Math 3350

1 Chapter 1

Reading assignment for Chapter 1: Study Sections 1.1 and 1.2.

1.1 Material for Section 1.1

An Ordinary Differential Equation (ODE) is a relation between an independent variable x and and a dependent variable y (i.e., $y = y(x)$ depends on x) and its derivatives $y^{(j)} = \dfrac{d^j y}{dx^j}$ for $j = 1, \cdots, n$. So it is an equation that can be written in the form

$$F(x, y, y^{(1)}, \cdots, y^{(n)}) = 0, \tag{1}$$

For example, in this chapter we will learn to *solve* equations like the following:

$$y' - x\cos(x^2) = 0,\ y' - \sin(x+y) = 0,\ \ y' - 2xy - x = 0, \text{ and } \ y' = \frac{x-y}{x+y}.$$

There are other types of differential equations most notably partial differential equations (PDEs). It is easy to distinguish an ODE from a PDE. An Ode always has a single independent variable while a PDE always has more that one independent variable and the equation involves partial derivatives. For example consider a PDE with independent variables x and y and dependent variable z so that z depends on x and y, i.e., $z = z(x, y)$. Then a general second order PDE would be an equation of the form

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}\right) = 0.$$

We study differential equations because many practical physical systems are govern (or described by) either ordinary or partial differential equations. Solving ODEs for explicit solutions can be very difficult or even impossible. But in this class we will focus on the solution of simple problems in order to give students some idea of what is involved in the more general case. The main tools needed by the students is a background in college algebra and calculus (differentiation and integration).

The goal in "solving" an equation is to find all functions $y(x)$ satisfying the equation (1). Unfortunately this objective is beyond our reach other than some very special cases. In this class we will study a few of these special cases.

Notation and Terminology: The following discussion may seem a bit over the top. The main purpose is to introduce the standard notation and terminology used in talking about ODEs. **DO NOT BE OVERWHELMED** they are just words. Also read the book where more examples are also given. The main words to understand are written in *italics* and underlined.

1. Chapter 1 is mostly concerned with notation and terminology. We will need this material so I will cover it fairly carefully. You need to learn these definitions and terminology.

2. The order of the highest order derivative, n in (1), is called **the order of the equation**.

3. If we can solve for the highest order derivative term, then we say the equation can be put in *normal form*:

$$y^{(n)} = f(x, y, y^{(1)}, \cdots, y^{(n-1)}).$$

   In chapter 2 we will consider only first order equations $y' = f(x, y)$.

4. An ODE is said to be *Linear* if it can be written in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{d^1 y}{dx^1} + a_0(x)y(x) = g(x). \qquad (2)$$

   In Chapters 3, 4 and 5 we will consider mostly linear equations. They are by far the easiest - but still not easy. Similarly a linear PDE in two independent variables is

$$a_{2,0}(x,y)\frac{\partial^2 z}{\partial x^2} + a_{0,2}(x,y)\frac{\partial^2 z}{\partial y^2} + a_{1,1}(x,y)\frac{\partial^2 z}{\partial x \partial y} + a_{1,0}(x,y)\frac{\partial z}{\partial x} + a_{0,1}(x,y)\frac{\partial z}{\partial y} + a_{0,0}z = 0.$$

   Notice the equation has independent variables x and y and dependent variable z.

   (a) A linear equation is is said to be *homogeneous* if $g(x) = 0$. If $g(x) \neq 0$ then the equation is called *non-homogeneous*.

(b) A solution $y(x)$ may only exist on a certain domain called the *interval of existence*. The interval of existence may be open, e.g., (a, b), closed, e.g., $[a, b]$ or it may be open on one end and closed on the other, e.g., $(a, b]$. It is possible that this interval is all of $(-\infty, \infty)$.

(c) Here are some examples

- $(1 + x)y'' - \cos(x^2)y' + e^x y = \cos(x)$ (second order, linear,ode)
- $(1 + y^2)y''' - x\sin(y + y') = 0$ (third order, nonlinear,ode)
- $y' - 2xy - y^2 = 0$ (first order, nonlinear,ode)
- $(y')^2 = \dfrac{x - y}{x + y}$ (first order, nonlinear,ode)
- $x^3y''' - x^2y'' + 2xy' - y = 0$ (third order, linear,ode)
- $z_{xx} + z_{yy} = 0$ (second order, linear, pde)

5. Sometimes it is possible that $y = 0$ is a solution. In this case we call this the *trivial solution*.

6. If we have a solution given in the form $y = y(x)$ then we say that $y(x)$ is an *explicit solution*. But very often it is either difficult or possibly impossible to obtain an explicit solution even if we know that it exists. In this case we may still be able to find a so-called *implicit solution*. We define an implicit solution as follows:

   > A relation $G(x, y) = 0$ is called an implicit solution of the ODE (1) if by repeated implicit differentiation of $G(x, y) = 0$ with respect to x and algebraic simplification we can arrive at (1).

