The Normal Form of the Navier–Stokes equations in Suitable Normed Spaces

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Outline

Introduction

Navier-Stokes equations (NSE) in $\mathbb{R}^3$ with a potential body force

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u = -\nabla p - \nabla \phi, \\ \mathrm{div}\ u = 0, \\ \mathbf{u}(x,0) = u^0(x), \end{cases}$$

$\nu > 0$ is the kinematic viscosity,
$u = (u_1, u_2, u_3)$ is the unknown velocity field,
$p \in \mathbb{R}$ is the unknown pressure,
ϕ is the potential of the body force,
u^0 is the initial velocity.

Let $L > 0$ and $\Omega = (0, L)^3$. The L-periodic solutions:

$$u(x + Le_j) = u(x) \text{ for all } x \in \mathbb{R}^3, j = 1, 2, 3,$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$.
Zero average condition

$$\int_\Omega u(x)dx = 0,$$

Throughout $L = 2\pi$ and $\nu = 1$.

The Stokes operator:

$$Au = -\Delta u \text{ for all } u \in \mathcal{D}_A.$$

The bilinear mapping:

$$B(u, v) = P_L(u \cdot \nabla v) \text{ for all } u, v \in \mathcal{D}_A.$$

P_L is the Leray projection from $L^2(\Omega)$ onto H.
Spectrum of A:

$$\sigma(A) = \{|k|^2, 0 \neq k \in \mathbb{Z}^3\}.$$

If $N \in \sigma(A)$, denote by $R_N H$ the eigenspace of A corresponding to N.
Otherwise, $R_N H = \{0\}$.

Denote by $\mathcal{R}$ the set of all initial data $u^0 \in V$ such that the solution is regular for all times $t > 0$. In particular $u(t) \in \mathcal{D}_A$ for all $t > 0$.
The functional form of the NSE:

$$\frac{du(t)}{dt} + Au(t) + B(u(t), u(t)) = 0, \ \ t > 0,$$

$$u(0) = u^0 \in \mathcal{R},$$

where the equation holds in $\mathcal{D}_A$ for all $t > 0$ and $u(t)$ is continuous from $[0, \infty)$ into V.

Asymptotic expansion of regular solutions

Asymptotic expansion of $u(t) = u(t, u^0)$ (Foias-Saut)

$$u(t) \sim q_1(t)e^{-t} + q_2(t)e^{-2t} + q_3(t)e^{-3t} + \ldots,$$

where $q_j(t) = W_j(t, u^0)$ is a polynomial in t of degree at most $(j-1)$ and with values are trigonometric polynomials. This means that for any $N \in \mathbb{N}$,

$$|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}| = O(e^{-(N+\varepsilon)t}) \text{ for } t \to \infty,$$

with some $\varepsilon = \varepsilon_N > 0$. Moreover (Guillope), for $m \in \mathbb{N}$,

$$\|u(t) - \sum_{j=1}^{N} q_j(t)e^{-jt}\|_{H^m(\Omega)} = O(e^{-(N+\varepsilon)t})$$

as $t \to \infty$, for some $\varepsilon = \varepsilon_{N,m} > 0$

Normalization map

Let

$$W(u^0) = W_1(u^0) \oplus W_2(u^0) \oplus \cdots ,$$

where $W_j(u^0) = R_j q_j(0)$, for $j = 1, 2, 3...$ Then W is an one-to-one analytic mapping from $\mathcal{R}$ to the Frechet space

$$S_A = R_1 H \oplus R_2 H \oplus \cdots .$$

Constructions of polynomials $q_j(t)$

If $u^0 \in \mathcal{R}$ and $W(u^0) = (\xi_1, \xi_2, ...)$, then q_j's are the unique polynomial solutions to the following equations

$$q_j' + (A - j)q_j + \beta_j = 0,$$

with $R_j q_j(0) = \xi_j$, where β_j's are defined by

$$\beta_1 = 0 \text{ and for } j > 1,\ \beta_j = \sum_{k+l=j} B(q_k, q_l).$$

Explicitly, these polynomials $q_j(t)$'s are recurrently given by

$$q_j(t) = \xi_j - \int_0^t R_j \beta_j(\tau) d\tau + \sum_{n \geq 0} (-1)^{n+1} [(A - j)(I - R_j)]^{-n-1} (\frac{d}{dt})^n (I - R_j)\beta_j,$$

where $[(A - j)(I - R_j)]^{-n-1} u(x) = \sum_{|k|^2 \neq j} \frac{a_k}{(|k|^2 - j)^{n+1}} e^{ik \cdot x}$, for $u(x) = \sum_{|k|^2 \neq j} a_k e^{ik \cdot x} \in \mathcal{V}$.

