An explicit Poincaré–Dulac normal form for Navier–Stokes equations

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February 9, 2011
Applied Math Seminar
Texas Tech University, Lubbock

Outline

Introduction

Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p - \nabla \phi, \\ \mathrm{div}\ u = 0, \\ \mathbf{u}(x,0) = u^0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
ϕ is the potential of the body force,
u^0 is the initial velocity.

Let $L > 0$ and $\Omega = (0, L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

Functional setting

Let $\mathcal{V}$ be the set of $\mathbb{R}^3$-valued L-periodic trigonometric polynomials which are divergence-free and satisfy the zero average condition.
We define

$$\begin{aligned} H &= \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3 = H^0(\Omega)^3, \\ V &= \text{ closure of } \mathcal{V} \text{ in } H^1(\Omega)^3, \\ \mathcal{D}(A) &= \text{ closure of } \mathcal{V} \text{ in } H^2(\Omega)^3. \end{aligned}$$

Norm on H: $|u| = \|u\|_{L^2(\Omega)}$,
Norm on V: $\|u\| = |\nabla u|$,
Norm on $\mathcal{D}(A)$: $|\Delta u|$.

The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}(A).$$

The bilinear mapping:

$$B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}(A).$$

P_L is the Leray projection from $L^2(\Omega)$ onto H.
Spectrum of A:

$$\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}.$$

If $N \in \sigma(A)$, denote by $R_N H$ the eigenspace of A corresponding to N.
Otherwise, $R_N H = \{0\}$.

Functional form of NSE

Denote by $\mathcal{R}$ the set of all initial data $u^0 \in V$ such that the solution $u(t)$ is regular for all $t > 0$. The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \ \ t > 0,$$

$$u(0) = u^0 \in \mathcal{R},$$

where the equation holds in $\mathcal{D}(A)$ for all $t > 0$ and $u(t)$ is continuous from $[0, \infty)$ into V.

Poincaré–Dulac theory for ODE

Consider an ODE in $\mathbb{R}^n$ of in the formal series form:

$$\frac{dx}{dt} + Ax + \Phi^{[2]}(x) + \Phi^{[3]}(x) + \ldots = 0,\ x \in \mathbb{R}^n,$$

- A is a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$
- each $\Phi^{[d]}$ is a homogeneous polynomial of degree d from $\mathbb{R}^n$ to $\mathbb{R}^n$

Then by an iteration of particular formal changes of variable, there exists a formal series $y = x + \sum_{d=1}^{\infty} \Psi^{[d]}(x)$, where $\Psi^{[d]}$ is a homogeneous polynomial of degree d from $\mathbb{R}^n$ to $\mathbb{R}^n$, which transforms the above ODE into an equation

$$\frac{dy}{dt} + Ay + \Theta^{[2]}(y) + \Theta^{[3]}(y) + \ldots = 0,\ y \in \mathbb{R}^n,$$

where all monomials of each $\Theta^{[d]}$ are resonant.

Poincaré–Dulac normal form for NSE

Functional form of NSE:

$$\frac{du}{dt} + Au + B(u,u) = 0.$$

A differential equation in E^∞

$$\frac{d\xi}{dt} + A\xi + \sum_{d=2}^{\infty} \Phi^{[d]}(\xi) = 0 \qquad (\star)$$

is a Poincaré–Dulac normal form for the NSE if
(i) Each $\Phi^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$ and $\Phi^{[d]}(\xi) = \sum_{k=1}^{\infty} \Phi_k^{[d]}(\xi)$, where all $\Phi_k^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$ are resonant monomials,
(ii) Equation $(\star)$ is obtained from NSE by a formal change of variable $u = \sum_{d=1}^{\infty} \Psi^{[d]}(\xi)$ where $\Psi^{[d]} \in \mathcal{H}^{[d]}(E^\infty)$.

Asymptotic expansion of regular solutions

For $u_0 \in \mathcal{R}$, the solution $u(t)$ has an asymptotic expansion: [Foias-Saut]

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \dots,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in t of degree at most $(j-1)$ and with values are trigonometric polynomials. This means that for any $N \in \mathbb{N}$, $m \in \mathbb{N}$,

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$

Normalization map

Let

$$W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots ,$$

where $W_j(u^0) = R_j q_j(0)$, for $j = 1, 2, 3...$ Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Fréchet space

$$S_A = R_1 H \oplus R_2 H \oplus \cdots .$$

Also, $W'(0) = Id$ meaning

$$W'(0)u^0 = R_1 u^0 \oplus R_2 u^0 \oplus R_3 u^0 \oplus \cdots .$$

Constructions of polynomials $q_j(t)$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, ...)$, then q_j's are the unique polynomial solutions to the following equations

$$q_j' + (A - j)q_j + \beta_j = 0,$$

with $R_j q_j(0) = \xi_j$, where β_j's are defined by

$$\beta_1 = 0 \text{ and for } j > 1,\ \beta_j = \sum_{k+l=j} B(q_k, q_l).$$

Explicitly, these polynomials $q_j(t)$'s are recurrently given by

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau \\ + \sum_{n \geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} (\frac{d}{dt})^n (I - R_j)\beta_j,$$

where $[(A - j)(I - R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik\cdot x}$, for $u(x) = \sum_{|k|^2 \neq j} a_k e^{ik\cdot x} \in \mathcal{V}$.

