j=1}^3 \int_0^\infty \int_U u_j u_i \partial_j \varphi_i dxdt = \int_0^\infty \int_U p \nabla \cdot \varphi dxdt + \int_0^\infty \int_U f \cdot \varphi dxdt,$$

and for any $\phi(x,t) \in C_c^\infty(\Omega \times (0,\infty))$

$$\int_0^\infty \int_U u \cdot \nabla \phi dxdt = 0.$$

Existence and Uniqueness

Denote by N the outward normal vector to the boundary. Define

$$H = \{u \in L^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega,\ u \cdot N = 0 \text{ on } \partial\Omega\},$$

$$V = \{u \in H \cap H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

[Leray 1933, 1934] Suppose $f = f(x) \in L^2(\Omega)$.

- If $u_0 \in H$, then there exists a weak solution on $[0,\infty)$:

$$u \in C([0,\infty); H_{weak}) \cap L^\infty(0,\infty; H) \cap L^2(0,\infty; V).$$

Question 1: Is this weak solution unique?

- If $u_0 \in V$, then there exists a unique strong solution on $[0,T)$ for some $T > 0$: $u \in C([0,\infty); V) \cap L^\infty(0,\infty; V) \cap L^2(0,\infty; H^2(\Omega))$.

Question 2: Can it be $T = \infty$?

In the 2D case, Questions 1 and 2 have affirmative answers.

In the 3D case, still open!

Small data results: If $\|u_0\|_{H^1(\Omega)}$ and $\|f\|_{L^2(\Omega)}$ are small then the strong solution exists for all $t > 0$.

Rotating Fluids

NSE with Coriolis force:

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u = -\nabla p + K e_3 \times u + f.$$

- $\Omega = \mathbb{R}^n$: dispersive effect.
- [Babin-Mahalov-Nicolaenko 1990s, early 2000s]

Domain: $\Omega = \mathbb{T}^2 \times (0,1)$ (this is an example, their result holds for more general domains.)
Stress-free condition on the top and bottom boundaries:

$$u_3 = 0, \quad \partial_3 u_1 = \partial_3 u_2 = 0, \quad x_3 = 0, 1.$$

Theorem

Given $M > 0$, there exists $K_0(M, \nu, f) > 0$ such that if $\|u_0\|_{H^1} < M$ and $K > K_0$ then the strong solution exists for all time.

Boussinesq Approximation

Gravity force: $-\rho g e_3$.
Boussinesq approximation: $\rho = \rho_0(1 - \beta(T - T_0))$.
Boussinesq equations for the ocean:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u = -\nabla p + K e_3 \times u + K_T \, T \, e_3, \\ \partial_t T + (u \cdot \nabla)T - \kappa_T \Delta T = f_T, \\ \mathrm{div}\ u = 0, \\ u(x,0) = u_0(x), \quad T(x,0) = T_0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity, $\kappa_T > 0$,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
T: unknown temperature,
$p \in \mathbb{R}$ is the unknown pressure,
(u_0, T_0) are the initial data.

Hydrostatic Assumption and Primitive Equations (PE)

Notation: $v = (u_1, u_2)$, $w = u_3$, $\nabla_2 = (\partial_1, \partial_2)$.
From the equation for the 3rd component: horizontal scale/velocity v.s. vertical scale/velocity give

$$K_T T = -\partial_3 p,$$

$$\partial_t v + ((v, w) \cdot \nabla)(v, w) - \nu \Delta u = -\nabla_2 p + K e_3 \times v,$$

$$\partial_t T + ((v, w) \cdot \nabla) T - \kappa_T \Delta T = f_T,$$

$$\mathrm{div}\ (v, w) = 0,$$

$$v(x, 0) = v_0(x), \quad T(x, 0) = T_0(x).$$

Domain: $\Omega = \Omega_2 \times (-h, 0)$, $h = constant$.
By the divergence free condition, we have

$$w(x_1, x_2, x_3) = w(x_1, x_2, -h) - \int_0^{x_3} \nabla_2 \cdot v(x_1, x_2, y) dy.$$

Boundary conditions: Dirichlet, Neumann or Robin.