Normal form of the Navier–Stokes equations

The S_A-valued function $\xi(t) = (\xi_n(t))_{n=1}^{\infty} = (W_n(u(t)))_{n=1}^{\infty} = W(u(t))$ satisfies the following system of differential equations

$$\frac{d\xi_1(t)}{dt} + A\xi_1(t) = 0,$$
$$\frac{d\xi_n(t)}{dt} + A\xi_n(t) + \sum_{k+j=n} R_n B(q_k(0,\xi(t)), q_j(0,\xi(t)) = 0, \;\; n > 1.$$

The solution of the above system with initial data $\xi^0 = (\xi_n^0)_{n=1}^{\infty} \in S_A$ is precisely $(R_n q_n(t,\xi^0)e^{-nt})_{n=1}^{\infty}$.

A construction of regular solutions

Split the initial data u^0 in V as $u^0 = \sum_{n=1}^{\infty} u_n^0$.
We find the solution $u(t)$ of the form $u(t) = \sum_{n=1}^{\infty} u_n(t)$, where for each n,

$$\frac{du_n(t)}{dt} + Au_n(t) + B_n(t) = 0, \quad t > 0,$$

with initial condition

$$u_n(0) = u_n^0,$$

where

$$B_1(t) \equiv 0, \quad B_n(t) = \sum_{j+k=n} B(u_j(t), u_k(t)), \ n > 1.$$

We call the above system the extended Navier–Stokes equations.

Existence theorems

Theorem (2006)

Let $S^0 = \sum_{n=1}^{\infty} \|u_n^0\| < \varepsilon_0$ *and* $u(t) = \sum_{n=1}^{\infty} u_n(t)$.
If S^0 *is small then* $u(t), t \geq 0$, *is the unique solution of the Navier–Stokes equations where* $u^0 = \sum_{n=1}^{\infty} u_n^0 \in V$ *and*

$$\sum_{n=1}^{\infty} \|u_n(t)\| \leq 2S^0 e^{-t}, \quad t > 0.$$

If $S^0 = \sum_{n=1}^{\infty} \|u_n^0\| < \infty$, *then* $u(t) = \sum_{n=1}^{\infty} u_n(t)$ *is the regular solution in* $(0, T)$ *for some* $T > 0$.

Connection to the asymptotic expansions

Theorem (2006)

Suppose $\sum_{n=1}^{\infty} \|W_n(0, u^0)\| < \varepsilon_0$*, then* $u(t, u^0) = \sum_{n=1}^{\infty} W_n(t, u^0) e^{-nt}$ *is the regular solution to the Navier–Stokes equations for all* $t > 0$*,*

Theorem (2006)

Suppose $\limsup_{n \to \infty} \|W_n(0, u^0)\|^{1/n} < \infty$*. Then there is* $T > 0$ *such that*

$$v(t) = \sum_{n=1}^{\infty} W_n(t, u^0) e^{-nt}$$

is absolutely convergent in V*, uniformly in* $t \in [T, \infty)$*,*
$\sum_{n=1}^{\infty} W_n(t, u^0) e^{-nt}$ *is the asymptotic expansion of* $v(t)$*, and*

$$u(t, u^0) = v(t) \text{ for all } t \in [T, \infty).$$

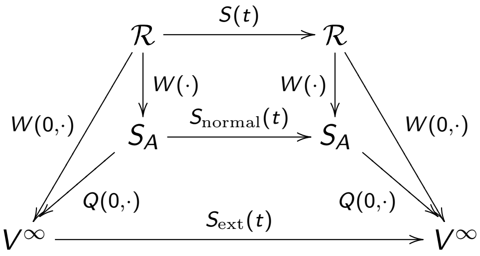
Algebraic relations

Let $V^\infty = V \oplus V \oplus V \oplus \cdots$. Define

$$W(t,\cdot) : u \in \mathcal{R} \mapsto (W_n(t,u)e^{-nt})_{n=1}^\infty \in V^\infty,$$

$$Q(t,\cdot) : \bar{\xi} \in S_A \mapsto (q_n(t,\bar{\xi})e^{-nt})_{n=1}^\infty \in V^\infty.$$