Normal form in S_A

The S_A-valued function $\xi(t) = (\xi_n(t))_{n=1}^{\infty} = (W_n(u(t)))_{n=1}^{\infty} = W(u(t))$ satisfies the following system of differential equations

$$\frac{d\xi_1(t)}{dt} + A\xi_1(t) = 0,$$
$$\frac{d\xi_j(t)}{dt} + A\xi_j(t) + \sum_{k+l=j} R_j B(\mathcal{P}_k(\xi(t)), \mathcal{P}_l(\xi(t)) = 0, \; n > 1,$$

where $P_j(\xi) = q_j(0, \xi)$ for $\xi \in S_A$. This system is the normal form (in S_A) of the Navier–Stokes equations associated with the asymptotic expansions of regular solutions.

The solution of the above system with initial data $\xi^0 = (\xi_n^0)_{n=1}^{\infty} \in S_A$ is precisely $(R_n q_n(t, \xi^0) e^{-nt})_{n=1}^{\infty}$. Then the algorithm producing the polynomials $q_j(t)$ yields the normal form and its solutions.

Main Result

Notation: For any polynomial Q in ξ regardless if it depends on t, we denote $Q^{[d]}$, for $d \geq 0$, the sum of all its monomials of degree d, i.e., the homogeneous part of degree d of Q.

For $d \geq 1$, let

$$\mathcal{P}^{[d]}(\xi) = \sum_{j=d}^{\infty} \mathcal{P}_j^{[d]}(\xi) = \sum_{j=d}^{\infty} q_j^{[d]}(0, \xi).$$

For $d \geq 2$, let

$$\mathcal{B}^{[d]}(\xi) = \sum_{j=1}^{\infty} \mathcal{B}_j^{[d]}(\xi) = \sum_{j=1}^{\infty} \sum_{k+l=j} \sum_{m+n=d} R_j B\big(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi)\big).$$

Rewrite the normal form:

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0.$$

Questions

- Convergence of $\mathcal{P}^{[d]}(\xi)$, $\mathcal{B}^{[d]}(\xi)$?
- What is the framework for the normal form: Sobove spaces, space of smooth functions, ...?
- Is it a Poincaré-Dulac normal form, that is, the power series form of which each monomial is resonant? If so, what is the change of variable $u = T(\xi)$ that transforms (formally) the Navier–Stokes equations into its normal form?

Answers

Let E^∞ be the Fréchet space $C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V$.
Then the above normal form is a Poincaré–Dulac normal form in E^∞ for the Navier–Stokes equations obtained by the formal change of variable

$$u = \xi + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}(\xi).$$

Along the way, $\mathcal{P}^{[d]}(\xi)$, $\mathcal{B}^{[d]}(\xi)$ are proved to converge in appropriate Sobolev spaces (depending on d).

Utilities

Set of (general) indices: $GI = \bigcup_{n=1}^{\infty} GI(n)$, where for $n \geq 1$,

$$GI(n) = \{\bar{\alpha} = (\alpha_k)_{k=1}^{\infty},\ \alpha_k \in \{0, 1, 2, \ldots\},\ \alpha_k = 0 \text{ for } k > n\}.$$

For $\bar{\alpha} \in GI$, define

$$|\bar{\alpha}| = \sum_{k=1}^{\infty} \alpha_k \text{ and } \|\bar{\alpha}\| = \sum_{k=1}^{\infty} k\alpha_k.$$

For $d, n \geq 1$, define the set of special multi-indices:

$$SI(d, n) = \Big\{\bar{\alpha} = (\alpha_k)_{=1}^{\infty} \in GI, |\bar{\alpha}| = d,\ \|\bar{\alpha}\| = n\Big\};$$

note $1 \leq d \leq n$ hence $SI(d, n) \subset GI(n)$. Also, for $n \geq d \geq 1$ and $n' \geq d' \geq 1$ we have

$$SI(d, n) + SI(d', n') \subset SI(d + d', n + n').$$

Homogeneous gauge

Let $\xi = (\xi_k)_{k=1}^{\infty} \in S_A$ and $\bar{\alpha} = (\alpha_k)_{k=1}^{\infty} \in GI(n)$, define

$$\big[\,\xi\,\big]^{\bar{\alpha}} = |\xi_1|^{\alpha_1}|\xi_2|^{\alpha_2}\ldots|\xi_n|^{\alpha_n}.$$