   In the first order case we will say that $G(x, y) = 0$ is an implicit solution of $y' = F(x, y)$ if when we differentiate $G(x, y) = 0$ implicitly with respect to x and solve for y' we obtain $y' = F(x, y)$. Here is an example. We claim an implicit solution of $y' = -x/y$ is $G(x, y) = x^2 + y^2 - 1 = 0$. Notice in this case we cannot solve the equation $G(x, y) = 0$ for a single function $y = f(x)$ since we arrive at $y = \pm\sqrt{1 - x^2}$ which is not a function. But if we differentiate $x^2 + y^2 - 1 = 0$ with respect to x we arrive at $2x + 2yy' = 0$ which when we solve for y' gives $y' = -x/y$.

7. Generally speaking an nth order ODE has an n-parameter family of solutions. That is to say that the solution depends on n arbitrary constants (constants of integration). Thus we would write an implicit solution as $G(x, y, c_1, c_2, \cdots, c_n) = 0$ where $c_1, c_2, \cdots, c_n$ are arbitrary parameters. If we can find an implicit or explicit solution containing n arbitrary parameters then we call the solution the *general solution*. Here is an example. Consider the differential equation $y'' = 0$. we have not yet introduced methods to solve a Differential Equation (DE) but from calculus we know that if we integrate both sides with respect to x we get $y' = C_1$ where C_1 is an arbitrary constant. Then we can integrate both sides again with respect to x and we arrive at $y = C_1x+C_2$ where C_2 is another arbitrary constant. We claim that the $y = C_1x + C_2$ is a general solution of $y'' = 0$.

   For example, a general explicit solution of $y^{(3)} = 0$ is $y = c_1 + c_2x + c_3x^2$.

Examples from Section 1.1

1. Given a set of equations and a set of solutions find the best answer (most complete answer) matching the equation with the solution.

	A. $y = e^{3x}$
1. ☐ $y'' + y = 0$	B. $y = \sin(x)$
2. ☐ $y' + 3y = 0$	C. $y = e^{-3x}$
3. ☐ $y' - 3y = 0$	D. $y = e^{3x},\ y = e^{-3x}$
4. ☐ $y'' - 9y = 0$	E. $y = \cos(x)$
	F. $y = \sin(x)$ and $y = \cos(x)$

   We need to match the best answer on the right with the equation on the left. You do this by trial and error. For example, if we differentiate $y = e^{3x}$ we get $y' = 3e^{3x} = 3y$ or $y' - 3y = 0$ which is exactly equation 4. You can do this a bit more systematically as follows (notice for parts with more than one function you need to consider taking two derivatives and for sines and cosines you must also differentiate twice)

   A. $y = e^{3x} \Rightarrow y' = 3e^{3x} = 3y$ or $y' - 3y = 0$.

B. $y = \sin(x) \Rightarrow y' = \cos(x) \Rightarrow y'' = -\sin(x) = -y \Rightarrow y'' + y = 0$ which is equation is equation 1. But see answer D below.

C. $y = e^{-3x} \Rightarrow y' = -3e^{-3x} = -3y$ or $y' + 3y = 0$ which is exactly equation 3.

D. $y = e^{3x},\ y = e^{-3x}$ In this case we have (as above) $y = e^{3x}$ satisfies $y' - 3y = 0$ and similarly $y = e^{-3x}$ satisfies $y' + 3y = 0$ but if we differentiate both of these twice we see they both satisfy $y'' - 9y = 0$ which is equation 5.

E. $y = \cos(x) \Rightarrow y' = -\sin(x) \Rightarrow y'' = -\cos(x) = -y \Rightarrow y'' + y = 0$ which is equation is equation 1. So both B and D satisfy equation 1.