We primarily have



Constructed normed spaces

Let $(\tilde{\kappa}_n)_{n=2}^{\infty}$ be a fixed sequence of real numbers in the interval $(0,1]$ satisfying

$$\lim_{n\to\infty} (\tilde{\kappa}_n)^{1/2^n} = 0.$$

We define the sequence of positive weights $(\rho_n)_{n=1}^{\infty}$ by

$$\rho_1 = 1, \quad \rho_n = \tilde{\kappa}_n \gamma_n \rho_{n-1}^2, \; n > 1,$$

where $\gamma_n \in (0,1]$ are known and decrease to zero faster than n^{-n}.
For $\bar{u} = (u_n)_{n=1}^{\infty} \in V^{\infty}$, let

$$\|\bar{u}\|_\star = \sum_{n=1}^{\infty} \rho_n \|u_n\|_{H^1(\Omega)},$$

Define $V^\star = \{\, \bar{u} \in V^\infty : \|\bar{u}\|_\star < \infty \,\}, \quad S_A^\star = S_A \cap V^\star.$
Clearly $V^\star$ and $S_A^\star$ are Banach spaces.

Main Results

We summarize our results in the commutative diagram

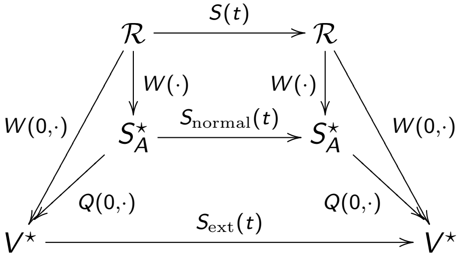


Figure: Commutative diagram

where all mappings are continuous.

Determining the weights

Recursive estimates: $\rho_n \| W_n(u^0) \| \le d_n$ where

$$d_1 = \rho_1 \| u^0 \|, \quad d_n = \rho_n \| u^0 \| + \kappa_n g_0^n \{ X^2 + (\sum_{k=1}^{n-1} d_k)^2 \}, \; n > 1$$

where g_0, X are positive numbers depending on u^0, κ_n can be chosen to be small.
Question: For which κ_n that $\sum_{n=1}^{\infty} d_n$ is finite?
We find decreasing ρ_n such that $\rho_n \le \kappa_n \rho_{n-1}^2$.

Numeric series

Lemma

Let $(a_n)_{n=1}^{\infty}$ *and* $(k_n)_{n=2}^{\infty}$ *be two sequences of positive numbers. Let* $d_1 = a_1$ *and* $d_n = a_n + k_n(\sum_{k=1}^{n-1} d_k)^2$*, for* $n > 1$*. Suppose*

$$\lim_{n\to\infty} k_n^{1/2^n} = 0.$$

If $\sum_{n=1}^{\infty} a_n$ *is finite, so is* $\sum_{n=1}^{\infty} d_n$*. More precisely,*

$$\sum_{n=1}^{\infty} d_n \leq \sum_{n=1}^{\infty} a_n + \alpha^2 \sum_{n=1}^{\infty} k_n M^{2(2^n-1)} < \infty,$$

where $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ *and* $M = 3\sup\{1, \alpha, k_n\alpha : n > 1\}$.