For $n \geq d \geq 1$, define

$$[[\,\xi\,]]_{d,n} = \left(\sum_{\bar{\alpha}\in SI(d,n)} \big[\,\xi\,\big]^{2\bar{\alpha}}\right)^{1/2} = \left(\sum_{|\bar{\alpha}|=d,\|\bar{\alpha}\|=n} \big[\,\xi\,\big]^{2\bar{\alpha}}\right)^{1/2}.$$

We have the following properties

$$\big[\,\xi\,\big]^{\bar{\alpha}}\big[\,\xi\,\big]^{\bar{\alpha}'} = \big[\,\xi\,\big]^{\bar{\alpha}+\bar{\alpha}'},$$

$$\big[\,\xi\,\big]^{r\bar{\alpha}} = (\big[\,\xi\,\big]^{\bar{\alpha}})^r \text{ for } r = 0,1,2,\ldots,$$

$$\sum_{|\bar{\alpha}|=d} \big[\,\xi\,\big]^{2\bar{\alpha}} = |\xi|^{2d}.$$

$$[[\,\xi\,]]_{d,n} \leq \Big(\sum_{\bar{\alpha}\in GI(n),|\bar{\alpha}|=d} \big[\,\xi\,\big]^{2\bar{\alpha}}\Big)^{1/2} \leq |P_n\xi|^d.$$

Multiplicative inequality

Lemma

Let $\xi \in S_A$, $n \ge d \ge 1$ and $n' \ge d' \ge 1$. Then

$$[[\xi]]_{d,n} \cdot [[\xi]]_{d',n'} \le e^{d+d'} [[\xi]]_{d+d',n+n'} ,$$

Note: The constant on the RHS is independent of n, n'.
Proof.

$$[[\xi]]^2_{d,n} \cdot [[\xi]]^2_{d',n'} = \left(\sum_{\bar\alpha \in SI(d,n)} \left[\xi\right]^{2\bar\alpha}\right)\left(\sum_{\bar\alpha' \in SI(d',n')} \left[\xi\right]^{2\bar\alpha'}\right)$$
$$= \sum_{\bar\alpha \in SI(d,n), \bar\alpha' \in SI(d',n')} \left[\xi\right]^{2(\bar\alpha+\bar\alpha')}$$

For above $\bar\alpha, \bar\alpha'$, the index $\bar\gamma = \bar\alpha + \bar\alpha'$ belongs to $SI(d+d', n+n')$. We need to compare the above sum to $\sum_{\bar\gamma \in SI(d+d',n+n')} \left[\xi\right]^{2\bar\gamma}$.
Let $\bar\gamma = (\gamma_k)_{k=1}^\infty \in SI(d+d', n+n')$.

Suppose $\bar{\gamma} \in SI(d,n) + SI(d',n')$. We count the number of ways to write each $\bar{\gamma}$ as the sum $\bar{\alpha} + \bar{\alpha}'$. If $k > n$ or $k > n'$ then $\alpha_k = 0, \alpha'_k = \gamma_k$ or $\alpha'_k = 0, \alpha_k = \gamma_k$, hence one way.
Let $k \le \min\{n, n'\}$. Counting via α_k: the set of possible values for α_k is $\{0, 1, 2, \dots, \gamma_k\}$, hence at most $\gamma_k + 1$ values. Thus the number of repetition of $\bar{\gamma}$ as the sum $\bar{\alpha} + \bar{\alpha}'$ is at most

$$N = N(\bar{\gamma}) = (\gamma_1+1)(\gamma_2+1)\dots(\gamma_n+1) \le (\gamma_1+1)(\gamma_2+1)\dots(\gamma_{n+n'}+1).$$

By generalized Young's inequality:

$$\begin{aligned} N &\le \left(\frac{(\gamma_1+1)+(\gamma_2+1)+\dots+(\gamma_{n+n'}+1)}{n+n'}\right)^{n+n'} \\ &= \left(\frac{d+d'+n+n'}{n+n'}\right)^{n+n'} \\ &= \left(1+\frac{d+d'}{n+n'}\right)^{n+n'} \le e^{d+d'}. \quad \square \end{aligned}$$

Poincaré inequality for homogeneous gauges

Lemma

For any $\xi \in S_A$*, any numbers* $\alpha, s \geq 0$ *and* $n \geq d \geq 1$*, one has*

$$[[A^\alpha \xi]]_{d,n} \leq \left(\frac{d}{n}\right)^s [[A^{\alpha+s}\xi]]_{d,n} \leq \left(\frac{d}{n}\right)^s |P_n A^{\alpha+s}\xi|^d.$$