Global well-posedness for PE

Theorem (Cao-Titi 2007)

For any v_0 and T_0 in $H^1(\Omega)$ that satisfy some boundary conditions, there exists a unique strong solution (v, T) for all time:

$$(v, T) \in C([0, \infty); H^1) \cap L^2_{loc}([0, \infty); H^2).$$

For domains with non-flat bottom: [Kukavica-Ziane 2007,2008].

NSE in thin domains

- Damped hyperbolic equations in thin domains: Hale-Raugel 1992
- NSE on thin domains: Raugel-Sell 1993, 1994
- Spherical domains: Temam-Ziane 1996
- Later: Avrin, Chueshov, Hu, Iftimie, Kukavica, Rekalo, Raugel, Sell, Ziane, LH, ...

Note: There were papers on PE in thin domains before Cao-Titi's result.

Theorem

Consider NSE in $\Omega = \mathbb{T}^2 \times (0, \varepsilon)$.
If $\|u_0\|_V, \ \|f\|_H = o(\varepsilon^{-1/2})$ *as* $\varepsilon \to 0$, *then the strong solution exists for all time when* ε *is sufficiently small.*

NSE in two-layer domains

Flat boundaries: Chueshov-Raugel-Rekalo 2005, Hu 2011.
In this talk:

- Thin two-layer domains with non-flat boundaries
- Navier boundary conditions
- Interface boundary conditions (with stress)

The domain

Let

$$\Omega_\varepsilon^+ = \{(x_1,x_2,x_3) : (x_1,x_2) \in \mathbb{T}^2, h_0(x_1,x_2) < x_3 < h_+(x_1,x_2)\},$$

$$\Omega_\varepsilon^- = \{(x_1,x_2,x_3) : (x_1,x_2) \in \mathbb{T}^2, h_-(x_1,x_2) < x_3 < h_0(x_1,x_2)\},$$

where $h_0 = \varepsilon g_0, \quad h_+ = \varepsilon(g_0 + g_+)$, and $h_- = \varepsilon(g_0 - g_-)$.
The two functions $g_+(x_1,x_2)$ and $g_+(x_1,x_2)$ are strictly positive.
The top and bottom boundaries are

$$\Gamma_+ = \{(x_1,x_2,x_3) : (x_1,x_2) \in \mathbb{T}^2, x_3 = h_+(x_1,x_2)\},$$

$$\Gamma_- = \{(x_1,x_2,x_3) : (x_1,x_2) \in \mathbb{T}^2, x_3 = h_-(x_1,x_2)\},$$

while the interface boundary is

$$\Gamma_0 = \{(x_1,x_2,x_3) : (x_1,x_2) \in \mathbb{T}^2, x_3 = h_0(x_1,x_2)\}.$$

Let $\Omega_\varepsilon = [{\Omega_\varepsilon}^+, {\Omega_\varepsilon}^-]$, $\partial\Omega_\varepsilon = [\partial{\Omega_\varepsilon}^+, \partial{\Omega_\varepsilon}^-]$, $\Gamma = [\Gamma_+, \Gamma_-]$, $\Gamma_{00} = [\Gamma_0, \Gamma_0]$.

Equations

Consider the Navier–Stokes equations in Ω_ε

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p + f, \\ \operatorname{div}\ u = 0, \\ u(x,0) = u^0(x), \end{cases}$$

$u = [u_+, u_-]$ defined on $\Omega_\varepsilon = [{\Omega_\varepsilon}^+, {\Omega_\varepsilon}^-]$ is the unknown velocity field,
$p = [p_+, p_-]$ is the unknown pressure,
$f = [f_+, f_-]$ is the body force,
$\nu = [\nu_+, \nu_-] > 0$ is the kinematic viscosity,
$u^0(x) = [u^0_+, u^0_-]$ is the known initial velocity field.
That means the equations hold on each domain ${\Omega_\varepsilon}^+, {\Omega_\varepsilon}^-$ for each u_+, u_- with corresponding viscosity, body force and initial data.

Boundary conditions

Let N_+, N_- be the outward normal vector to the boundary of $\Omega_\varepsilon{}^+, \Omega_\varepsilon{}^-$, respectively.