F. $y = \sin(x)$ and $y = \cos(x)$ The best answer for equation 1 is E since both functions satisfy the equation.

So we have the best answer

1. $\boxed{\text{E}}$ $y'' + y = 0$
2. $\boxed{\text{C}}$ $y' + 3y = 0$
3. $\boxed{\text{A}}$ $y' - 3y = 0$
4. $\boxed{\text{D}}$ $y'' - 9y = 0$

2. Consider the problem of finding all solutions in the form $y = e^{mx}$ of $y'' - 3y' + 2y = 0$.

   Substituting $y = e^{mx}$, $y' = me^{mx}$ and $y'' = m^2e^{mx}$ into the equation we have

$$m^2e^{mx} - 3me^{mx} + 2e^{mx} = 0 \quad \Rightarrow \quad m^2 - 3m + 2 = 0.$$

   We can easily solve this quadratic equation to obtain $m = 1,\ m = 2$. So we have solutions $y = e^{x}$ and $y = e^{2x}$.

1.2 Material for Section 1.2

The *Initial Value Problem* (IVP) is to find a unique solution of (1) which we rewrite here

$$F(x, y, y^{(1)}, \cdots, y^{(n)}) = 0$$

satisfying the n constraints (called *initial conditions*)

$$y(x_0) = y_1, \quad y'(x_0) = y_2, \quad \cdots, \quad y^{(n-1)}(x_0) = y_n. \tag{3}$$

Remark 1.1. In order to solve the IVP we first find a general solution which depends on n parameters $c_1, c_2, \cdots, c_n$. We then apply the n constraints (3) to obtain a system of n equations in n unknowns to evaluate the constants c_j, $j = 1, \cdots, n$ and thus to obtain a unique solution.

Example 1.1. In the example worked above we found two solutions of $y'' - 3y' + 2y = 0$ given by e^x and e^{2x}. We claim that $y = c_1e^x + c_2e^{2x}$ is a general solution. This will be done in Chapter 3 but for now let us assume that it is a general solution.

Let us finish this example by solving the IVP

$$y'' - 3y' + 2y = 0 \text{ with } y(0) = 0 \text{ and } y'(0) = 1 \ .$$

We have $y = c_1e^x + c_2e^{2x}$ and $y'(x) = c_1e^x + 2c_2e^{2x}$. Substituting $x = 0$ into the formulas for y and y' and using the initial conditions $y(0) = 0$ and $y'(0) = 1$ we have

$$0 = y(0) = c_1 + c_2 \text{ and } 1 = y'(0) = c_1 + 2c_2.$$

So we have $\begin{cases} c_1 + c_2 = 0 \\ c_1 + 2c_2 = 1 \end{cases} \Rightarrow c_1 = -1 \quad c_2 = 1$

Therefore the unique solution of the IVP is $y(x) = e^{2x} - e^x$.

Example 1.2. By simple integration we see that the general solution to the differential equation $y' = 3x^2$ is $y(x) = x^3 + c$ where c is an arbitrary constant of integration. The unique solution satisfying the initial condition $y(2) = -1$ is $y(x) = x^3 - 9$.

Two very important questions that arise in studying the solution of an IVP is (1) do solutions exist and (2) are they unique. The answer to this question leads us to the statement for a first order IVP of *The Fundamental Existence and Uniqueness Theorem*.

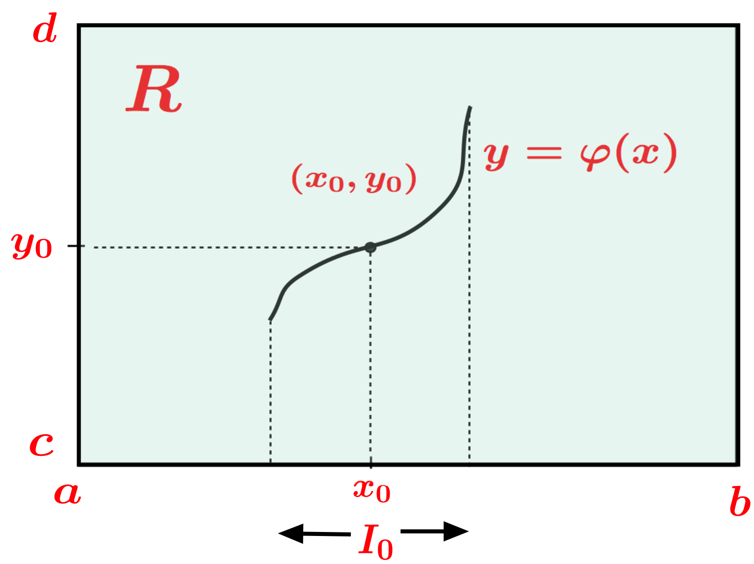
In this section we will consider first order equations and, for the most part, we are interested in those equations that can be put in the normal form

$$y' = f(x, y) \tag{4}$$

and the associated IVP

$$y' = f(x, y), \quad y(x_0) = y_0. \tag{5}$$

Theorem 1.1 (Fundamental Existence Uniqueness Theorem (FEUT))**.** *Let R be a rectangular region $R = \{(x, y) : a \leq x \leq b, \quad c \leq y \leq d\}$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous in R, then there exists an interval $I_0 = (x_0 - h, x + h) \subset (a, b)$ for a number $h > 0$, and a unique function $y = \varphi(x)$ defined on I_0 that solves the IVP (5) on I_0.*



