Forchheimer Equations in Porous Media - Part II

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Introduction

- Darcy's Law:
$$\alpha u = -\Pi \nabla p,$$
- the "two term" law
$$\alpha u + \beta |u|\, u = -\nabla p,$$
- the "power" law
$$c^n |u|^{n-1}\, u + au = -\nabla p,$$
- the "three term" law
$$\mathcal{A}u + \mathcal{B}\,|u|\,u + \mathcal{C}|u|^2\,u = -\nabla p.$$

Here α, β, c, $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are empirical positive constants.

General Forchheimer equations

Generalizing the above equations as follows

$$g(x, |u|)u = -\nabla p.$$

Let $G(s) = sg(s)$. Then $G(|u|) = |\nabla p| \Rightarrow |u| = G^{-1}(|\nabla p|)$. Hence

$$u = -\frac{\nabla p}{g(G^{-1}(|\nabla p|))} = -K(|\nabla p|)\nabla p,$$

where

$$K(\xi) = K_g(\xi) = \frac{1}{g(s)} = \frac{1}{g(G^{-1}(\xi))}, \quad sg(s) = \xi.$$

We derived non-linear Darcy equations from Forchheimer equations.

Equations of Fluids

Let ρ be the density. Continuity equation

$$\frac{d\rho}{dt} = -\nabla \cdot (\rho u),$$

For slightly compressible fluid it takes

$$\frac{d\rho}{dp} = \frac{1}{\kappa}\rho,$$

where $\kappa \gg 1$. Substituting this into the continuity equation yields

$$\frac{d\rho}{dp}\frac{dp}{dt} = -\rho\nabla \cdot u - \frac{d\rho}{dp} u \cdot \nabla p,$$

$$\frac{dp}{dt} = -\kappa\nabla \cdot u - u \cdot \nabla p.$$

Since $\kappa \gg 1$, we neglect the second term in continuity equation

$$\frac{dp}{dt} = -\kappa\nabla \cdot u\,.$$

Non-dimensional Equations and Boundary Conditions

Combining the equation of pressure and the Forchheimer equation, one gets after scaling:

$$\frac{dp}{dt} = \nabla \cdot \left(K(\nabla p|) \, \nabla p \right).$$

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Consider the equation on a bounded domain U in $\mathbb{R}^3$. The boundary of U consists of two connected components: exterior boundary Γ_e and interior (accessible) boundary Γ_i.

- On Γ_e:

$$u \cdot N = 0 \Leftrightarrow \frac{\partial p}{\partial N} = 0.$$

Non-dimensional Equations and Boundary Conditions

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- On Γ_e:

$$u \cdot N = 0 \Leftrightarrow \frac{\partial p}{\partial N} = 0.$$

- Dirichlet condition on Γ_i: $p(x,t) = \phi(x,t)$ which is known for $x \in \Gamma_i$.
- Total flux condition on Γ_i:

$$\int_{\Gamma_i} u \cdot N d\sigma = Q(t) \Leftrightarrow \int_{\Gamma_i} K(|\nabla p|) \nabla p \cdot N d\sigma = -Q(t),$$

  where $Q(t)$ is known.

Class (GPPC)

We introduce a class of "generalized polynomials with positive coefficients"

Definition

A function $g(s)$ is said to be of class (APPC) if

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + a_2 s^{\alpha_2} + \ldots + a_k s^{\alpha_k} = \sum_{j=0}^{k} a_j s^{\alpha_j},$$

where $k \geq 0$, $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_k$, and $a_0, a_1, \ldots, a_k$ are positive coefficients. Notation $\alpha_k = \deg(g)$.

Lemma

Let $g(s)$ be a function of class (GPPC). Then the function K exists Moreover, for any $\xi \geq 0$, one has

$$\frac{C_1}{(1+\xi)^a} \leq K(\xi) \leq \frac{C_2}{(1+\xi)^a},$$

$$-\theta \frac{K(\xi)}{\xi} \leq K'(\xi) \leq 0,$$

$$(K(\xi)\xi^n)' \geq K(\xi)\xi^{n-1}\left(n - \frac{1}{1+\lambda}\right) \geq 0,$$

where $a = \alpha_k/(1+\alpha_k)$, $0 < \theta < 1$, $n \geq 1$, $\lambda = 1/\alpha_k$.