Proof. For $|\bar{\alpha}| = d$ and $\|\bar{\alpha}\| = n$ we have

$$\left[\,\xi\,\right]^{2\bar{\alpha}} = \prod_k |\xi_k|^{2\alpha_k} = \prod_k \frac{|k^s \xi_k|^{2\alpha_k}}{k^{2\alpha_k s}}$$

$$= \frac{\prod_k |k^s \xi_k|^{2\alpha_k}}{(\prod_k k^{\alpha_k})^{2s}} = \frac{\left[\,A^s \xi\,\right]^{2\bar{\alpha}}}{(\prod_k k^{\alpha_k})^{2s}}.$$

Let $k_0 = \max\{k : \alpha_k \neq 0\}$. Then $n = \sum k\alpha_k \leq k_0(\sum \alpha_k) = k_0 d$. Hence $k_0 \geq n/d$ and

$$\prod_k k^{\alpha_k} \geq k_0^{\alpha_{k_0}} \geq k_0 \geq n/d.$$

Therefore

$$\left[\,\xi\,\right]^{2\bar{\alpha}} \leq (d/n)^{2s} \left[\,A^s \xi\,\right]^{2\bar{\alpha}}.$$

Summing over $\bar{\alpha} \in SI(n, d)$ one obtains

$$[[\,\xi\,]]_{d,n} \leq (d/n)^s \, [[\,A^s \xi\,]]_{d,d} \leq (d/n)^s |P_n A^s \xi|^d.$$

Then replace ξ by $A^\alpha \xi$. □

Simple nonlinear estimate

Lemma

For $\alpha \geq 1/2$ one has

$$|A^\alpha B(u,v)| \leq K^\alpha |A^{\alpha+1/2}u|\,|A^{\alpha+1/2}v|,$$

for all $u, v \in \mathcal{D}(A^{\alpha+1/2})$, where $K > 1$.

Note: This inequality is symmetric in u and v.

Degrees in t and ξ

Write

$$q_j(t,\xi) = \sum_{m=0}^{j-1} q_{j,m}(\xi) t^m = \sum_{m=0}^{j-1} \sum_{d=1}^{j} q_{j,m}^{[d]}(\xi) t^m = \sum_{d=1}^{j} q_j^{[d]}(t,\xi),$$

where $q_{j,m}(\xi)$ is a polynomial in ξ, and $q_{j,m}^{[d]}(\xi)$ and $q_j^{[d]}(t,\xi)$ are homogeneous polynomials in ξ of degree d.

Also, $q_j^{[d]}(t,\xi) = \sum_{|\bar{\alpha}|=d} q_j^{[d],(\bar{\alpha})}(t,\xi)$, where $\bar{\alpha} = (\alpha_k)_{k=1}^{\infty} \in GI$ and $q_j^{[d],(\bar{\alpha})}(t,\xi)$ is the sum of all monomials of $q_j^{[d]}(t,\xi)$ having degree α_k in ξ_k for all $k \geq 1$. Similarly,

$$\beta_j(t,\xi) = \sum_{m=0}^{j-2} \beta_{j,m}(\xi) t^m = \sum_{m=0}^{j-2} \sum_{d=1}^{j} \beta_{j,m}^{[d]}(\xi) t^m = \sum_{d=1}^{j} \beta_j^{[d]}(t,\xi),$$

where $\beta_{1,m}(\xi) = \beta_{1,m}^{[d]}(\xi) = \beta_1^{[d]}(t,\xi) = 0$ for all m, d, t and ξ,

$$\beta_{j,m}(\xi) = \sum_{l+l'=j} \sum_{r+r'=m} B(q_{l,r}(\xi), q_{l',r'}(\xi)),$$

$$\beta_{j,m}^{[d]}(\xi) = \sum_{l+l'=j} \sum_{r+r'=m} \sum_{s+s'=d} B(q_{l,r}^{[s]}(\xi), q_{l',r'}^{[s']}(\xi)),$$

for $j \geq 2$ and $0 \leq m \leq j-2$, $\beta_j^{[d]}(t,\xi) = \sum_{|\bar{\alpha}|=d} \beta_j^{[d],(\bar{\alpha})}(t,\xi)$, where

$$\beta_j^{[d],(\bar{\alpha})}(t,\xi) = \sum_{l+l'=j} \sum_{k+k'=d} \sum_{\bar{\gamma}+\bar{\gamma}'=\bar{\alpha}} B(q_l^{[k],(\bar{\gamma})}(t,\xi), q_{l'}^{[k'],(\bar{\gamma}')}(t,\xi)).$$

Lemma

(i) $\deg_t q_j(t,\xi) \le j-1$, $\deg_t q_j^{[d]}(t,\xi) \le d-1$.
(ii) If $q_j^{[d],(\bar\alpha)} \neq 0$ *then* $\bar\alpha \in SI(d,j)$.
(iii) Consequently, for each (non-zero) monomial of $\mathcal{P}_j(\xi)$, $j \ge 1$, *having degree* α_k *in* ξ_k, $k \ge 1$, *one has* $\bar\alpha = (\alpha_k)_{k=1}^\infty$ *belongs to* $SI(d,j)$ *where* $d = |\bar\alpha|$. *Also, for each (non-zero) monomial of* $B(\mathcal{P}_m(\xi), \mathcal{P}_n(\xi))$, *having degree* α_k *in* ξ_k, $k \ge 1$, *one has* $\bar\alpha = (\alpha_k)_{k=1}^\infty$ *belongs to* $SI(d, m+n)$ *where* $d = |\bar\alpha|$.