As an example let us consider the IVP $y' = -y^2$, $y(0) = 1$. Here $f(x, y) = -y^2$, $f_y(x, y) = -2y$ are both continuous functions everywhere in $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. Therefore Theorem 1.1 implies that the IVP has a unique solution. Indeed, the general solution to the ODE is $y = (c + x)^{-1}$ where c is an arbitrary constant. Then the unique solution to the IVP is $y = (1 + x)^{-1}$. Notice that this solution exists for all $x > -1$. But it does not exist at $x = -1$.

Theorem 1.2 (Fundamental Existence Uniqueness Theorem First Order Linear)**.** *Consider* *the first order linear IVP*

$$a_1(x)y' + a_0(x)y = g(x), \quad y(x_0) = y_0, \tag{6}$$

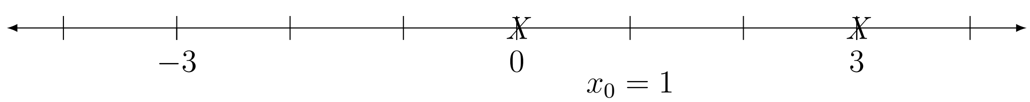
on an interval $I = (a, b) = \{x : a < x < b\}$ *with* $x_0 \in I$*. If the functions* $a_1(x)$*,* $a_0(x)$ *and* $g(x)$ *are all continuous on* I *and* $a_1(x) \neq 0$ *on all of* I*, then* (6) *has a unique soltuion that exists on the whole interval* I*.*

Here is an example

Example 1.3. *Consider the first order linear IVP*

$$(x^2 - 9)y' + x\cos(x)y = \frac{x+1}{x} \text{ with } y(1) = 5.$$

In this case $a_1(x) = (x^2 - 9)$*,* $a_0(x) = x\cos(x)$ *and* $g(x) = \dfrac{x+1}{x}$*. The functions* a_1 *and* a_0 *are continuous on the whole real line but* $g(x)$ *has a discontinuity at* $x = 0$*. Also the leading coefficient* $a_1(x) = 0$ *at both* $x = \pm 3$*. The number line is then broken into* 4 *parts:* $-\infty < x < -3$*,* $-3 < x < 0$*,* $0 < x < 3$ *, and* $3 < x < \infty$*. Since the initial point* $x_0 = 1$ *we see that the solution is guaranteed to exist on the interval* $0 < x < 3$*.*



Examples from Section 1.2

1. Given the IVP $y' = -2xy^2$ with $y(0) = 1$ and given the general solution $y(x) = \dfrac{1}{x^2 + C}$ find the unique solution to the IVP.

$$1 = y(0) = \frac{1}{0^2 + C} = \frac{1}{C} \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad y(x) = \frac{1}{x^2 + 1}.$$

2. Given the IVP $y'' + 16y = 0$ with $y(\pi/2) = -2$ and $y'(\pi/2) = 4$ and given the general solution $y(x) = C_1\cos(4x) + C_2\sin(4x)$ find the unique solution to the IVP. In this

case we have $y'(x) = -4C_1 \sin(4x) + 4C_2 \cos(4x)$ so the pair of equations obtained by setting $x = \pi/2$ are

$$-2 = C_1 \cos(4\ \pi/2) + C_2 \sin(4\ \pi/2) = C_1 \cos(2\pi) + C_2 \sin(2\pi) = C_1$$
$$4 = -4C_1 \sin(4\ \pi/2) + 4C_2 \cos(4\ \pi/2) = -4C_1 \sin(2\pi) + 4C_2 \cos(2\pi) = 4C_2$$

Therefore $C_1 = -2$ and $C_2 = 1$ and the solution is $y(x) = -2\cos(4x) + \sin(4x)$

3. Given the equation $y' = y^{2/3}$ find all (x_0, y_0) in the plane for which the FEUT guarantees the existence of a unique solution in a neighborhood of x_0.