• The slip boundary condition on $\partial\Omega_\varepsilon$ is

$$u_+ \cdot N_+ = 0 \text{ on } \Gamma_+, \Gamma_0; \qquad u_- \cdot N_- = 0 \text{ on } \Gamma_-, \Gamma_0.$$

• The Navier friction conditions on the top and bottom boundaries are

$$[\nu_+(Du_+)\ N_+]_{\tan} + \gamma_+ u_+ = 0 \quad \text{on } \Gamma_+,$$

$$[\nu_-(Du_-)\ N_-]_{\tan} + \gamma_- u_- = 0 \quad \text{on } \Gamma_-,$$

where $\gamma_+, \gamma_- > 0$.

• The interface boundary condition on Γ_0 is

$$[\nu_+(Du_+)N_+]_{\tan} + \gamma_0(u_+ - u_-) = 0,$$

$$[\nu_-(Du_-)N_-]_{\tan} + \gamma_0(u_- - u_+) = 0,$$

where $\gamma_0 > 0$.

Remarks on Navier friction boundary conditions

On the boundary Γ:

$$u \cdot N = 0,$$

$$\nu [D(u)N]_{\tan} + \gamma u = 0,$$

- $\gamma = \infty$: Dirichlet condition.
- $\gamma = 0$: Navier boundary conditions (without friction/free slip). One-layer domain: flat bottom [Iftimie-Raugel-Sell 2005], non-flat top and bottom [LH-Sell 2010]
- $\gamma \neq 0$, one-layer domain [LH 2010]
- If the boundary is flat, say, part of $x_3 = \mathit{const}$, then the conditions become the Robin conditions

$$u_3 = 0, \quad \gamma u_1 + \nu \partial_3 u_1 = \gamma u_2 + \nu \partial_3 u_2 = 0.$$

See [Hu 2007].

Assumptions on Coefficients

There exist $\delta_-, \delta_0, \delta_+ \in [0,1]$ such that

$$0 < \liminf_{\varepsilon \to 0} \varepsilon^{-\delta_\iota} \gamma_\iota \le \limsup_{\varepsilon \to 0} \varepsilon^{-\delta_\iota} \gamma_\iota < \infty, \quad \iota = +, -,$$

$$0 < \liminf_{\varepsilon \to 0} \varepsilon^{-\delta_0} \gamma_0 \le \limsup_{\varepsilon \to 0} \varepsilon^{-\delta_0} \gamma_0 < \infty.$$

Here

$$\delta_0 = 1 \text{ and } 2/3 \le \delta_+ = \delta_- = \delta \le 1.$$

Hence

$$\gamma_+, \gamma_-, \gamma_0 = O(\varepsilon^\delta) \text{ as } \varepsilon \to 0.$$

Therefore, without loss of generality, we assume that

$$C^{-1}\varepsilon^\delta \le \gamma_-, \gamma_+ \le C\varepsilon^\delta, \quad C^{-1}\varepsilon^{\delta_0} \le \gamma_0 \le C\varepsilon^{\delta_0},$$

for all $\varepsilon \in (0,1]$, where $C > 0$.

The Stokes operator

- Leray-Helmholtz decomposition

$$L^2(\Omega_\varepsilon)^3 = H \oplus H^\perp,$$

where $H = \{u \in L^2(\Omega_\varepsilon)^3 : \nabla \cdot u = 0 \text{ in } \Omega_\varepsilon,\ u \cdot N = 0 \text{ on } \Gamma\}$,

$$H^\perp = \{\nabla\phi : \phi \in H^1(\Omega_\varepsilon)\}.$$

- Let V be the closure in $H^1(\Omega_\varepsilon, \mathbb{R}^3)$ of the subspace of $u \in C^\infty(\overline{\Omega_\varepsilon}, \mathbb{R}^3) \cap H$ that satisfies the boundary conditions.
- Let P denotes the (Leray) projection from $L^2(\Omega_\varepsilon)$ onto H.
- Then the Stokes operator is: $Au = -P\Delta u, \quad u \in D_A$, where

$$D_A = \{u \in H^2(\Omega_\varepsilon)^3 \cap V : \text{ u satisfies the boundary conditions}\}.$$