Recursive estimates

Sketch: Given $u^0 \in \mathcal{R}$, the asymptotic expansion of $u(t)$ is

$$u(t) \sim \sum u_n(t) = \sum W_n(t, u^0) e^{-nt} \quad \text{as} \quad t \to \infty.$$

For $n \geq 2$, denote $\tilde{u}_n(t) = u(t) - \sum_{k=1}^{n-1} u_k(t)$.
Suppose we have estimates for $\xi_j = W_j(u^0), q_j(\zeta) = W_j(\zeta, u^0)$ for $j = 1, \ldots, n-1$ and $\tilde{u}_j(\zeta)$ for $j = 2, \ldots, n$ for ζ in some domain of analyticity.

- Estimate $W_n(u^0) = \xi_n$ using

$$W_n(u^0) = R_n \tilde{u}_n(0) - \int_0^\infty e^{n\tau} \sum_{\substack{k,j \leq n-1 \\ k+j \geq n+1}} R_n B(u_k, u_j) d\tau$$
$$- \int_0^\infty e^{n\tau} R_n \big[B(u, \tilde{u}_n) + B(\tilde{u}_n, u) - B(\tilde{u}_n, \tilde{u}_n) \big] d\tau.$$

  for $n \in \sigma(A)$ and $n \geq 2$.

- Estimate $q_n(0, \xi_1, \ldots, \xi_{n-1})$.
- Using extended NSE with initial data $u_n(0)$ being the above $q_n(0)$ to bound $\rho_n \| W_n(\zeta, u^0) e^{-n\zeta} \| \leq M_n e^{-\mathbf{Re}\zeta}$. Then use Fragmen-Linderlöf type esitimate to obtain exact rate of decay.
- Using Navier–Stokes equations and Phragmen-Linderlöf type esitimate to bound bound $\| \tilde{u}_{n+1}(\zeta) \|$.

Above, we need to complexify NSE as well as extended NSE.

Formula of $q_n(0, \xi_1, \ldots, \xi_{n-1})$

Recall: $q_n(t)$ is the polynomial solution of

$$q_n' + (A - n)q_n + \beta_n = 0, \quad R_n q_n(0) = \xi_n,$$

$$\beta_n = \sum_{k+j=n} B(q_k, q_j).$$

Then

$$\begin{aligned} R_n q_n(0) &= \xi_n \\ P_{n-1} q_n(0) &= \int_0^\infty e^{\tau(A-n)P_{n-1}} P_{n-1} \beta_n(\tau) d\tau \\ (I - P_n) q_n(0) &= -\int_{-\infty}^0 e^{\tau(A-n)(P_{n^2}-P_n)} (P_{n^2} - P_n) \beta_n(\tau) d\tau. \end{aligned}$$

Extended Navier-Stokes Equations

Let $(\rho_n)_{n=1}^{\infty}$ be a sequence of positive numbers satisfying

$$\rho_n = \kappa_n \min\{\rho_k \rho_j : k+j=n\}, \quad \kappa_n \in (0,1], \quad n \geq 2.$$

with $\lim_{n\to\infty} \kappa_n^{1/n} = 0$.

Theorem

If $\bar{u}^0 \in V^\star$, then $S_{\text{ext}}(t)\bar{u}^0 \in V^\star$ for all $t > 0$. More precisely,

$$\|S_{\text{ext}}(t)\bar{u}^0\|_\star \leq M e^{-t}, \ t > 0,$$

where $M = \|\bar{u}^0\|_\star + C_1 \sum_{n=2}^{\infty} \kappa_n (n-1) M_0^n$,
$M_0 = \max\{1, 2C_1\kappa_n(n-1)\} \max\{1, 2\|\bar{u}^0\|_\star\}$.

Theorem

$S_{\mathrm{ext}}(t)$ is continuous from $V^\star$ to $V^\star$, for $t \in [0, \infty)$. More precisely, for any $\bar{u}^0 \in V^\star$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|S_{\mathrm{ext}}(t)\bar{v}^0 - S_{\mathrm{ext}}(t)\bar{u}^0\|_\star < \varepsilon e^{-t},$$

for all $\bar{v}^0 \in V^\star$ satisfying $\|\bar{v}^0 - \bar{u}^0\|_\star < \delta$ and for all $t \geq 0$.

Phragmen-Linderlöf type estimates.