   Here we have $F(x, y) = y^{2/3}$ in the FEUT. Thus we have $F_y = (2/3)y^{-1/3}$. Notice that since $F(x, y)$ only depends on y all values of x_0 are okay. Also we see that $F(x, y) = y^{2/3}$ is continuous for all y. But $F_y = (2/3)y^{-1/3}$ does not exist for $y = 0$ (why?). But it is continuous for all $y > 0$ and all $y < 0$. So the answer is

   The IVP has a unique solution in a neighborhood of any (x_0, y_0) in the plane for which x_0 is arbitrary and $y_0 > 0$ or $y_0 < 0$.

4. Given the equation $xy' = y$ find all (x_0, y_0) in the plane for which the FEUT guarantees the existence of a unique solution in a neighborhood of x_0.

   Here we have $F(x, y) = y/x$ (here we divided both sides by x) in the FEUT. Thus we have $F_y = 1/x$. Notice that $F(x, y)$ depends on both x and y all values of y_0 are okay since it is continuous for all y as long as $x \neq 0$. The same is true of $F_y = 1/x$ which does not exist for $x = 0$ (why?). But it is continuous for all $x > 0$ and all $x < 0$ for all y. So the answer is

   The IVP has a unique solution in a neighborhood of any (x_0, y_0) in the plane for which y_0 is arbitrary and $x_0 > 0$ or $x_0 < 0$.

5. Let us reconsider the previous example considered as a first order linear IVP. The equation can be written as $xy' - y = 0$ which is in the form $a_1(x)y' + a_0(x)y = 0$ (see equation (2)) so the leading coefficient is $a_1 = x$ and the other coefficients are both constants, $a_0 = -1$ and $g(x) = 0$. Since all the coefficients are continuous

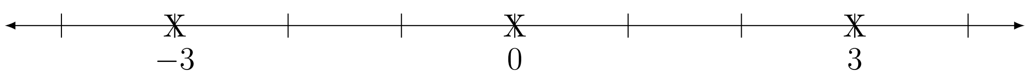
everywhere we only need to consider where the leading coefficient can be equal to zero. $a_1(x) = x = 0$ when $x = 0$ so we can only choose $x_0 > 0$ or $x_0 < 0$. Appealing to Theorem 1.2 we see that:

(a) If $x_0 > 0$ and y_0 is arbitrary, a unique solution will exist for all $x > 0$.

(b) If $x_0 < 0$ and y_0 is arbitrary, a unique solution will exist for all $x < 0$.

We note that the solution of this problem is $y = x$ which actually exists for all x. Notice that Theorem 1.2 is much stronger than Theorem 1.1.

6. Given the linear IVP $(x^2 - 9)y' + \frac{1}{x}y = \cos(x)$ find the interval of existence guaranteed by Theorem 1.2. The coefficients $a_1(x) = (x^2 - 9)$ and $g(x) = \cos(x)$ are continuous functions for all real numbers. The function $a_0(x) = 1/x$ is continuous for all $x \neq 0$. Finally, the leading coefficient $a_1(x) = (x^2 - 9) = 0$ when $x = \pm 3$. So we must exclude $x = 0, -3, 3$



Using this number line we can give the interval of existence determined by the initial condition $y(x_0) = y_0$

(a) $y(-5) = 17$ solution exists on $-\infty < x < -3$

(b) $y(-1) = 12$ solution exists on $-3 < x < 0$

(c) $y(2) = -17$ solution exists on $0 < x < 3$

(d) $y(7) = 22$ solution exists on $3 < x < \infty$

7. Given the linear IVP $(x^2 - 9)y' + x\cos(x)y = \frac{(x+1)}{x}$ and IC $y(1) = 7$ find the interval of existence guaranteed by Theorem 1.2. The coefficients $a_1(x) = (x^2 - 9)$ and $a_0(x) = x\cos(x)$ are continuous functions for all real numbers. The function $a_0(x) = (x+1)/x$ is continuous for all $x \neq 0$. Finally, the leading coefficient $a_1(x) = (x^2 - 9) = 0$ when $x = \pm 3$. So we must exclude $x = 0, -3, 3$. So we have the same number line picture as above. And, if $x_0 = 1$ then the solution exits on the interval $0 < x < 3$.

2 Chapter 2

Reading assignment: Your need to study Chapter 2, Sections 2.1 through 2.5.

2.1 Autonomous Equations

The first order autonomous equations $y' = f(y)$ are particularly interesting and the behavior of solutions can be described rather nicely even without solving the equation.