Vertical averaging of horizontal components

Let $\phi = [\phi_+, \phi_-]$. For $x' \in \mathbb{T}^2$, define

$$M_0\phi(x') = \Big[\frac{1}{\varepsilon g_+(x')}\int_{h_0(x')}^{h_+(x')} \phi_+(x', x_3)dx_3, \frac{1}{\varepsilon g_-(x')}\int_{h_-(x')}^{h_0(x')} \phi_-(x', x_3)dx_3\Big].$$

Let $u = (u_1, u_2, u_3)$ with $u_i = [u_i^+, u_i^-]$.
Define

$$\overline{M}u = (M_0u_1, M_0u_2, 0).$$

Also, denote

$$\|A^{1/2}u\|_{L^2}^2 = 2\int_{\Omega_\varepsilon} \nu|Du|^2dx + 2\int_\Gamma \gamma|u|^2d\sigma + 2\gamma_0\int_{\Gamma_0}|u^+ - u^-|^2d\sigma.$$

Main results

Theorem

There are positive numbers ε_* *and* κ *such that if* $\varepsilon < \varepsilon_*$ *and* $u_0 \in V$, $f \in L^\infty(0,\infty; L^2(\Omega_\varepsilon)^3)$ *satisfy that*

$$\|\overline{M}u_0\|_{L^2}^2, \quad \varepsilon\|A^{\frac{1}{2}}u_0\|_{L^2}^2, \quad \varepsilon^{1-\delta}\|\overline{M}Pf\|_{L^\infty L^2}^2, \quad \varepsilon\|Pf\|_{L^\infty L^2}^2 \le \kappa,$$

then the strong solution $u(t)$ *of the Navier–Stokes equations exists for all* $t \ge 0$. *Moreover,*

$$\|u(t)\|_{L^2}^2 \le C\kappa, \quad t \ge 0,$$

$$\|A^{\frac{1}{2}}u(t)\|_{L^2}^2 \le C\varepsilon^{-1}\kappa, \quad t \ge 0,$$

where $C > 0$ *is independent of* ε, u_0, f.

Remark: The condition on u_0 is acceptable.

A Green's formula

If v is tangential to the boundary $\partial\Omega_\varepsilon{}^\iota$, $\iota = +, -$, then

$$-\int_{\Omega_\iota} \nu_\iota \Delta u_\iota \cdot v dx = 2\int_{\Omega_\iota} \nu_\iota (Du_\iota : Dv) dx + 2\gamma_0 \int_{\Gamma_0} (u_\iota - u_{-\iota}) \cdot v d\sigma + 2\gamma_\iota \int_{\Gamma_\iota} u_\iota \cdot v d\sigma.$$

Taking $v = u_\iota$ and summing up for $\iota = +, -$, we obtain

$$-\int_\Omega \nu \Delta u \cdot u dx = E(u),$$

where

$$E(u) = 2\int_\Omega \nu |Du|^2 dx + 2\int_\Gamma \gamma |u|^2 d\sigma + 2\gamma_0 \int_{\Gamma_0} |u_+ - u_-|^2 d\sigma.$$

Define the bi-linear form on V:

$$E(u, v) = 2\int_{\Omega_\varepsilon} \nu Du : Dv dx + 2\int_\Gamma \gamma u \cdot v d\sigma + 2\gamma_0 \int_{\Gamma_0} (u_+ - u_-) \cdot (v_+ - v_-) d\sigma.$$

Need Korn's inequality: $\|u\|^2_{H^1(\Omega_\varepsilon)} \leq C_\varepsilon E(u)$.

Uniform Korn inequality

Lemma

There is $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $u \in H^1(\Omega_\varepsilon)$ and u is tangential to the boundary of Ω_ε, one has

$$C\|u\|_{H^1}^2 \le E(u) \le C'(\|\nabla u\|_{L^2} + \varepsilon^{\delta-1}\|u\|_{L^2}),$$

where C, C' are positive constants independent of ε.

Then we can rewrite the NSE in the functional form

$$u_t + Au + B(u,u) = Pf \text{ in } V',$$

where $B(u,v) = P((u \cdot \nabla)v)$. We skip the details.

