

Fall 2025. MATH3350.
Concise Lecture Notes
(Supplements to Dr. Gilliam's notes)
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CHAPTER 6

The Laplace Transform

6.1. Definition of the Laplace Transform

Definition:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

We have: for any integer $n \geq 0$ and real numbers a, b :

$$\mathcal{L}\{1\}(s) = \frac{1}{s}, \quad \mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a},$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2 + b^2}, \quad \mathcal{L}\{\sin(bt)\} = \frac{s}{s^2 + b^2}.$$

Factorials:

$$0! = 1, \quad 1! = 1, \quad 2! = 1(2) = 2, \quad 3! = 1(2)(3) = 6, \dots$$

$$n! = (n-1)!n = 1(2) \dots (n-1)(n).$$

Linear property:

$$\mathcal{L}\{\alpha f + \beta g\}(s) = \alpha \mathcal{L}\{f\}(s) + \beta \mathcal{L}\{g\}(s).$$

6.2. The Inverse Laplace Transform and Transforms of Derivatives

6.2.1. Inverse Laplace transforms. Partial fractions: Prof. Gilliam's notes.

EXAMPLE 6.1.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2!} \cdot \frac{2!}{s^3}\right\} = \frac{3}{2!} \mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{3}{2} t^2.$$

EXAMPLE 6.2.

$$F(s) = \frac{4s}{s^2 + 2s - 3} = \frac{4s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}.$$

Then

$$\mathcal{L}^{-1}\{F\} = Ae^{-3t} + Be^t.$$

Find A, B .

(a) Direct method:

$$\frac{4s}{(s+3)(s-1)} = \frac{A(s-1) + B(s+3)}{(s+3)(s-1)}.$$

Then

$$4s = A(s-1) + B(s+3).$$

Either equate coefficients or select values.

For the latter method:

$$s = -3: 4(-3) = A(-3-1), \text{ hence } A = 3.$$

$$s = 1: 4(1) = B(1+3), \text{ hence } B = 1.$$

(b) Cover-up:

$$\frac{4s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}.$$

For A , take $s = -3$:

$$A = \frac{4(-3)}{(-3-1)} = 3$$

For B , take $s = 1$,

$$B = \frac{4(1)}{(1+3)} = 1.$$

EXAMPLE 6.3.

$$\frac{5}{s^3 + 5s} = \frac{5}{s(s^2 + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 5} = \frac{A(s^2 + 5) + (Bs + C)s}{s(s^2 + 5)}.$$

Then

$$5 = A(s^2 + 5) + (Bs + C)s.$$

Take $s = 0$: $A = 1$.

Take $s = 1$: $5 = 6A + B + C$, $B + C = -1$.

Take $s = -1$: $5 = 6A - (C - B)$, $B - C = -1$.

Hence $C = 0$, $B = -1$.

EXAMPLE 6.4.

$$F(s) = \frac{2s}{(s-2)(s-3)(s-6)} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6}.$$

Then

$$\mathcal{L}^{-1}\{F\} = Ae^{2t} + Be^{3t} + Ce^{6t}.$$

Cover-up method:

$$A = \frac{2(2)}{(2-3)(2-6)} = \frac{4}{(-1)(-4)} = 1,$$

$$B = \frac{2(3)}{(3-2)(3-6)} = \frac{6}{-3} = -2,$$

$$C = \frac{2(6)}{(6-2)(6-3)} = \frac{12}{(4)(3)} = 1.$$

EXAMPLE 6.5.

$$F(s) = \frac{2s-4}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}.$$

Then

$$\mathcal{L}^{-1}\{F\} = A + Be^{-t} + C \cos(t) + D \sin(t).$$

Find A, B, C, D . We have

$$2s-4 = A(s+1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s+1).$$

Take $s = 0$: $-4 = A$.

Take $s = -1$: $-6 = B(-1)(2) = -2B$, then $B = 3$.

Take $s = 1$:

$$-2 = -4(2)(2) + 3(1)(2) + (C+D)(1)(2) = -16 + 6 + 2(C+D) = -10 + 2(C+D),$$

then

$$C + D = 4.$$

Take $s = 2$:

$$0 = -4(3)(5) + 3(2)(5) + (2C+D)(2)(3) = -60 + 30 + 6(2C+D) = -30 + 6(2C+D).$$

Then

$$2C + D = 5.$$

We obtain $C = 1, D = 3$.

Another way, after A, B .

For s^3 : $0 = A + B + C = -4 + 3 + C$. Then $C = 1$.

For s : $2 = A + B + D = -4 + 3 + D$. Then $D = 3$.