The Monotonicity

Proposition

Let g be a GPPC. Then g satisfies the **Lambda-Condition**:

$$g(s) \geq \lambda s g'(s), \quad \text{where} \quad \lambda = 1/\deg(g) > 0.$$

The function $F(y) = K(|y|)y$ is strictly monotone on bounded sets. More precisely,

$$(F(y) - F(y')) \cdot (y - y') \geq \frac{\lambda}{\lambda + 1} K(\max\{|y|, |y'|\})|y' - y|^2.$$

IBVP type (S)

On Γ_i the solution has the form:

$$p(x,t) = \gamma(t) + \varphi(x) \quad \text{on} \quad \Gamma_i, \quad t > 0,$$

where the function $\varphi(x)$ is defined for $x \in \Gamma_i$ and satisfies

$$\int_{\Gamma_i} \varphi(x) d\sigma = 0.$$

A priori Estimates - I

Theorem

Let $p(x,t)$ be a solution of IBVP-I(S) with the bounadry profile $(\gamma(t), \varphi(x))$. Then one has for all $t \geq 0$ that

$$\int_U K\left(|\nabla p(x,t)|\right)|\nabla p(x,t)|^2\,dx \leq 2\int_U K\left(|\nabla p(x,0)|\right)|\nabla p(x,0)|^2\,dx + |U|\int_0^t \left(\gamma\prime(\tau)\right)^2 d\tau,$$

If, in addition, $g(s)$ belongs to class (GPPC), then one has

$$\int_U |\nabla p(x,t)|^{2-a} dx \leq C_1|U| + C_2\int_U |\nabla p(x,0)|^{2-a} dx + C_3|U|\int_0^t \left(\gamma\prime(\tau)\right)^2 d\tau.$$

A priori Estimates - I: Proof

Define $H(x,t)$ by:

$$H(x,t) = \int_0^{|\nabla p(x,t)|^2} K(\sqrt{s})ds,$$

for $(x,t) \in U \times [0,\infty)$.
Multiplying the equation by $\partial p/\partial t$ and integrate over U:

$$\int_U \left(\frac{\partial p}{\partial t}\right)^2 dx = -\int_U K(|\nabla p|)\nabla p \cdot \frac{\partial}{\partial t}(\nabla p)dx - Q(t)\cdot\gamma\prime(t)$$
$$= \frac{1}{2}\frac{\partial}{\partial t}H(x,t) - Q(t)\cdot\gamma\prime(t).$$

Note

$$Q^2(t) = \left(\int_U \frac{\partial p}{\partial t}dx\right)^2 \le |U| \int_U \left(\frac{\partial p}{\partial t}\right)^2 dx.$$

A priori Estimates - I: Proof

Applying Cauchy's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_U H(t,x)dx \le \frac{|U|}{2}\cdot|\gamma\prime(t)|^2 - \frac{Q^2(t)}{2|U|},$$

and thus

$$\int\limits_U H(t,x)dx \le \int\limits_U H(0,x)dx + |U|\int\limits_0^t |\gamma\prime(\tau)|^2 d\tau - \frac{1}{|U|}\int_0^t Q^2(\tau)d\tau.$$

Then use relations

$$K(|\nabla p(x,t)|)|\nabla p(x,t)|^2 \le H[p](x,t) \le 2K(|\nabla p(x,t)|)|\nabla p(x,t)|^2,$$

$$K(|\nabla p(x,t)|)|\nabla p(x,t)|^2 \sim |\nabla p(x,t)|^{2-a}.$$

A priori Estimates - II

Theorem

Let $p(x,t)$ be a solution of IBVP-II(S) with total flux $Q(t)$. Assume that $Q(t) \in C^1([0,\infty))$. Then for any $\delta > 0$, one has

$$\int_U |\nabla p(x,t)|^{2-a} dx \le \Lambda^*(t) - C_1 h_2(t) + \delta \int_0^t e^{\delta(t-\tau)} \Big(\Lambda^*(\tau) - C_1 h_2(\tau) \Big) d\tau,$$

for any $t \ge 0$, where

$$\Lambda^*(t) = L_2 + L_0 h_0(t) + L_1 h_1(t) + 2h_0(t)h_1(t) + C_1 |Q(t)|^{\frac{2-a}{1-a}} + C_\delta \int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}} d\tau,$$

with the postive numbers L^*, L_0, L_1 depending on the initial data; while C, C_δ independent of the initial data;