Autonomous Equations

In contrast to the general case above for which it can be very difficult to find explicit solutions, there is a class of problems for which it is possible to obtain a somewhat more detailed description of the behavior of the solutions without actually finding the solution. These are the first order autonomous equations $y' = f(y)$ (notice the independent variable x does not appear on the right hand side) which are particularly interesting and the behavior of solutions can be described rather nicely even without solving the equation.

In particular, it is possible to visualize the qualitative properties of the solution using the so called *Phase Line* which consists of a one dimensional plot of the critical points (also called the equilibria points or nodes) together with arrows depicting whether a solution is increasing or decreasing.

We begin our analysis by looking for solutions of the equation that are very simple. Namely we look for what are called *equilibrium* solutions which are solutions that don't depend on the independent variable, in other words they are constants. If a solution is a constant then it is of the form $y(x) = y_0$ where y_0 is a real number. But then we see that $y'(x) = 0$ for all x so the left side of the equation is zero, which in turn means the right hand side must be zero. In other words we need $f(y_0) = 0$. These real numbers are the equilibrium points (also called fixed point, critical points or nodes).

Qualitative information about the equilibrium points of the differential equation $y' = f(y)$ can be obtained from special diagrams called phase diagrams.

A phase line diagram for the autonomous equation $y' = f(y)$ is a line segment with labels for all the nodes, i.e., one for each root of $f(y) = 0$, i.e. each equilibrium.

We can classify the possible behavior of a solution that begins in between two nodes according to a simple classification of the three types of nodes: Asymptotically Stable (AS), Unstable (US) or Semi-Stable (SS).

1. An equilibrium y_0 is called *Asymptotically Stable* if solutions that begin near y_0 at $x = 0$ approach y_0 as $x \to \infty$. Such an equilibrium is also called a *sink* because it attracts nearby solutions, i.e., if $y(0)$ is close to y_1, then $|y(x) - y_0| \xrightarrow{x\to\infty} 0$. In other words $y(x)$ moves toward y_1.

2. An equilibrium y_0 is called *Unstable* if solutions that begin near y_0 at $x = 0$ move away from y_0 as $x \to \infty$. Such an equilibrium y_0 is said to repel nearby solutions and it is called a source, i.e., if $y(0)$ is close to y_0, then $|y(x) - y_0|$ increases as $x \to \infty$. In other words $y(x)$ moves away from y_0.

3. An equilibrium y_0 which is neither a sink or a source is called a *Semi-Stable* node. In this case solutions that begin near y_0 on one side (either greater than or less than y_0) will approach y_0 on one side but move away on the other side.

Classification of Nodes

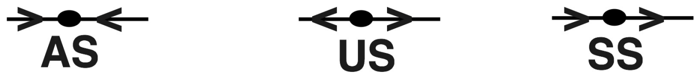
In order to determine where a node is AS, US or SS we only need to determine the sign of $f(y)$ for y near the node.

1. AS: In this case we must have $f(y) > 0$ for $y < y_0$ and $f(y) < 0$ for $y > y_0$. Here $f(y) > 0$ indicates that $y'(x) = f(y(x)) > 0$ which means that $y(x)$ is increasing (or moving to the right on the number line). And, $y'(x) = f(y(x)) < 0$ means that $y(x)$ is decreasing (or moving to the left on the number line).

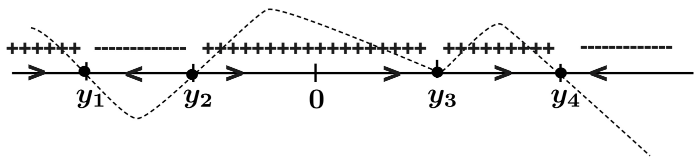
2. US: In this case we must have $f(y) < 0$ for $y < y_1$ and $f(y) > 0$ for $y > y_1$.

3. SS: In this case there are two possibilities: we could have $f(y) < 0$ for $y < y_2$ and $f(y) < 0$ for $y > y_2$. Or we could have, $f(y) > 0$ for $y < y_2$ and $f(y) > 0$ for $y > y_2$. In other words the sign of $f(y)$ is the same on both sides of y_2.

The following figure gives the important distinctions using arrows indicating whether the solution y is increasing $>$ or decreasing $<$.