6.2.2. Laplace transform of derivatives.

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - 1 \cdot f(0),$$

$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - s \cdot f(0) - 1 \cdot f'(0),$$

$$\mathcal{L}\{f'''\}(s) = s^3\mathcal{L}\{f\}(s) - s^2 \cdot f(0) - s \cdot f'(0) - 1 \cdot f''(0),$$

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) - \dots - s \cdot f^{(n-2)}(0) - 1 \cdot f^{(n-1)}(0).$$

Application to solving ODE.

$$Ay'' + By' + Cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Setp 1. Let $Y(s) = \mathcal{L}\{y\}(s)$, $F(s) = \mathcal{L}\{f\}(s)$.

Applying $\mathcal{L}$ to the ODE gives

$$\mathcal{L}\{Ay'' + By' + Cy\} = \mathcal{L}\{f(t)\},$$

$$A\mathcal{L}\{y''\} + B\mathcal{L}\{y'\} + C\mathcal{L}\{y\} = F(s),$$

$$A(s^2Y - sy(0) - y'(0)) + B(sY - y(0)) + CY = F(s),$$

$$(As^2 + Bs + C)Y - A(sy_0 + y_1) - By_0 = F(s).$$

Setp 2. Solve for $Y(s)$.

Setp 3. Solution of the ODE is $y(t) = \mathcal{L}^{-1}\{Y\}$. Calculatte $y(t)$.

6.3. Shift/Translation Theorems (Operational Properties I)

Let

$$\mathcal{L}\{f(t)\}(s) = F(s).$$

6.3.1. Shift in s -variable. Then

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a),$$

$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at}f(t).$$

EXAMPLE 6.6.

$$\mathcal{L}\{t^3\}(s) = \frac{3!}{s^4}.$$

Then

$$\mathcal{L}\{e^{5t}t^3\}(s) = \mathcal{L}\{t^3\}(s-5) = \frac{3!}{(s-5)^4},$$

$$\mathcal{L}\{e^{-2t}t^3\}(s) = \mathcal{L}\{t^3\}(s+2) = \frac{3!}{(s+2)^4}.$$

EXAMPLE 6.7.

$$\mathcal{L}\{\cos(2t)\}(s) = \frac{s}{s^2+4}.$$

Then

$$\mathcal{L}\{e^{3t}\cos(2t)\}(s) = \mathcal{L}\{\cos(2t)\}(s-3) = \frac{(s-3)}{(s-3)^2+4},$$

$$\mathcal{L}\{e^{-4t}\cos(2t)\}(s) = \mathcal{L}\{\cos(2t)\}(s+4) = \frac{(s+4)}{(s+4)^2+4}.$$

EXAMPLE 6.8.

$$\mathcal{L}\{\sin(5t)\}(s) = \frac{5}{s^2+25}.$$

Then

$$\mathcal{L}\{e^t\sin(5t)\}(s) = \mathcal{L}\{\sin(5t)\}(s-1) = \frac{5}{(s-1)^2+25},$$

$$\mathcal{L}\{e^{-7t}\sin(5t)\}(s) = \mathcal{L}\{\sin(5t)\}(s+7) = \frac{5}{(s+7)^2+25}.$$

EXAMPLE 6.9.

$$s^2 + 2s + 5 = s^2 + 2s + 1^2 - 1^2 + 5 = (s+1)^2 + 4.$$

$$s^2 + 6s + 34 = s^2 + 6s + 3^2 - 3^2 + 34 = (s+3)^2 + 25.$$

$$\begin{aligned} F(s) &= \frac{2s+5}{(s+3)^2+5^2} = \frac{2[(s+3)-3]+5}{(s+3)^2+5^2} = \frac{2(s+3)-6+5}{(s+3)^2+5^2} = \frac{2(s+3)-1}{(s+3)^2+5^2} \\ &= \frac{2(s+3)}{(s+3)^2+5^2} - \frac{1}{5} \cdot \frac{5}{(s+3)^2+5^2}. \end{aligned}$$

Then

$$\mathcal{L}^{-1}\{F\}(t) = 2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t).$$

6.3.2. Shift in t -variable. Unit step/Heaviside function $u(t-a)$. Then

$$\mathcal{L}\{f(\textcolor{red}{t}-a)u(\textcolor{red}{t}-a)\}(s) = e^{-as}\mathcal{L}\{f(t)\}(s),$$

$$\mathcal{L}\{f(\textcolor{red}{t})u(t-a)\}(s) = e^{-as}\mathcal{L}\{f(\textcolor{red}{t}+a)\}(s),$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(\textcolor{red}{t}-a)u(t-a).$$

Note

$$\mathcal{L}\{u(t-a)\}(s) = e^{-as}\mathcal{L}\{1\}(s) = \frac{e^{-as}}{s}.$$

EXAMPLE 6.10. $f(t) = 2 + (t-2)u(t-3) + (e^t - t)u(t-4)$.