Theorem

Let $f(\zeta)$ be analytic on the right half plane H_0, bounded by a constant M and

$$\sup_{x>0} e^{\alpha x}|f(x)| < \infty,$$

where α is a positive number. Then

$$|f(\zeta)| \leq Me^{-\alpha \mathbf{Re}\zeta}, \ \zeta \in H_0.$$

Our domain of analyticity when $\|u^0\|$ is small

$$D = \{\tau + i\sigma : \tau > 0, |\sigma| < c\tau e^{\alpha\tau}\},$$

where $c, \alpha > 0$.

Lemma

Let $c \geq \sqrt{2}, \alpha > 0$, then the transformation

$$\phi(\zeta) = \zeta - \frac{1}{\alpha} \log(1 + \alpha\zeta)$$

conformally maps D to a set containing the right half plane. Moreover, $\phi([0, \infty)) = [0, \infty)$.

Corollary

Suppose $u(\zeta)$ is analytic in $D(c, \alpha)$ where $c \geq \sqrt{2}, \alpha > 0$,

$$|u(\zeta)| \leq M, \quad \zeta \in D(c, \alpha),$$

$$\sup_{t>0} e^{nt} |u(t)| < \infty, \quad t > 0,$$

where n is a positive constant. Then

$$|u(\zeta)| \leq M e^{-n\mathbf{Re}\zeta} |1 + \alpha\zeta|^{n/\alpha}, \quad \zeta \in \phi_\alpha^{-1}(H_0).$$

Corollary

Let $q(\zeta)$ be a polynomial of degree less than or equal to p and

$$|e^{-N\zeta}q(\zeta)| \le M, \quad \zeta \in D.$$

Then

$$|q(\zeta)| \le M|1+\alpha\zeta|^{N/\alpha}, \quad \zeta \in \phi^{-1}(H_0),$$

$$|q(\zeta)| \le M(p+1)(1+\alpha a+\alpha r_a)^{N/\alpha}\left(\frac{|\zeta|+a}{r_a}\right)^p, \quad \zeta \in \mathbb{C}.$$

The range of the normalization map

Let $u^0 \in \mathcal{R}$, estinate $\|W(u^0)\|_\star = \sum_{n=1}^{\infty} \rho_n \|W_n(u^0)\|_{H^1(\Omega)}$.

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Estimates when $\|u^0\|$ is large. Combine above estimates on $[t_0, \infty)$, when t_0 is large, with the energy estimate on $[0, t_0)$.

Continuity of the normalization map, etc.

Similar to the estimates for the range. Final form: Given $u^0, v^0 \in \mathcal{R}$, with $\|u^0 - v^0\| < 1$. Let $w^0 = u^0 - v^0$, $w(t) = u(t) - v(t)$. Then

$$\rho_n \|W_n(u^0) - W_n(v^0)\| \leq y_n,$$

$$y_1 = \rho_1 \|w^0\|,$$

$$y_n = \rho_n \|w^0\| + \kappa_n M^n \Big(|w^0| + \|w(t_0)\| + \sum_{k=1}^{n-1} y_k \Big),$$

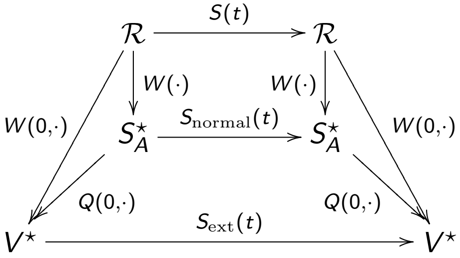
where M depends on u^0, positive t_0 is fixed.

Lemma

Given $\varepsilon > 0$, there is $\delta = \delta(u^0) > 0$ such that if $\|u^0 - v^0\| < \delta$, then $\sum_{n=1}^{\infty} y_n < \varepsilon$.

Summary

We have proved the commutative diagram



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Open problems

- Find u^0 such that $\sum_{n=1}^{\infty} \|W_n(0,u^0)\| < \varepsilon_0$ or $\limsup_{n\to\infty} \|W_n(0,u^0)\|^{1/n} < \infty$.
- Relations between the classical Leray weak solutions and the solutions to the extended Navier–Stokes equations.
- More properties and applications of the normalization map.