Convention $0/0 = 0$, shorthand notation

$$j\big|_d = \min\{j, d-1\} \text{ for all } j, d.$$

It is clear from the above Lemma that $q_{j,m}^{[d]} = 0$ for $m > (j-1)\big|_d$, and $\beta_{j,m}^{[d]} = 0$ for $m > (j-2)\big|_{d-1}$.

Recursive formulas

For $m = 0$:

$$R_k q_{j,0} = R_k \xi_j + \sum_{n=0}^{j-2} \Big(\frac{(-1)^{n+1} n!}{(k-j)^{n+1}} R_k (I - R_j) \beta_{j,n} \Big);$$

for $m = 1, \ldots, j-2$:

$$R_k q_{j,m} = -\frac{R_k R_j \beta_{j,m-1}}{m} + \sum_{n=0}^{j-2-m} \Big(\frac{(-1)^{n+1}}{(k-j)^{n+1}} \frac{(m+n)!}{m!} R_k (I - R_j) \beta_{j,m+n} \Big);$$

and for $m = j-1$:

$$R_k q_{j,j-1} = -\sum_{m=1}^{j-1} \frac{R_k R_j \beta_{j,j-2}}{j-1}.$$

Recursive formulas for homogeneous polynomials in ξ:

$$R_k q_{j,0}^{[d]} = R_k \xi_j^{[d]} + \sum_{n=0}^{j-2} \Big(\frac{(-1)^{n+1} n!}{(k-j)^{n+1}} R_k (I - R_j) \beta_{j,n}^{[d]} \Big)$$

$$= R_k \xi_j^{[d]} + \sum_{n=0}^{(j-2)|_{d-1}} \Big(\frac{(-1)^{n+1} n!}{(k-j)^{n+1}} R_k (I - R_j) \beta_{j,n}^{[d]} \Big),$$

$$R_k q_{j,m}^{[d]} = -\frac{R_k R_j \beta_{j,m-1}^{[d]}}{m}$$

$$+ \sum_{n=0}^{j-2-m} \Big(\frac{(-1)^{n+1}}{(k-j)^{n+1}} \frac{(m+n)!}{m!} R_k (I - R_j) \beta_{j,m+n}^{[d]} \Big)$$

$$= -\frac{R_k R_j \beta_{j,m-1}^{[d]}}{m} + \sum_{n=m}^{(j-2)|_{d-1}} \Big(\frac{(-1)^{n-m+1}}{(k-j)^{n-m+1}} \frac{n!}{m!} R_k (I - R_j) \beta_{j,n}^{[d]} \Big)$$

for $m = 1, \ldots, (j-2)\big|_d$, and $R_k q_{j,j-1}^{[d]} = -\frac{R_k R_j \beta_{j,j-2}^{[d]}}{j-1}$.

Recursive estimates

Lemma

For $j \geq 2$, $d \geq 1$, $\alpha \geq 0$ and $\xi \in S_A$, one has

$$|A^\alpha q_{j,0}^{[d]}(\xi)|^2 \leq 2(d!)(d-1)!\Big(|A^\alpha \xi_j^{[d]}|^2 + \sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha (I-R_j)\beta_{j,n}^{[d]}(\xi)|^2\Big);$$

$$|A^\alpha q_{j,m}^{[d]}(\xi)|^2 \leq (d!)(d-1)!\Big(\frac{|A^\alpha R_j \beta_{j,m-1}^{[d]}(\xi)|^2}{m^2} + \frac{1}{m!^2}\sum_{n=0}^{(j-2)|_{d-1}} |A^\alpha (I-R_j)\beta_{j,n}^{[d]}(\xi)|^2\Big)$$

for $m = 1, \ldots, (j-2)\big|_d$; and $|A^\alpha q_{j,j-1}^{[d]}(\xi)|^2 = \frac{|A^\alpha R_j \beta_{j,j-2}^{[d]}(\xi)|^2}{(j-1)^2}$.