We have

$$\mathcal{L}\{2\} = \frac{2}{s},$$

$$\mathcal{L}\{(t-2)u(t-3)\}(s) = e^{-3s}\mathcal{L}\{(\textcolor{red}{t}+3)-2\}(s) = e^{-3s}\mathcal{L}\{t+1\}(s) = e^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right),$$

$$\begin{aligned}\mathcal{L}\{(e^t - t)u(t-4)\}(s) &= e^{-4s}\mathcal{L}\{e^{\textcolor{red}{t}+4} - (\textcolor{red}{t}+4)\}(s) = e^{-4s}\left(e^4\mathcal{L}\{e^t\} - \mathcal{L}\{t\} - 4\mathcal{L}\{1\}\right) \\ &= e^{-4s}\left(\frac{e^4}{s-1} - \frac{1}{s^2} - \frac{4}{s}\right).\end{aligned}$$

Hence,

$$\mathcal{L}\{f\}(s) = \frac{2}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-4s}\left(\frac{e^4}{s-1} + \frac{1}{s^2} - \frac{4}{s}\right).$$

EXAMPLE 6.11.

$$F(s) = \frac{e^{-2s}}{s^2 + 9}.$$

Let

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\} = \frac{1}{3}\sin(3t).$$

Then

$$\mathcal{L}^{-1}\{F(s)\}(t) = g(\textcolor{red}{t} - 2)u(\textcolor{red}{t} - 2) = \frac{1}{3} \sin(3(\textcolor{red}{t} - 2))u(t - 2) = \frac{1}{3} \sin(3t - 6)u(t - 2).$$

EXAMPLE 6.12 (Example 4.35 in Prof. Gilliam's notes).

$$Y = \frac{s^2 + 27}{s^2(s - 3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 3} + \frac{D}{(s - 3)^2} = \frac{As(s - 3)^2 + B(s - 3)^2 + Cs^2(s - 3) + Ds^2}{s^2(s - 3)^2},$$

then

$$s^2 + 27 = As(s - 3)^2 + B(s - 3)^2 + Cs^2(s - 3) + Ds^2.$$

Take $s = 0$:

$$27 = 9B, \quad B = 3.$$

Take $s = 3$:

$$9 + 27 = 9D, \quad D = 4.$$

Take $s = 1$:

$$28 = 4A + 4B - 2C + D = 4A + 12 - 2C + 4, \quad 2A - C = 6.$$

Take $s = -1$:

$$28 = -16A + 16B - 4C + D = -16A + 48 - 4C + 4, \quad 4A + C = 6.$$

Hence $A = 2$, $C = -2$. Thus,

$$Y = \frac{2}{s} + \frac{3}{s^2} - \frac{2}{s - 3} + \frac{4}{(s - 3)^2}.$$

EXAMPLE 6.13 (Example 4.36 in Prof. Gilliam's notes).

$$Y = \frac{2}{s(s^2 - 2s + 2)} = \frac{2}{s((s - 1)^2 + 1)} = \frac{A}{s} + \frac{B(s - 1) + C}{(s - 1)^2 + 1} = \frac{A[(s - 1)^2 + 1] + s[B(s - 1) + C]}{s((s - 1)^2 + 1)},$$

then

$$2 = A[(s - 1)^2 + 1] + s[B(s - 1) + C].$$

Take $s = 0$: $2 = 2A$, then $A = 1$.

Take $s = 1$:

$$2 = A + C = 1 + C, \quad C = 1.$$

Take $s = 2$:

$$2 = 2A + 2(B + C), \quad 1 = A + B + C = 2 + B, \quad B = -1.$$

Then

$$Y = \frac{1}{s} - \frac{(s-1)}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1}.$$

Solution

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = 1 - e^t \cos(t) + e^t \sin(t).$$

6.4. Additional Operational Properties (Operational Properties II)

6.4.1. Derivative of the Laplace transform.

$$\frac{d}{ds} \mathcal{L}\{f(t)\}(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt = \int_0^\infty (-t) e^{-st} f(t) dt = -\mathcal{L}\{tf(t)\}(s).$$