A priori Estimates - II

Theorem

and the functions $h_i(t)$, $i = 0, 1, 2$ are defined by

$$h_0(t) = \int_0^t |Q(\tau)| d\tau, \quad h_1(t) = \int_0^t |Q'(\tau)| d\tau.$$

A priori Estimates - III

Theorem

Let $g(s)$ be of class (GPPC). Let $p_\gamma(x,t)$ be a solution of IBVP-I(S) with known total flux $Q(t)$ and known boundary profile $(\gamma(t), \varphi(x))$. Let $p(x,t)$ be a solution of IBVP-II(S) with total flux $Q(t)$ and boundary profile $(B(t), \varphi(x))$, where $B(t)$ is not given but bounded from above. Suppose that $Q \in C^1([0,\infty))$, $Q'(t) \geq 0$ and $B(t) \leq B_0 < \infty$. Then

$$\int_U |\nabla p(x,t)|^{2-a} dx \leq C_1 \Big(\int_U |\nabla p_\gamma(x,t)|^{2-a} dx + |Q(t)|^{\frac{2-a}{1-a}} + |Q(t)||\gamma(t)| \Big) + L_0,$$

the positive number L_0 depends on the initial data of p_γ and p; and C_1, C_2 are positive constants.

A priori Estimates - III

Theorem

Consequently,

$$\int_U |\nabla p(x,t)|^{2-a} dx \le C_1\Big(\int_0^t |\gamma'(\tau)|^2 d\tau + |Q(t)|^{\frac{2-a}{1-a}} + |Q(t)||\gamma(t)|\Big) + L_0.$$

Asymptotic stability for IBVP-I(S)

Lemma

Let the function g be of the class (GPPC). For any functions f, p_1 and p_2, and for $1 \le q < 2$, one has

$$\left(\int_U |f|^q dx\right)^{2/q} \le C\left(\int_{U_1} K(|\nabla p_1|)\,|f|^2 dx + \int_{U_2} K(|\nabla p_2|)\,|f|^2 dx\right) \times \left\{1 + \max\left(\|\nabla p_1\|_{L^{\frac{aq}{2-q}}(U)}, \|\nabla p_2\|_{L^{\frac{aq}{2-q}}(U)}\right)\right\}^a,$$

where

$$U_1 = \{x : |\nabla p_1(x)| \ge |\nabla p_2(x)|\}, \quad U_2 = \{x : |\nabla p_1(x)| < |\nabla p_2(x)|\}.$$

Lemma

Consequently

$$\int_U (K(|\nabla p_1|)\nabla p_1 - K(|\nabla p_2|)\nabla p_2)\cdot(\nabla p_1 - \nabla p_2)dx$$
$$\geq C\left(\int_U |\nabla(p_1 - p_2)|^q dx\right)^{2/q}$$
$$\times\left\{1 + \max\left(\|\nabla p_1\|_{L^{\frac{aq}{2-q}}(U)}, \|\nabla p_2\|_{L^{\frac{aq}{2-q}}(U)}\right)\right\}^{-a}.$$

Theorem

Assume that $\deg(g) \le \frac{4}{d-2}$. *Suppose* $p_1(x,t)$, $p_2(x,t)$ *are two solutions of IBVP-I(S) with the same boundary profile* $(\gamma(t), \varphi(x))$. *Then*

$$\|p_1(\cdot,t) - p_2(\cdot,t)\|_{L_2(U)} \le \|p_1(\cdot,0) - p_2(\cdot,0)\|_{L_2(U)} \cdot \exp\left[-C\int_0^t \Lambda^{-b}(\tau)d\tau\right],$$

for all $t \ge 0$, *where*

$$\Lambda(t) = 1 + \int_U |\nabla p_1(x,0)|^{2-a}dx + \int_U |\nabla p_2(x,0)|^{2-a}dx + \int_0^t |\gamma'(\tau)|^2 d\tau.$$

Corollary

If $\int_0^t |\gamma'(\tau)|^2 d\tau = O(t^r)$ *as* $t \to \infty$*, for some* $0 < r < 1/b$*, then*

$$\|z(\cdot,t)\|_{L^2(U)} \le C_1 e^{-C_2 t^{\varepsilon}} \|z(\cdot,0)\|_{L^2(U)},$$

where $\varepsilon = 1 - rb > 0$*.*

Example

Suppose $\gamma(t) = a_0 + a_1 t^{\beta}$, where $a_1 \neq 0$, for all $t > T$, where $T > 0$. Then

(i) if $\beta < 1/a$ then (12) holds for $\varepsilon = (1-2\beta)b + 1$;

(ii) if $\beta = 1/a$ then $\|z(\cdot,t)\|_{L^2} \le C(1+t)^{-s}\|z(\cdot,0)\|_{L^2}$ for some number $s > 0$.