Estimates of homogeneous polynomials

Proposition

For $j \geq d \geq 1$ *and* $0 \leq m \leq (j-1)\big|_d$*, one has*

$$|A^\alpha q_{j,m}^{[d]}(\xi)| \leq c(\alpha, d) \left[\left[A^{\alpha+\frac{3}{2}(d-1)}\xi \right]\right]_{d,j},$$

for all $\xi \in S_A$ *and* $\alpha \geq 1/2$*, where the positive number* $c(\alpha, d)$ *is*

$$c(\alpha, d) = (M_d)^{(\alpha+\tau_d)(d-1)},$$

with

$$M_d = K^2 + d^6 e^{2d}(d!)^2 \quad \text{and} \quad \tau_d = (d-1)/2.$$

In particular, when $m = 0$ *one has*

$$|A^\alpha \mathcal{P}_j^{[d]}(\xi)| \leq c(\alpha, d) \left[\left[A^{\alpha+\frac{3}{2}(d-1)}\xi \right]\right]_{d,j}.$$

Proof.

By induction in j and the use of Multiplicative Inequality:

$$[[\,\xi\,]]_{d,n} \cdot [[\,\xi\,]]_{d',n'} \le e^{d+d'} [[\,\xi\,]]_{d+d',n+n'} \,.$$

Convergence of homogeneous polynomials

Theorem

Let $\alpha \geq 1/2$, $d \geq 1$ *and* $\xi \in \mathcal{D}(A^{\alpha+3d/2})$.
(i)Then $\mathcal{P}^{[d]}(\xi)$ *converges absolutely in* $\mathcal{D}(A^{\alpha})$ *and satisfies*

$$|A^{\alpha}\mathcal{P}^{[d]}(\xi)| \leq \sum_{j=d}^{\infty} |A^{\alpha}\mathcal{P}_j^{[d]}(\xi)| \leq M(\alpha, d)|A^{\alpha+3d/2}\xi|^d,$$

where $M(\alpha, d) > 0$. *Moreover,* $\mathcal{P}^{[d]}(\xi)$ *is a continuous homogeneous polynomial of degree* d *from* $\mathcal{D}(A^{\alpha+3d/2})$ *to* $\mathcal{D}(A^{\alpha})$.
(ii) Similarly, $\mathcal{B}^{[d]}(\xi)$, $d \geq 2$, *is a continuous homogeneous polynomial of degree* d *in* ξ *mapping* $\mathcal{D}(A^{\alpha+3d/2})$ *into* $\mathcal{D}(A^{\alpha})$ *for all* $\alpha \geq 1/2$, *and satisfies*

$$|A^{\alpha}\mathcal{B}^{[d]}(\xi)| \leq \sum_{n=1}^{\infty} |A^{\alpha}\mathcal{B}_n^{[d]}(\xi)| \leq C(\alpha, d)|A^{\alpha+3d/2}\xi|^d.$$

Proof.

$$
\begin{aligned}
\sum_{j=1}^{\infty} |A^{\alpha} q_{j,m}^{[d]}(\xi)| &\le \sum_{j=d}^{\infty} c(\alpha, d) \left[\left[A^{\alpha+(3/2)(d-1)}\xi \right]\right]_{d,j} \\
&\le \sum_{j=d}^{\infty} c(\alpha, d) \left(\frac{d}{j}\right)^{3/2} |A^{\alpha+(3/2)(d-1)+3/2}\xi|^{d} \\
&= M(\alpha, d) |A^{\alpha+3d/2}\xi|^{d}.
\end{aligned}
$$

Explicit change of variable

Formally, $u = \sum_j \sum_d q_j^{[d]}(0,\xi) = \sum_d \sum_j q_j^{[d]}(0,\xi)$, hence

$$u = \mathcal{P}(\xi) \stackrel{\text{def}}{=\!=} \xi + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}(\xi) = \sum_{d=1}^{\infty} \mathcal{P}^{[d]}(\xi).$$

Note that this expansion, in fact, is the formal inverse of the normalization map W and hence is our natural choice.
This power series has the formal inverse of the form

$$\xi = \tilde{\mathcal{P}}(u) \stackrel{\text{def}}{=\!=} u + \sum_{d=2}^{\infty} \tilde{\mathcal{P}}^{[d]}(u) = \sum_{d=1}^{\infty} \tilde{\mathcal{P}}^{[d]}(u),$$

where each $\tilde{\mathcal{P}}^{[d]}(u)$, $d \geq 1$, is a homogeneous polynomial of degree d, particularly, $\tilde{\mathcal{P}}^{[1]}(u) = \mathcal{P}^{[1]}(u) = u$.