Norm Relations

If $u \in V = D_{A^{1/2}}$, then

$$\|A^{1/2}u\|_{L^2}^2 = E(u).$$

Proposition

For $\varepsilon \in (0, \varepsilon_0]$, one has the following:
(i) If $u \in V$ then

$$\|u\|_{L^2} \le c_2 \varepsilon^{(1-\delta)/2} \|A^{\frac{1}{2}} u\|_{L^2}, \quad \|u\|_{H^1} \le c_2 \|A^{\frac{1}{2}} u\|_{L^2},$$

$$\|A^{\frac{1}{2}} u\|_{L^2} \le c_3 (\|\nabla u\|_{L^2} + \varepsilon^{(\delta-1)/2} \|u\|_{L^2}).$$

(ii) If $u \in D_A$ then

$$\|A^{\frac{1}{2}} u\|_{L^2} \le c_2 \varepsilon^{(1-\delta)/2} \|Au\|_{L^2}, \quad \|u\|_{L^2} \le c_2 \varepsilon^{1-\delta} \|Au\|_{L^2}.$$

Interpreting the Navier boundary conditions

Lemma

Let τ be a tangential vector field on the boundary. If u satisfies the Navier friction boundary conditions then one has on Γ that

$$\frac{\partial u}{\partial \tau} \cdot N = -u \cdot \frac{\partial N}{\partial \tau},$$
$$\frac{\partial u}{\partial N} \cdot \tau = u \cdot \left\{ \frac{\partial N}{\partial \tau} - \gamma \tau \right\}.$$

One also has [Chueshov-Raugel-Rekalo]

$$N \times (\nabla \times u) = 2N \times \{N \times ((\nabla N)^* u) - \gamma N \times u\}.$$

Interpreting the interface boundary conditions

Lemma

Let $u = [u_+, u_-]$ satisfy the slip and interface boundary conditions on Γ_0. Suppose τ is a tangential vector field on Γ_0. Let $\check{N} \in C^1(\overline{\Omega}, \mathbb{R}^3)$ such that $\check{N}\big|_{\Gamma_0} = \pm N_\iota$, then one has on Γ_0 that

$$\frac{\partial u_\iota}{\partial N_\iota} \cdot \tau = u_\iota \cdot \frac{\partial N_\iota}{\partial \tau} - 2\gamma_0 (u_\iota - u_{-\iota}) \cdot \tau,$$

and

$$N_\iota \times (\nabla \times u_\iota) = 2N_\iota \times \left\{ \check{N} \times ((\nabla \check{N})^* u_\iota) - \gamma_0 N_\iota \times (u_\iota - u_{-\iota}) \right\},$$

$$N_\iota \times (\nabla \times u_\iota) = 2N_\iota \times \left\{ \check{N} \times ((\nabla \check{N})^* u_\iota) - \gamma_0 N_\iota \times u_\iota \right\} - 2\gamma_0 u_{-\iota}.$$

Linear Estimates

Proposition

If $u \in D_A$, the

$$\|Au + \Delta u\|_{L^2} \leq C\varepsilon \|\nabla^2 u\|_{L^2} + C\|\nabla u\|_{L^2} + C\varepsilon^{\delta - 1}\|u\|_{L^2},$$

$$C\|Au\|_{L^2} \leq \|u\|_{H^2} \leq C'\|Au\|_{L^2}.$$

Note: for one-layer domain:

$$\|Au + \Delta u\|_{L^2} \leq C\varepsilon^{\delta}\|\nabla u\|_{L^2} + C\varepsilon^{\delta - 1}\|u\|_{L^2},$$

Key Lemma

Lemma

Let $u = [u_+, u_-] \in D_A$ *and* $\Phi = [\Phi_+, \Phi_-] \in H^1(\Omega_\varepsilon)^3$*. One has*

$$\int_{\Omega_\varepsilon} (\nabla \times (\nabla \times u)) \cdot \Phi dx = \int_{\Omega_\varepsilon} (\nabla \times \Phi) \cdot (\nabla \times u - G(u)) dx$$
$$+ \int_{\Omega_\varepsilon} \Phi \cdot (\nabla \times G(u)) dx - 2\gamma_0 \sum_\iota \int_{\Gamma_0} u_\iota \cdot \Phi_{-\iota} d\sigma,$$

where $G(u)$ *satisfies*

$$|G(u)| \le C\varepsilon^\delta |u|, \quad |\nabla G(u)| \le C\varepsilon^\delta |\nabla u| + C\varepsilon^{\delta-1}|u|.$$