Thus,

$$\frac{d}{ds} \mathcal{L}\{f(t)\}(s) = -\mathcal{L}\{tf(t)\}(s).$$

Then

$$\begin{aligned} \frac{d^2}{ds^2} \mathcal{L}\{f(t)\}(s) &= \frac{d}{ds} \left(\frac{d}{ds} \mathcal{L}\{f(t)\}(s) \right) = -\frac{d}{ds} (\mathcal{L}\{tf(t)\}(s)) \\ &= -(-\mathcal{L}\{t \cdot tf(t)\}(s)) = 1 \cdot \mathcal{L}\{t^2 f(t)\}(s). \end{aligned}$$

Therefore,

$$\frac{d^2}{ds^2} \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{t^2 f(t)\}(s).$$

Repeating this gives

$$\begin{aligned} \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}(s) &= (-1)^n \mathcal{L}\{t^n f(t)\}(s), \\ \mathcal{L}\{t^n f(t)\}(s) &= (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(t)\}(s). \end{aligned}$$

6.4.2. Laplace transform of integrals.

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}(s) = \frac{1}{s} \mathcal{L}\{f(t)\}(s).$$

Later, we can use convolution:

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\}(s) = \mathcal{L}\{1 * f\} = \mathcal{L}\{1\} \mathcal{L}\{f\} = \frac{1}{s} \mathcal{L}\{f(t)\}(s).$$

Convolutions.

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau.$$

We have

$$(f * g)(t) = (g * f)(t) = \int_0^t f(t - \tau) g(\tau) d\tau.$$

Also,

$$(f * g) * h = f * (g * h).$$

Let $F(s) = \mathcal{L}\{f\}$ and $G(s) = \mathcal{L}\{g\}$. Then

$$\begin{aligned}\mathcal{L}\{f * g\} &= F \cdot G, \\ \mathcal{L}^{-1}\{F \cdot G\} &= f * g.\end{aligned}$$

EXAMPLE 6.14.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{14}{(s+2)(s-5)}\right\} &= 14\mathcal{L}^{-1}\left\{\frac{1}{s+2} \cdot \frac{1}{s-5}\right\} \\ &= 14\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = 14e^{-2t} * e^{5t} \\ &= 14 \int_0^t e^{-2\tau} e^{5(t-\tau)} d\tau = 14e^{5t} \int_0^t e^{-7\tau} d\tau \\ &= \frac{14e^{5t}}{-7}(e^{-7t} - 1) = -2(e^{-2t} - e^{5t}).\end{aligned}$$

Trigonometric identities.

$$\begin{aligned}\cos(a)\cos(b) &= \frac{1}{2}(\cos(a+b) + \cos(a-b)), \\ \sin(a)\sin(b) &= \frac{1}{2}(\cos(a-b) - \cos(a+b)), \\ \sin(a)\cos(b) &= \frac{1}{2}(\sin(a+b) + \sin(a-b)).\end{aligned}$$

EXAMPLE 6.15 (Example 4.48 in Prof. Gilliam's notes).

$$\begin{aligned}
\mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 1)^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \cdot \frac{1}{s^2 + 1} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 1} \right\} * \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} = (\cos t) * (\sin t) \\
&= \int_0^t \cos \tau \sin(t - \tau) d\tau = \frac{1}{2} \int_0^t (\sin t + \sin(t - 2\tau)) d\tau \\
&= \frac{1}{2} t \sin t - \frac{1}{2} \int_0^t \sin(2\tau - t) d\tau \\
&= \frac{1}{2} t \sin t + \frac{1}{2} \cdot \frac{1}{2} \cos(2\tau - t) \Big|_{\tau=0}^{\tau=t} \\
&= \frac{1}{2} t \sin t + \frac{1}{4} (\cos t - \cos(-t)) \\
&= \frac{1}{2} t \sin t.
\end{aligned}$$

EXAMPLE 6.16.

$$\begin{aligned}
\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 4)^2} \right\} &= \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \cdot \frac{2}{s^2 + 4} \right\} \\
&= \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} * \mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} = \frac{1}{4} \sin(2t) * \sin(2t) \\
&= \frac{1}{4} \int_0^t \sin(2\tau) \sin(2(t - \tau)) d\tau = \frac{1}{4} \cdot \frac{1}{2} \int_0^t (\cos(4\tau - 2t) - \cos(2t)) d\tau \\
&= \frac{1}{8} \left\{ \frac{1}{4} \sin(4\tau - 2t) \Big|_{\tau=0}^{\tau=t} - t \cos(2t) \right\} \\
&= \frac{1}{8} \left\{ \frac{1}{4} [\sin(2t) - \sin(-2t)] - t \cos(2t) \right\} \\
&= \frac{1}{8} \left\{ \frac{1}{2} \sin(2t) - t \cos(2t) \right\}.
\end{aligned}$$