Suppose for example that $y' = f(y)$ and $f(y) = 0$ has four nodes $y_1 < y_2 < y_3 < y_4$ and suppose when we plot these values on a phase line and sketch $f(y)$ we arrive at the following diagram



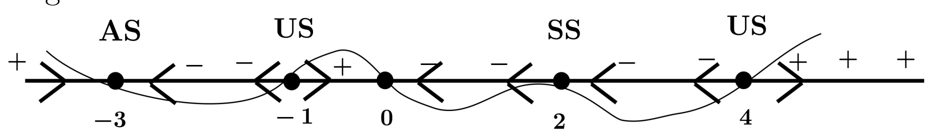
Notice that y_1 is AS, y_2 is US, y_3 is SS and y_4 is AS. We usually give the answer as a pair as

$$(y_1, AS), (y_2, US), (y_3, SS), (y_4, AS).$$

Example 2.1. Let us consider an example given by

$$y' = y(y+1)(y-2)^2(y+3)^3(y-4)^2.$$

So we have $f(y) = y(y+1)(y-2)^2(y+3)^3(y-4)$ which has critical points $y = -3, -1, 0, 2, 4$ and we want to classify each of the nodes. So we draw a phase line on which we plot all these critical points and indicate, using a simple sketch with $+$ and $-$ signs of where $f(y)$ is positive and negative.

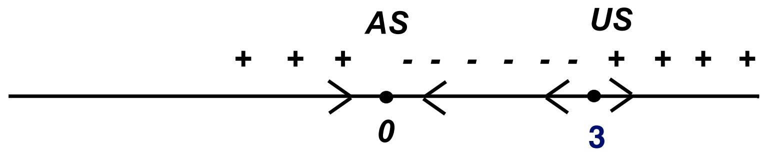


Finally then we have $(-3, AS), (-1, US), (0, AS), (2, SS), (4, US)$.

Some Examples of Autonomous Equations

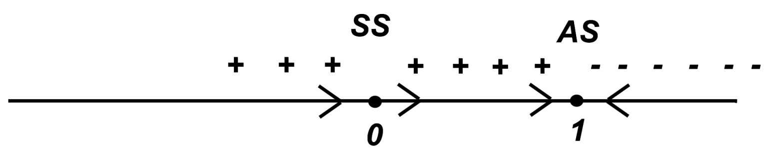
In these examples we consider an autonomous equation in the form $y' = f(y)$ where y is a function of the independent variable x (Note that the independent variable could be anything, for example it could be t). So the equation could read $x' = f(x)$. In our examples we will stick with the examples as they are in the book and use $y = y(x)$.

1. Given the equation $y' = y^2 - 3y$ we have $f(y) = y^2 - 3y = 0$ implies that $y = 0, 3$



So we see that $y = 0$ is an Asymptotically Stable node (sink) and $y = 3$ is an Unstable node (source).

2. Given the equation $y' = y^2 - y^3$ we have $f(y) = y^2 - y^3 = y^2(1 - y) = 0$ implies that $y = 0, 1$



So we see that $y = 0$ is a Semi-Stable node and $y = 1$ is an Asymptotically Stable node (sink).

2.2 Separable Equations

A very important class of problem (autonomous and nonautonomous) are ones that can be "separated." These are problems that can be written in the form

$$\frac{dy}{dx} = f(y)g(x).$$

In this case we can rewrite the problem in the form

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x)$$

or

$$\frac{1}{f(y)}\,\frac{dy}{dx} = g(x).$$

Integrating both sides we arrive at

$$\int \frac{1}{f(y)}\,\frac{dy}{dx}\,dx = \int g(x)\,dx + C$$

or

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx + C$$

and we say the problem is solved "up to quadrature." So to solve the problem we must evaluate the indefinite integrals.

This shows already how hard solving differential equations are since it is easy to write down functions that cannot be integrated exactly in terms of elementary functions. For example,

$$y' = e^{-x^2}$$

has solution which can be expressed as

$$y(x) = \int_0^x e^{-s^2}\,ds + C$$

but it is well known that this integral cannot be expressed "in closed form."

In spite of this example, the simplest class of separable first order equations are ones in the form $y' = f(x)$ which can be written in the separated form $dy = f(x)\,dx$. Therefore by the Fundamental Theorem of Calculus $y = \int f(x)\,dx + C$.

You will be required to make substitutions many times when doing integrals. Suppose, for example, you want to integrate $\int e^{3x}dx$ then you need to use the substitution

$$u = 3x \;\; => \;\; du = 3dx$$

and the integral becomes

$$1/3\int e^u du = 1/3e^u + C = 1/3e^{3x} + C.$$

More generally let us replace $\int e^{3x}dx$ by $\int e^{kx}dx$ and use the same type substitution

$$u = kx \;\; => \;\; du = kdx$$

so the integral becomes

$$1/k \int e^u du = 1/k e^u + C = 1/k e^{kx} + C.$$

Remark 2.1. Suppose in general you need to integrate $\int f(kx)dx$ and you know an antiderivative for $f(x)$ is $F(x)$, i.e., $F'(x) = f(x)$, then using the substitution $u = kx$ would give

$$\int f(kx)dx = 1/kF(kx) + C.$$

We will use this idea maybe even a hundred times this semester so I would learn it. Just remember this simple fact and you won't have to make all these trivial substitutions.