Let $\hat{\mathcal{P}}^{[d]}$ be a symmetric d-linear mapping representing $\mathcal{P}^{[d]}$,

$$\begin{aligned}\tilde{\mathcal{P}}^{[d]}(u) &= -\sum_{m=2}^{d}\Big(\sum_{k_1+\ldots+k_m=d}\hat{\mathcal{P}}^{[m]}(\tilde{\mathcal{P}}^{[k_1]}u,\ldots,\tilde{\mathcal{P}}^{[k_m]}u)\Big)\\ &= -\mathcal{P}^{[d]}(u) - \sum_{m=2}^{d-1}\Big(\sum_{k_1+\ldots+k_m=d}\hat{\mathcal{P}}^{[m]}(\tilde{\mathcal{P}}^{[k_1]}u,\ldots,\tilde{\mathcal{P}}^{[k_m]}u)\Big)\end{aligned}$$

for $d \geq 2$. In particular, when $d = 2$, $\tilde{\mathcal{P}}^{[2]}(u) = -\mathcal{P}^{[2]}(u)$.

Proposition

All $\tilde{\mathcal{P}}^{[d]}(u)$, $d \geq 1$, are continuous homogeneous polynomials of degree d in E^{∞}.

NSE under the change of variable

Let $u(t)$ be a regular solution of Navier–Stokes equations. Then $u(t) \in E^\infty$ for all $t > 0$. We make a formal change of variable using $u = \mathcal{P}(\xi)$, or equivalently, $\xi = \tilde{\mathcal{P}}(u) = u + \sum_{d=2}^\infty \tilde{P}^{[d]}(u)$. Taking the derivative in t formally, we obtain

$$\begin{aligned}\frac{d}{dt}\xi &= \frac{d}{dt}u + \sum_{d=2}^\infty D\tilde{\mathcal{P}}^{[d]}(u)\frac{d}{dt}u \quad \text{(then use } \frac{d}{dt}u = -Au - B(u,u)\text{)}\\ &= -A\xi - \sum_{d=2}^\infty A\mathcal{P}^{[d]}(\xi) - \sum_{k,l=1}^\infty B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \ldots,\end{aligned}$$

here, notation D denotes the Fréchet derivative operator. We then derive

$$\frac{d}{dt}\xi + \sum_{d=1}^\infty Q^{[d]}(\xi) = 0,$$

where each $Q^{[d]}(\xi)$, $d \geq 1$, is a homogeneous polynomial of degree d.

Computing $Q^{[d]}(\xi)$

Obviously, we have $Q^{[1]}(\xi) = A\xi$. Up to degree $d \geq 2$ in ξ, knowing the differential equation for ξ, we formally calculate

$$\frac{d}{dt}u = \boxed{\frac{d}{dt}\xi} + \sum_{m\geq 2} D\mathcal{P}^{[m]}(\xi)\frac{d}{dt}\xi$$

$$= \boxed{-A\xi - \sum_{d\geq 2} Q^{[d]}(\xi)} - \sum_{k\geq 2} D\mathcal{P}^{[k]}(\xi)\Big(A\xi + \sum_{l\geq 2} Q^{[l]}(\xi)\Big).$$

Therefore we obtain the recursive formula for $d \geq 2$:

$$Q^{[d]}(\xi) = H_A^{(d)}\mathcal{P}^{[d]}(\xi) + \sum_{k+l=d} B(\mathcal{P}^{[k]}(\xi), \mathcal{P}^{[l]}(\xi)) - \sum_{\substack{2\leq k,l\leq d-1\\ k+l=d+1}} D\mathcal{P}^{[k]}(\xi)(Q^{[l]}(\xi)),$$

where $H_A^{(d)}\mathcal{P}^{[d]}(\xi) = A\mathcal{P}^{[d]}(\xi) - D\mathcal{P}^{[d]}(\xi)A\xi$ ($H_A^{(d)}$ is the Poincaré homology operator).

Now the Navier–Stokes equations after the change of variable is

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} Q^{[d]}(\xi) = 0,$$

where the polynomials $Q^{[d]}(\xi)$, $d \geq 2$, are given explicitly.

Poincaré's homology operator

Lemma

For $\alpha \geq 1/2$, $d \geq 1$, *then* $H_A^{(d)}\mathcal{P}^{[d]}(\cdot)$ *maps* $\mathcal{D}(A^{\alpha+3d})$ *into* $\mathcal{D}(A^{\alpha})$ *and one has*

$$H_A^{(d)}\mathcal{P}^{[d]}(\xi) = \sum_{j=1}^{\infty}(A-j)\mathcal{P}_j^{[d]}(\xi), \text{ for all } \xi \in \mathcal{D}(A^{\alpha+3d}).$$

Resonant monomials

Denote by $\mathcal{H}^{[d]}(E^\infty)$ the space of homogeneous polynomials on E^∞ of degree d.