Perturbed Boundary Value Problems - IBVP-I(S)

Solutions p_k with Dirichlet data $(\gamma_k(t), \varphi(x))$ and flux $Q_k(t)$, for $k = 1, 2$. We assume that for $k = 1, 2$:

$$\int_0^t |\gamma_k'(\tau)|^2 d\tau \le \lambda_0(t),$$

where $\lambda_0 \in C([0, \infty))$. Let

$$\Lambda_0(t) = (1 + \lambda_0(t))^b.$$

where $b = a/(2 - a)$,

$$Z(t) = \int_U z^2(x, t) dx, \quad F_1(t) = e^{-C_0 A_1 \int_0^t \Lambda_0^{-1}(\tau) d\tau},$$

where $C_0 > 0$ is a constant independent of the solutions.

Theorem

Assume $\deg(g) \le \frac{4}{d-2}$. *Let* $\overline{p}_k(x,t) = p_k(x,t) - \gamma_k(t)$ *for* $k = 1, 2$. *Let*

$$\overline{z}(x,t) = \overline{p}_1(x,t) - \overline{p}_2(x,t), \quad \text{and} \quad \overline{Z}(t) = \int_U \overline{z}^2(x,t)dx.$$

Then one has for all $t \ge 0$ *that*

$$\overline{Z}(t) \le F_1(t)\overline{Z}(0) + C_1 A_1^{-1} F_1(t) \int_0^t \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2 F_1^{-1}(\tau)d\tau.$$

Consequently,

$$Z(t) \le 2F_1(t)\overline{Z}(0) + 2C_2 A_1^{-1} F_1(t) \int_0^t \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2 F_1^{-1}(\tau)d\tau + 2|\Delta_\gamma(t)|^2.$$

Corollary

Assume that

$$\int_0^\infty \Lambda_0^{-1}(\tau)d\tau = \infty,$$
$$\int_0^\infty \Lambda_0(\tau)(\Delta'_\gamma(\tau))^2 F_1^{-1}(\tau)d\tau = \infty,$$
$$\lim_{t\to\infty} \Lambda_0(t)\Delta'_\gamma(t) = \lambda_1 \in \mathbb{R}.$$

Then

$$\overline{Z}(t) \le F_1(t)\overline{Z}(0) + C_1 C_0^{-1} A_1^{-2}\lambda_1^2 + \epsilon(t),$$

where $\epsilon(t) \to 0$ *as* $t \to \infty$*.*
Consequently, if $\lambda_1 = 0$ *then*

$$\lim_{t\to\infty} \overline{Z}(t) = 0.$$

Example

Suppose $\gamma_k(t) = a_{0,k} + a_{1,k} t^{\beta_k}$, where $a_{i,k} \neq 0$, for $i = 0, 1$, and $k = 1, 2$. Let $\beta = \max\{\beta_1, \beta_2\}$ and $\alpha = \deg(g)$. If $\beta < 2(\alpha + 1)/(3\alpha + 2)$ then $\lim_{t\to\infty} \overline{Z}(t) = 0$.

Example

Suppose $\gamma_1(t) = a_{0,1} + a_{1,1} t^{\beta}$ and $\gamma_2(t) = \gamma_1(t) + \Delta_\gamma(t)$, where $\Delta_\gamma(t) = a_{0,3} + a_{1,3} t^r$, with $r < \beta$ and $a_{1,3} \neq 0$. Then $\lim_{t\to\infty} \overline{Z}(t) = 0$ if $\beta < 1/a$ and $r < 1 - (2\beta - 1)b$.

Let p_1 and p_2 be two solutions of IBVP-II(S). Let $\delta > 0$ be fixed, and let

$$\Lambda_k^*(t) = \Big(1 + \int_0^t |Q_k(\tau)| d\tau\Big)\Big(1 + \int_0^t |Q_k'(\tau)| d\tau\Big) + |Q(t)|^{\frac{2-a}{1-a}} + \int_0^t |Q'(\tau)|^{\frac{2-a}{1-a}} d\tau.$$

We assume that

$$e^{-\delta t}\Lambda_k^*(t) + \delta \int_0^t e^{-\delta\tau}\Lambda_k^*(\tau) d\tau \le \widetilde{\lambda}_0(t), \quad t \ge 0, \quad k = 1, 2,$$

where the function $\widetilde{\lambda}_0(t)$ is known and belongs to $C([0,\infty))$.