EXAMPLE 6.17.

$$\begin{aligned}
\mathcal{L}^{-1} \left\{ \frac{1}{s^3(s + 3)} \right\} &= \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^3} \cdot \frac{1}{s + 3} \right\} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^3} \right\} * \mathcal{L}^{-1} \left\{ \frac{1}{s + 3} \right\} \\
&= \frac{1}{2} t^2 * e^{-3t} = \frac{1}{2} \int_0^t \tau^2 e^{-3(t-\tau)} d\tau
\end{aligned}$$

Using integration by parts

alternate sign	derivative in τ	anti-derivative in τ
+	τ^2	$e^{-3(t-\tau)}$
-	2τ	$\frac{1}{3}e^{-3(t-\tau)}$
+	2	$\frac{1}{9}e^{-3(t-\tau)}$
-	0	$\frac{1}{27}e^{-3(t-\tau)}$

Then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+3)}\right\} &= \frac{1}{2}\left\{\tau^2\frac{1}{3}e^{-3(t-\tau)}\Big|_{\tau=0}^{\tau=t} - 2\tau\frac{1}{9}e^{-3(t-\tau)}\Big|_{\tau=0}^{\tau=t} + \frac{2}{27}e^{-3(t-\tau)}\Big|_{\tau=0}^{\tau=t} - 0\right\} \\ &= \frac{1}{2}\left\{\frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27}(1 - e^{-3t})\right\}.\end{aligned}$$

EXAMPLE 6.18.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{2}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\ &= \frac{1}{2}t * \sin(2t) = \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)d\tau\end{aligned}$$

Using integration by parts

alternate sign	derivative in τ	anti-derivative in τ
+	$t - \tau$	$\sin(2\tau)$
-	-1	$-\frac{1}{2}\cos(2\tau)$
+	0	$-\frac{1}{4}\sin(2\tau)$

Then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} &= \frac{1}{2}\left\{-\frac{1}{2}(t-\tau)\cos(2\tau)\Big|_{\tau=0}^{\tau=t} - \frac{1}{4}\sin(2\tau)\Big|_{\tau=0}^{\tau=t} + 0\right\} \\ &= \frac{1}{2}\left\{\frac{1}{2}t - \frac{1}{4}(\sin(2t) - 0)\right\}.\end{aligned}$$

Volterra Integral Equations.

EXAMPLE 6.19. Solve for $y(t)$:

$$y(t) = -t + \int_0^t y(t-\tau)\sin(\tau)d\tau.$$

Write

$$y(t) = -t + y(t) * \sin(t).$$

Take Laplace transform of the equation:

$$Y = \mathcal{L}\{y(t)\} = -\mathcal{L}\{t\} + \mathcal{L}\{y(t) * \sin(t)\} = -\frac{1}{s^2} + \mathcal{L}\{y(t)\} \mathcal{L}\{\sin(t)\} = -\frac{1}{s^2} + Y \cdot \frac{1}{s^2 + 1}.$$

Solve for $Y(s)$. Then solution is $y(t) = \mathcal{L}^{-1}\{Y\}(t)$.

EXAMPLE 6.20. Solve for $y(t)$:

$$y(t) + \int_0^t (t - \tau)y(\tau)d\tau = t.$$

Write

$$y(t) + g(t) * y(t) = t,$$

where $g(\textcolor{red}{t} - \tau) = \textcolor{red}{t} - \tau$. Then $g(\textcolor{red}{t}) = \textcolor{red}{t}$. Thus,

$$y(t) + t * y(t) = t,$$

Take Laplace transform of the equation:

$$\mathcal{L}\{y(t)\} + \mathcal{L}\{t * y(t)\} = \mathcal{L}\{t\},$$

$$Y + \mathcal{L}\{t\} \mathcal{L}\{y(t)\} = \frac{1}{s^2},$$

$$Y + \frac{1}{s^2} \cdot Y = \frac{1}{s^2}.$$

Solve for $Y(s)$. Then solution is $y(t) = \mathcal{L}^{-1}\{Y\}(t)$.