Estimate of $\|Au+\Delta u\|_{L^2}$

Let $\Phi = Au + \Delta u = -P\Delta u + \Delta u$. We have $\Phi \in H^\perp$, $\Phi = \nabla q$, hence $\nabla \times \Phi = 0$, and

$$\|\Phi\|_{L^2}^2 = \int_{\Omega_\varepsilon} |\Phi|^2 dx = \int_{\Omega_\varepsilon} \Delta u \cdot \Phi dx = -\int_{\Omega_\varepsilon} (\nabla \times \omega)\cdot \Phi dx$$
$$\le \left|\int_{\Omega_\varepsilon} \Phi \cdot \nabla \times G(u) dx\right| + 2\gamma_0 \left|\sum_\iota \int_{\Gamma_0} [N_\iota \times (N_\iota \times u_\iota)]\cdot \Phi_{-\iota} d\sigma\right| = I_1 + I_2.$$

Then

$$I_1 \le C\int_{\Omega_\varepsilon} |\Phi|(\varepsilon^\delta |\nabla u| + \varepsilon^{\delta-1}|u|)dx \le C\|\Phi\|_{L^2}\{\varepsilon^\delta \|\nabla u\|_{L^2} + \varepsilon^{\delta-1}\|u\|_{L^2}\}.$$

Estimate of I_2 involves more boundary behaviors:

$$I_2 \le 2\gamma_0 \|\Phi\|_{L^2}\Big(\|\nabla^2 u\|_{L^2} + \varepsilon^{-1}\|\nabla u\|_{L^2} + \varepsilon\|u\|_{L^2}\Big)$$
$$\le C\|\Phi\|_{L^2}\Big(\varepsilon\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2} + \varepsilon^2\|u\|_{L^2}\Big).$$

Estimate of $\|\nabla^2 u\|_{L^2}$

Lemma

There is $\varepsilon_0 \in (0,1]$ such that if $\varepsilon < \varepsilon_0$ and $u \in H^2(\Omega_\varepsilon)^3$ satisfies the Navier friction boundary conditions, then

$$\|\nabla^2 u\|_{L^2} \le C\|\Delta u\|_{L^2} + C\|u\|_{H^1}.$$

Remarks on the proof. Integration by parts

$$\int_{\Omega_\varepsilon} |\nabla^2 u|^2 dx = \int_{\Omega_\varepsilon} |\Delta u|^2 dx + \int_{\partial\Omega_\varepsilon} \left(\frac{1}{2} \frac{\partial |\nabla u|^2}{\partial N} - \frac{\partial u}{\partial N} \cdot \Delta u \right) d\sigma.$$

- Remove the second derivatives in the boundary integrals
- Appropriate order for ε
- The role of the positivity of the coefficients γ_ι on Γ_ι
- The positivity of

$$\gamma_0 \left| \frac{\partial(u_+ - u_-)}{\partial \tau} \cdot \tau' \right|^2$$

on Γ_0, where τ and τ' are tangential vector fields to Γ_0.

Integration by parts on the boundary

We also use:

Lemma

Let $S \in \{\Gamma_0, \Gamma_+, \Gamma_-\}$. For two periodic vector fields u, v on S and a periodic, tangential vector field $a(x)$ to S, we have

$$\int_S \frac{\partial u(x)}{\partial a(x)} \cdot v(x) d\sigma = -\int_S \frac{\partial v(x)}{\partial a(x)} \cdot u(x) d\sigma + O(\varepsilon) \int_S |u \cdot v| d\sigma.$$

Then

$$\|\nabla^2 u\|_{L^2}^2 \leq \|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2 + C\varepsilon^2 \|\nabla^2 u\|_{L^2}^2.$$