Definition

Let $Q \in \mathcal{H}^{[d]}(E^\infty)$. Then $Q(\xi)$ ($\xi \in E^\infty$ and $\xi_j = R_j\xi$, $j \in \mathbb{N}$), is a monomial of degree $\alpha_{k_i} > 0$ in ξ_{k_i} where $i = 1, 2, \ldots, m$, $\alpha_{k_1} + \ldots + \alpha_{k_m} = d$ and $k_1 < k_2 < \ldots < k_m$, if it can be represented as

$$Q(\xi) = \tilde{Q}(\underbrace{\xi_{k_1}, \ldots, \xi_{k_1}}_{\alpha_{k_1}}, \underbrace{\xi_{k_2}, \ldots, \xi_{k_2}}_{\alpha_{k_2}}, \ldots, \underbrace{\xi_{k_m}, \ldots, \xi_{k_m}}_{\alpha_{k_m}}),$$

where $\tilde{Q}(\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(d)})$ is a continuous d-linear map from $(E^\infty)^d$ to E^∞.

The monomial $Q(\xi)$ with degree $d \geq 2$, is called *resonant* if $\sum_{i=1}^m \alpha_{k_i} k_i = j$ and $Q = R_j Q \neq 0$.

Partial symmetric representation

By the symmetrization of $\tilde{Q}$ in each group of variables, specifically, α_{k_1} variables of $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{\alpha_{k_1}}$, α_{k_2} variables of $\xi^{(\alpha_{k_1}+1)}$,$\xi^{(\alpha_{k_1}+2)}, \ldots,$ $\xi^{(\alpha_{k_1}+\alpha_{k_2})}$, ... , and α_{k_m} variables of $\xi^{(d-\alpha_{k_m}+1)}, \xi^{(d-\alpha_{k_m}+2)}, \ldots, \xi^{(d)}$, we will always assume without loss of generality that $\tilde{Q}$ is symmetric in each of these groups of variables.

Goal

Compare the normal formal

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0$$

with the NSE under an explicit change of variable

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} Q^{[d]}(\xi) = 0.$$

Desired relation

Theorem

$Q^{[d]}(\xi) = \mathcal{B}^{[d]}(\xi)$ for all $\xi \in E^\infty$ and $d \geq 2$.

Proof. Let $d \geq 2$. It was proved previously by Foias-Saut:

$$(A - j)\mathcal{P}_j(\xi) + \sum_{k+l=j} B(\mathcal{P}_k(\xi), \mathcal{P}_l(\xi)) = (D\mathcal{P}_j(\xi))(\sum_{k=2}^{j} \mathcal{B}_k(\xi)).$$

Collecting the homogeneous terms of degree d in ξ gives

$$(A - j)\mathcal{P}_j^{[d]}(\xi) + \sum_{m+n=d} \sum_{k+l=j} B(\mathcal{P}_k^{[m]}(\xi), \mathcal{P}_l^{[n]}(\xi))) - \sum_{\substack{2 \leq m,n \leq d-1 \\ m+n=d+1}} D\mathcal{P}_j^{[m]}(\xi)(\mathcal{B}^{[n]}(\xi)) = R_j \mathcal{B}^{[d]}(\xi).$$

Summing in j we obtain

$$\mathcal{B}^{[d]}(\xi) = H_A \mathcal{P}^{[d]}(\xi) + \sum_{m+n=d} B(\mathcal{P}^{[m]}(\xi), \mathcal{P}^{[n]}(\xi)))$$

$$- \sum_{\substack{2\le m,n\le d-1 \\ m+n=d+1}} \boxed{D\mathcal{P}^{[m]}(\xi)(\mathcal{B}^{[n]}(\xi))}.$$

Compare with

$$Q^{[d]}(\xi) = H_A^{(d)} \mathcal{P}^{[d]}(\xi) + \sum_{m+n=d} B(\mathcal{P}^{[m]}(\xi), \mathcal{P}^{[n]}(\xi))$$

$$- \sum_{\substack{2\le k,l\le d-1 \\ m+n=d+1}} \boxed{D\mathcal{P}^{[m]}(\xi)(Q^{[n]}(\xi))},$$

For $d = 2$:

$$\mathcal{B}^{[2]}(\xi) = H_A^{(d)} \mathcal{P}^{[d]}(\xi) + B(\mathcal{P}^{[1]}(\xi), \mathcal{P}^{[1]}(\xi))) = Q^{[2]}(\xi).$$

Then prove by induction in d. □

Summary

Theorem

The formal power series change of variable

$$u = \xi + \sum_{d=2}^{\infty} \mathcal{P}^{[d]}(\xi),$$

where $\xi \in E^\infty = C^\infty(\mathbb{R}^3, \mathbb{R}^3) \cap V$, *reduces the Navier–Stokes equations to a Poincaré–Dulac normal form*

$$\frac{d}{dt}\xi + A\xi + \sum_{d=2}^{\infty} \mathcal{B}^{[d]}(\xi) = 0.$$

THANK YOU FOR YOUR ATTENTION!