6.4.3. Laplace transform of periodic functions. Suppose f is T -periodic, that is $f(t + T) = f(t)$ for all $t \geq 0$.

Define function f_T by

$$f_T(t) = \begin{cases} f(t), & \text{if } 0 \leq t < T, \\ 0, & \text{if } t > T. \end{cases}$$

Then

$$f_T(t) = f(t) - f(t)u(t - T).$$

We have

$$\mathcal{L}\{f_T(t)\} = \mathcal{L}\{f(t)\} - e^{-Ts} \mathcal{L}\{f(\textcolor{red}{t} + \textcolor{red}{T})\} = \mathcal{L}\{f(t)\} - e^{-Ts} \mathcal{L}\{f(\textcolor{red}{t})\}.$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-Ts}}. \quad (6.1)$$

Note that $\mathcal{L}\{f_T(t)\}(s) = \int_0^T e^{-st} f(t) dt$, then

$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-Ts}}. \quad (6.2)$$

However, we will calculate $\mathcal{L}\{f_T(t)\}$ in (6.1) by different ways, rather than use the the integral in (6.2). (Read also another proof of (6.2) in the textbook or Prof. Gilliam's notes.)

EXAMPLE 6.21. $f(t) = t$ for $0 < t < 2$ and has period $T = 2$. Let

$$f_T(t) = \begin{cases} t, & \text{if } 0 \leq t < 2, \\ 0, & \text{if } t > 2. \end{cases}$$

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-2s}}.$$

Now calculate $\mathcal{L}\{f_T(t)\}(s)$. We have

$$f_T(t) = t - tu(t-2).$$

Then

$$\mathcal{L}\{f_T(t)\}(s) = \frac{1}{s^2} - e^{-2s} \mathcal{L}\{t+2\}(s) = \frac{1}{s^2} - e^{-2s} \left(\frac{1}{s^2} + \frac{2}{s} \right).$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = \frac{\frac{1}{s^2} - e^{-2s} \left(\frac{1}{s^2} + \frac{2}{s} \right)}{1 - e^{-2s}}.$$

EXAMPLE 6.22. Function $f(t)$ has period $T = 2$, and

$$f(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ -1, & \text{if } 1 < t < 2. \end{cases}$$

Let

$$f_T(t) = \begin{cases} f(t), & \text{if } 0 \leq t < 2, \\ 0, & \text{if } t > 2 \end{cases} = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ -1, & \text{if } 1 < t < 2, \\ 0, & \text{if } t > 2. \end{cases}$$

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-2s}}.$$

Now calculate $\mathcal{L}\{f_T(t)\}(s)$. We have

$$f_T(t) = 1 - 2u(t-1) + 1 \cdot u(t-2).$$

Then

$$\mathcal{L}\{f_T(t)\}(s) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}.$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = \frac{\frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}}{1 - e^{-2s}}.$$

EXAMPLE 6.23. Function $f(t)$ has period $T = 2\pi$, and

$$f(t) = \begin{cases} \sin(t), & \text{if } 0 \leq t < \pi, \\ 0, & \text{if } \pi < t < 2\pi. \end{cases}$$

Let

$$f_T(t) = \begin{cases} f(t), & \text{if } 0 \leq t < 2\pi, \\ 0, & \text{if } t > 2\pi \end{cases} = \begin{cases} \sin(t) & \text{if } 0 \leq t < \pi, \\ 0, & \text{if } \pi < t < 2\pi, \\ 0, & \text{if } t > 2\pi \end{cases} = \begin{cases} \sin(t) & \text{if } 0 \leq t < \pi, \\ 0, & \text{if } t > \pi. \end{cases}$$

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-2\pi s}}.$$

Now calculate $\mathcal{L}\{f_T(t)\}(s)$. We have

$$f_T(t) = \sin(t) - \sin(t)u(t-\pi).$$

Then

$$\begin{aligned} \mathcal{L}\{f_T(t)\}(s) &= \frac{1}{s^2 + 1} - e^{-\pi s} \mathcal{L}\{\sin(t+\pi)\} = \frac{1}{s^2 + 1} + e^{-\pi s} \mathcal{L}\{\sin(t)\} \\ &= \frac{1}{s^2 + 1} + \frac{e^{-\pi s}}{s^2 + 1} = \frac{1 + e^{-\pi s}}{s^2 + 1}. \end{aligned}$$

Therefore,

$$\mathcal{L}\{f\}(s) = \frac{1 + e^{-\pi s}}{(s^2 + 1)(1 - e^{-2\pi s})} = \frac{1 + e^{-\pi s}}{(s^2 + 1)(1 - e^{-\pi s})(1 + e^{-\pi s})} = \frac{1}{(s^2 + 1)(1 - e^{-\pi s})}.$$

6.5. The Dirac Delta Function

DEFINITION 6.5.1. *The Dirac delta function $\delta(t)$ is characterized by the following two properties.*

(i)

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0. \end{cases}$$

(ii)

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0),$$

for any function f which is continuous on an open interval containing 0.

In fact, $\delta(t)$ is not a function, but rather a *generalized function*.