Non-linear estimate

Proposition

Given $\alpha > 0$, there is $C_\alpha > 0$ such that for any $\varepsilon \in (0,1]$ and $u \in D_A$, we have

$$\left| \int_{\Omega_\varepsilon} (u \cdot \nabla) u \cdot Au dx \right| \le \alpha \|u\|_{H^2}^2 + C\varepsilon^{1/2} \|A^{1/2} u\|_{L^2} \|Au\|_{L^2}^2$$
$$+ C_\alpha \Big(\|u\|_{L^2}^2 + \big[\varepsilon \|u\|_{L^2}^2 \|A^{1/2} u\|_{L^2}^2 \big]^{1/3} \Big) (\varepsilon^{-1} \|A^{1/2} u\|_{L^2}^2).$$

where $C > 0$ is independent of ε and α.

Averaging operators

Let $\phi = [\phi_+, \phi_-]$. For $x' \in \mathbb{T}^2$, define

$$M_0\phi(x') = \Big[\frac{1}{\varepsilon g_+(x')}\int_{h_0(x')}^{h_+(x')} \phi_+(x', x_3)dx_3, \frac{1}{\varepsilon g_-(x')}\int_{h_-(x')}^{h_0(x')} \phi_-(x', x_3)dx_3\Big],$$

$$\overline{M}u = (M_0 u_1, M_0 u_2, 0),$$

$$v = Mu = (\bar{v}, v_3) = (M_0 u_1, M_0 u_2, \bar{v}\cdot\psi),$$

$$\psi = \frac{1}{\varepsilon g}\{(x_3 - h_0)\nabla_2 h_1 + (h_1 - x_3)\nabla_2 h_0\}.$$

If u is div-free and $u \cdot N = 0$ on Γ, so is v.

Let $u \in D_A$ and $w = u - Mu$. Then

- w satisfies "good" inequalities: Poincaré, Agmon, Ladyzhenskaya-Garliagdo-Nirenberg-Sobolev
- v is "almost" 2D

Let $u \in D_A$, $\omega = \nabla \times u$, and $\Phi = v \times \omega$.

$$|\langle u \cdot \nabla u, Au \rangle| = |\langle u \times \omega, Au \rangle| \le J_1 + J_2 + J_3.$$

$$J_1 \le |\langle w \times \omega, Au \rangle| \le C\varepsilon^{1/2} \|u\|_{H^1} \|u\|_{H^2}^2,$$

$$J_2 \le |\langle v \times \omega, (Au + \Delta u) \rangle| \le \alpha \|u\|_{H^2}^2 + C\varepsilon^{1/2} \|A^{1/2}u\|_{L^2} \|Au\|_{L^2}^2$$
$$+ C_\alpha \Big(\|u\|_{L^2}^2 + \big[\varepsilon \|u\|_{L^2}^2 \|A^{1/2}u\|_{L^2}^2 \big]^{1/3} \Big) (\varepsilon^{-1} \|A^{1/2}u\|_{L^2}^2),$$

$$J_3 \le |\langle \Phi, \Delta u \rangle| = |\langle \Phi, \nabla \times \omega \rangle|$$
$$\le |\langle \nabla \times \Phi, \omega \rangle| + |\langle \nabla \times \Phi, G(u) \rangle| + |\langle \Phi, \nabla \times G(u) \rangle|$$
$$+ 2\gamma_0 \sum_{\iota = +,-} \Big| \int_{\Gamma_0} u_\iota \cdot \Phi_{-\iota} d\sigma \Big|.$$

Lemma (Cao-Titi)

If f is 2D and g is 3D then

$$\|fg\|_{L^2} \le C\varepsilon^{-1/2} \|f\|_{L^2}^{1/2} \|f\|_{H^1}^{1/2} \|g\|_{L^2}^{1/2} \|g\|_{H^1}^{1/2}.$$