IBVP-II(S)

Theorem

Given δ, T *and a solution* $p_1(x,t)$ *with* $Q_1, Q_1' \in L^\infty_{loc}([0,\infty))$. *For any* $\varepsilon > 0$, *there is* $\sigma > 0$ *depending on* δ, T, $\|Q_1\|_{L^\infty(0,T)}$, $\|Q_1'\|_{L^\infty(0,T)}$ *and the initial data of* p_1, *such that if*

$$\int_U |z(x,0)|^2 dx, \int_U |\nabla z(x,0)|^{2-a} dx, \|\Delta_Q\|_{L^\infty(0,T)}, \|\Delta_Q'\|_{L^\infty(0,T)} < \sigma,$$

then

$$\int_U |z(x,t)|^2 dx < \varepsilon, \quad \textit{for all} \quad t \in [0,T].$$

Theorem (contd)

More specifically, there is $L > 0$ depending on δ, T, $\|Q_1\|_{L^\infty(0,T)}$, $\|Q_1'\|_{L^\infty(0,T)}$ and the initial data of p_1, such that

$$\sup_{t\in[0,T]} \int_U |p_1(x,t) - p_2(x,t)|^2 dx \le L\Big(\int_U |p_1(x,0) - p_2(x,0)|^2 dx + \big(\sup_{t\in[0,T]} |\Delta_Q(t)|\big)^2 \Big).$$

IBVP-II(S) with controlled Dirichlet data

Theorem

Let p_1 and p_2 be two solutions of IBVP-II(S) satisfying condition

$$\int_0^t |\gamma_k'(\tau)|^2 d\tau \le \lambda_0(t).$$

Let $\Lambda_0(t) = (1+\lambda_0(t))^b$. *Assume* $\deg(g) \le \frac{4}{d-2}$. *Let*

$$\bar{p}_k(x,t) = p_k(x,t) + |U|^{-1}\int_0^t Q_k(\tau)d\tau \quad \textit{for} \quad k = 1,2,$$

$$\bar{z}(x,t) = \bar{p}_1(x,t) - \bar{p}_2(x,t) - |U|^{-1}\int_U (p_1(x,0) - p_2(x,0))dx, \quad \textit{and}$$

$$\bar{Z}(t) = \int_U \bar{z}^2(x,t)dx.$$

Theorem

Then one has for all $t \geq 0$ that

$$\overline{Z}(t) \leq F_1(t)\overline{Z}(0) + C_2 A_1^{-1} F_1(t) \int_0^t \Lambda_0(\tau)(\Delta_Q(\tau))^2 F_1^{-1}(\tau) d\tau.$$

Define

$$I_Q(t) = \left(\int_0^t \Delta_Q(\tau)d\tau\right)^2 \quad \text{and} \quad I_z(t) = \left(\int_U z(x,t)dx\right)^2.$$

Theorem

Assume $\deg(g) < \frac{4}{d-2}$. *One has for all* $t \geq 0$ *that*

$$Z(t) \leq F_1(t)\Big[Z(0) + C_1 A_1 \int_0^t \frac{I_Q(\tau)}{F_1(\tau)\Lambda_0(\tau)}d\tau + C_2 A_1 I_z(0)\int_0^t \frac{1}{F_1(\tau)\Lambda_0(\tau)}d\tau + C_3 A_1^{-1}\int_0^t \frac{\Lambda_0(\tau)\Delta_Q^2(\tau)}{F_1(\tau)}d\tau\Big].$$

IBVP-I(S) with flux constraints

Theorem

Let p_1 and p_2 be two solutions to IBVP-I(S). Assume that

$$\Delta_Q^2(t) \le q_0 I_Q(t) + q_1 \Delta_Q^2(0) + q_2, \quad \textit{some} \quad q_0, q_1, q_2 \ge 0.$$

Then one has

$$Z(t) \le F_1(t)\Big[Z(0) + C_1(q_1\Delta_Q(0) + q_2 + I_z(0)) \int_0^t \frac{1}{F_1(\tau)\Lambda_0(\tau)} d\tau + C_2 \int_0^t \frac{\Lambda_0(\tau)\Delta_\gamma^2(\tau)}{F_1(\tau)} d\tau\Big].$$

Remark

From physical point of view, the above condition restricts the amplitude of possible spikes of the total flux from too large deviation, and this, in fact, is not stringent. Indeed, the condition implies

$$|\Delta_Q(t)| \le C_1 \int_0^t |\Delta_Q(\tau)| d\tau + C_2,$$

hence by Gronwall's inequality: $|\Delta_Q(t)| \le C_3 e^{C_1 t} + C_4$. It means that $|\Delta_Q(t)|$ cannot grow faster than exponential functions.