Let $a \in \mathbb{R}$. Shifting in t by a , we have $\delta(t - a)$. Then, similarly,

(i)

$$\delta(t - a) = \begin{cases} 0, & \text{if } t \neq a, \\ \infty, & \text{if } t = a. \end{cases}$$

(ii)

$$\int_{-\infty}^{\infty} f(t)\delta(t - a)dt = f(a),$$

for any function f which is continuous on an open interval containing a .

For $a \geq 0$,

$$\mathcal{L}\{\delta(t - a)\}(s) = \int_0^{\infty} e^{-st}\delta(t - a)dt = e^{-as}.$$

When $a = 0$, we have

$$\mathcal{L}\{\delta(t)\}(s) = 1.$$

In general,

$$\mathcal{L}\{f(t)\delta(t - a)\}(s) = \int_0^{\infty} e^{-st}f(t)\delta(t - a)dt = f(a)e^{-as}. \quad (6.3)$$

EXAMPLE 6.24. Find $\mathcal{L}\{e^{-2t}\delta(t - 3)\}(s)$.

Solution 1: Using translation in t ,

$$\mathcal{L}\{e^{-2t}\delta(t - 3)\}(s) = \mathcal{L}\{\delta(t - 3)\}(s + 2) = e^{-3(s+2)}.$$

Solution 2: By definition,

$$\mathcal{L}\{e^{-2t}\delta(t-3)\}(s) = \int_0^{\infty} e^{-st}e^{-2t}\delta(t-3)dt = e^{-3s}e^{-2(3)}.$$

Solution 3: By using (6.3),

$$\mathcal{L}\{e^{-2t}\delta(t-3)\}(s) = e^{-2(3)}e^{-3s}.$$

EXAMPLE 6.25.

$$\mathcal{L}\{\delta(t-1) - \delta(t-3)\} = e^{-s} - e^{-3s},$$

$$\mathcal{L}\{t\delta(t-1)\} = 1 \cdot e^{-s},$$

$$\mathcal{L}\{\delta(t-\pi)\sin t\} = (\sin \pi)e^{-\pi s},$$

or directly,

$$\mathcal{L}\{\delta(t-\pi)\sin t\} = \int_0^{\infty} e^{-st}\sin(t)\delta(t-\pi)dt = e^{-\pi s}(\sin \pi).$$

EXAMPLE 6.26 (Example 4.62 in Prof. Gilliam's notes).

$$\begin{aligned} Y &= \frac{2s+5}{(s+2)(s+3)} + \frac{e^{-2(s+1)}}{(s+2)(s+3)} \\ &= \frac{1}{s+2} + \frac{1}{s+3} + e^{-2}e^{-2s}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\}. \end{aligned}$$

Solution is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y\} \\ &= e^{-2t} + e^{-3t} + e^{-2}u(t-2)\left\{e^{-2(t-2)} - e^{-3(t-2)}\right\}. \end{aligned}$$

Remarks.

(a) Consider ODE:

$$ay'' + by' + c = \delta(t), \quad y(0) = 0 = y'(0),$$

where a, b, c are constants.

Let $Y = \mathcal{L}\{y(t)\}(s)$. Then we have

$$as^2Y + bsY + cY = \mathcal{L}\{\delta(t)\} = 1,$$

$$Y = \frac{1}{as^2 + bs + c}.$$

Set

$$H(s) = \frac{1}{as^2 + bs + c}, \quad h(t) = \mathcal{L}^{-1}\{H\}.$$

(H is called the transfer function.)

Then $Y = H$ and

$$y(t) = \mathcal{L}^{-1}\{Y\} = h(t).$$

The solution $h(t)$ is called the response to the impulse δ .

(b) Now, consider ODE:

$$ay'' + by' + c = f(t), \quad y(0) = 0 = y'(0),$$

where a, b, c are constants.

Let $Y = \mathcal{L}\{y(t)\}(s)$. Then we have

$$as^2Y + bsY + cY = \mathcal{L}\{f(t)\} = F(s),$$

$$Y = \frac{1}{as^2 + bs + c} \cdot F(s).$$

Set

$$H(s) = \frac{1}{as^2 + bs + c}, \quad h(t) = \mathcal{L}^{-1}\{H\}.$$

Then $Y = H \cdot F$ and

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{H \cdot F\} = h(t) * f(t).$$

The solution $y(t)$ is the convolution of the response $h(t)$ and the forcing function $f(t)$.