Poincaré-like inequalities:

$$\|(I - \overline{M})u\|_{L^2} \le C\varepsilon\|u\|_{H^1}, \quad \|u_3\|_{L^2} \le C\varepsilon\|u\|_{H^1}.$$

$$\begin{aligned} |\langle f, u\rangle| &= |\langle (I - \overline{M})Pf, (I - \overline{M})u\rangle + \langle \overline{M}Pf, \overline{M}u\rangle| \\ &\le C\varepsilon\|Pf\|_{L^2}\|u\|_{H^1} + \|\overline{M}Pf\|_{L^2}\|u\|_{L^2}. \end{aligned}$$

$$\|u_0\|_{L^2}^2 = \|\overline{M}u_0\|_{L^2}^2 + \|(I - \overline{M})u_0\|_{L^2}^2 \le \|\overline{M}u_0\|_{L^2}^2 + C\varepsilon^2\|u_0\|_{H^1}^2.$$

Taking inner product of NSE with u:

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|A^{1/2}u\|_{L^2}^2 \le \|\overline{M}Pf\|_{L^2}^2 + \varepsilon^2\|Pf\|_{L^2}^2.$$

$$\frac{d}{dt}\|u\|_{L^2}^2 + \frac{c_1}{\varepsilon^{1-\delta}}\|u\|_{L^2}^2 \le \|\overline{M}Pf\|_{L^2}^2 + \varepsilon^2\|Pf\|_{L^2}^2.$$

Hence Gronwall's inequality yields

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\le \|u_0\|_{L^2}^2 e^{-c_1 t/\varepsilon^{1-\delta}} + C\varepsilon^2\int_0^t e^{-c_1(t-\tau)/\varepsilon^{1-\delta}}\|Pf(\tau)\|_{L^2}^2 d\tau \\ &\quad + C\int_0^t e^{-c_1(t-\tau)/\varepsilon^{1-\delta}}\|\overline{M}Pf(\tau)\|_{L^2}^2 d\tau \\ &\le \|u_0\|_{L^2}^2 e^{-c_1 t} + C\varepsilon^{1-\delta}\varepsilon^2\|Pf\|_{L^\infty L^2}^2 + C\varepsilon^{1-\delta}\|\overline{M}Pf\|_{L^\infty L^2}^2.\end{aligned}$$

$$\|u(t)\|_{L^2}^2, \int_t^{t+1}\|A^{1/2}u\|_{L^2}^2 ds \le C\kappa.$$

H^1-Estimate

Taking inner product of NSE with Au:

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \|Au\|_{L^2}^2 \leq C\|Pf\|_{L^2}\|Au\|_{L^2} + \{\frac{1}{4} + d_1\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\}\|Au\|_{L^2}^2 + C\{\|u\|_{L^2}^2 + \left(\varepsilon\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2\right)^{1/3}\}\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2.$$

$$\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \{1 - 2d_1\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\}\|Au\|_{L^2}^2 \leq d_2\{\|u\|_{L^2}^2 + \left(\varepsilon\|u\|_{L^2}^2\|A^{1/2}u\|_{L^2}^2\right)^{1/3}\}\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + d_3\|Pf\|_{L^2}^2).$$

As far as $\{1 - 2d_1\varepsilon^{1/2}\|A^{1/2}u\|_{L^2}\} \geq 1/2$ equiv. $\|A^{1/2}u\|_{L^2} \leq d\varepsilon^{-1/2}$,

$$\frac{d}{dt}\|A^{1/2}u\|_{L^2}^2 + \frac{1}{2}\|Au\|_{L^2}^2 \leq h = d_4\varepsilon^{-1}\|A^{1/2}u\|_{L^2}^2 + d_3\|Pf\|_{L^2}^2).$$

We have $\int_{t-1}^{t} h \leq \varepsilon^{-1}k$, where $k = k(\kappa)$ is small.
Using uniform Gronwall's inequality

$$\|A^{1/2}u(t)\|_{L^2}^2 \leq \left(\int_{t-1}^{t} \|A^{1/2}u(s)\|_{L^2}^2 ds + \int_{t-1}^{t} h(s)ds\right),$$

we obtain

$$\|A^{1/2}u(t)\|_{L^2}^2 \leq \varepsilon^{-1}k.$$

Take $k < d^2$, then the bound for $\|A^{1/2}u(t)\|_{L^2}$ persists for all time.

THANK YOU FOR YOUR ATTENTION.