If initial data is $y(0)$ and $y'(0)$ are not specified, then solution

$$y_p = h(t) * f(t)$$

plays the role of a particular solution. Recall the general solution is

$$y = y_p + y_c.$$

(c) Finally, relation between unit step function and Dirac delta function is

$$u'(t - a) = \delta(t - a).$$

APPENDIX A

Some problems

Review p. 18. Find

$$f(t) = \mathcal{L}^{-1} \left\{ e^{-2s} \left(\frac{1}{s^3} - \frac{2s}{(s-1)^2 + 9} \right) \right\}.$$

Solution. Let

$$g(t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\}, \quad h(t) = \mathcal{L}^{-1} \left\{ \frac{2s}{(s-1)^2 + 9} \right\}.$$

Then

$$f(t) = u(t-2) \left(g(t-2) - h(t-2) \right).$$

Calculations

$$g(t) = \frac{1}{2}t^2,$$

$$h(t) = \mathcal{L}^{-1} \left\{ \frac{2(s-1) + 2}{(s-1)^2 + 9} \right\} = 2e^t \cos(3t) + \frac{2}{3}e^t \sin(3t).$$

Then

$$f(t) = u(t-2) \left(\frac{1}{2}(t-2)^2 - 2e^{t-2} \cos(3(t-2)) - \frac{2}{3}e^{t-2} \sin(3(t-2)) \right).$$

HW 9, pb 13. Solve equation

$$y'' + 16y = \begin{cases} 16, & 0 \leq t < 3, \\ 0, & 3 \leq t, \end{cases} \quad y(0) = 3, \quad y'(0) = 4.$$

Solution. Write equation as

$$y'' + 16y = 16 - 16u(t-3).$$

Denote $Y = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation gives

$$\begin{aligned}(s^2Y - 3s - 4) + 16Y &= \frac{16}{s} - 16\frac{e^{-3s}}{s}, \\(s^2 + 16)Y &= 3s + 4 + \frac{16}{s} - 16\frac{e^{-3s}}{s}, \\Y &= \frac{3s + 4}{s^2 + 16} + \frac{16}{s(s^2 + 16)} - \frac{16e^{-3s}}{s(s^2 + 16)} \\&= \frac{3s + 4}{s^2 + 16} + G(s) - G(s)e^{-3s},\end{aligned}$$

where

$$G(s) = \frac{16}{s(s^2 + 16)}.$$

Let

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{16}{s(s^2 + 16)}\right\}.$$

Solution is

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{3s + 4}{s^2 + 16}\right\} + g(t) - u(t - 3)g(t - 3) \\&= 3\cos(4t) + \sin(4t) + g(t) - u(t - 3)g(t - 3).\end{aligned}$$

Using partial fraction:

$$G(s) = \frac{16}{s(s^2 + 16)} = \frac{A}{s} + \frac{Bs + 4C}{s^2 + 16} = \frac{A(s^2 + 16) + (Bs + 4C)s}{s(s^2 + 16)}.$$

Then

$$16 = A(s^2 + 16) + (Bs + 4C)s.$$

Take $s = 0$: $16 = 16A$. Then $A = 1$.

Take $s = 1$: $16 = 17A + B + 4C$, hence $B + 4C = -1$.

Take $s = -1$: $16 = 17A - (-B + 4C)$, hence $-B + 4C = 1$.

Solving for B, C gives $C = 0$, $B = -1$.

Thus,

$$g(t) = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs + 4C}{s^2 + 16}\right\} = A + B\cos(4t) + C\sin(4t) = 1 - \cos(4t).$$

Therefore, the solution is

$$\begin{aligned}y(t) &= 3\cos(4t) + \sin(4t) + 1 - \cos(4t) - u(t - 3)(1 - \cos(4(t - 3))) \\&= 1 + 2\cos(4t) + \sin(4t) - u(t - 3)(1 - \cos(4t - 12)).\end{aligned}$$

HW 10, pb 5. Solve equation

$$y(t) + 9 \int_0^t (t-v)y(v)dv = -3t.$$

Solution. Write equation as

$$y(t) + 9t * y(t) = -3t.$$

Denote $Y = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation gives

$$\mathcal{L}\{y(t)\} + 9\mathcal{L}\{t * y(t)\} = -3\mathcal{L}\{t\},$$

$$Y + 9\mathcal{L}\{t\}\mathcal{L}\{y\} = -3\mathcal{L}\{t\},$$

$$Y + \frac{9}{s^2}Y = -\frac{3}{s^2},$$

$$\frac{s^2+9}{s^2}Y = -\frac{3}{s^2},$$

$$Y = -\frac{3}{s^2+9}.$$

Solution is

$$y(t) = \mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = -\sin(3t).$$