Example 2.2. *Consider the differential equation $y' = (x+1)\cos(x^2+2x)$. We separate the equation and integrate to find the solution.*

$$dy = (x+1)\cos(x^2+2x)dx$$

so that

$$\int dy = \int (x+1)\cos(x^2+2x)dx + C.$$

By the power rule the integral of dy is y so we have

$$y = \int (x+1)\cos(x^2+2x)dx + C.$$

To carry out the integral on the right we need to use a simple substitution $u = x^2 + 2x$ which implies $du = 2(x+1)dx$ and we have

$$y = \int (x+1)\cos(x^2+2x)dx + C = \frac{1}{2}\int \cos(u)\, du + C = \frac{1}{2}\sin(x^2+2x) + C.$$

Example 2.3. *Consider the differential equation $y' = -2xy^2$. We separate the dependent and independent variables and integrate to find the solution.*

$$\frac{dy}{dx} = -2xy^2$$

$$\begin{aligned}
&y^{-2}\,dy = -2x\,dx\\
&\int y^{-2}\,dy = -2\int x\,dx + c\\
&-y^{-1} = -x^2 + c\\
&y = \frac{1}{x^2 - c}
\end{aligned}$$

But since c is an arbitrary constant we can just as easily write $y = \dfrac{1}{x^2 + c}$.

Example 2.4. *The equation $y' = y - y^2$ is separable and we find*

$$\frac{dy}{y - y^2} = dx$$

On the left we expand in partial fractions and integrate.

$$\begin{aligned}
&\int \left(\frac{1}{y} - \frac{1}{y-1}\right) dy = \int dx + c\\
&\ln|y| - \ln|y-1| = x + c
\end{aligned}$$

This is an implicit solution for $y(x)$. We can solve for $y(x)$ to obtain an explicit solution as follows.

$$\begin{aligned}
&\ln\left|\frac{y}{y-1}\right| = x + c \quad \Rightarrow \quad \left|\frac{y}{y-1}\right| = e^{x+c} \Rightarrow \quad \frac{y}{y-1} = \pm e^{x+c}\\
&\frac{y}{y-1} = c_1 e^x \quad \textit{(Here we have substituted } c_1 \textit{ for } \pm e^c.)\\
&y = \frac{-c_1 e^x}{c_1 e^x - 1} \quad \textit{(which is the same as)}\\
&y = \frac{1}{1 + 1/c_1 e^{-x}} \quad \textit{(divided top and bottom by } -c_1 \textit{ and } e^x.)\\
&y = \frac{1}{1 + Ce^{-x}} \quad \textit{(finally we set } C = 1/c_1.)
\end{aligned}$$

Here are a list of some separable equations with partial details.

Supplemental Separable Equations

1. $y' = \dfrac{1 - x^2}{y^2} \quad \Rightarrow \quad y^2 dy = (1 - x^2)dx \quad \Rightarrow \quad y = (3x - x^3 + k)^{1/3}$

2. $y' = 3yx^2 \quad \Rightarrow \quad \dfrac{dy}{y} = 2x^2 dx \quad \Rightarrow \quad y = ke^{x^3}$

3. $xy' = \dfrac{1-y^2}{2y} \quad \Rightarrow \quad \dfrac{2y}{y^2-1} = -\dfrac{dx}{x} \quad \Rightarrow \quad y = \pm(1+kx^{-1})^{1/2}$

4. $y' = \dfrac{\cos^2(y)x}{1+x^2} \quad \Rightarrow \quad \sec^2(y)dy = \dfrac{xdx}{1+x^2} \quad \Rightarrow \quad y = \tan^{-1}(\ln(1+x^2)^{1/2}+k)$

5. $y' = 4x^3(1-y), \quad y(0) = 3 \quad \Rightarrow \quad \dfrac{dy}{1-y} = 4x^3 dx \quad \Rightarrow \quad y = ke^{x^4}+1, \quad y = 2e^{x^4}+1$

6. $y' = 2\sqrt{y+1}\cos(x), \quad y(\pi) = 0 \ \Rightarrow 1/2(y+1)^{-1/2}dy = \cos(x)dx \quad \Rightarrow$
$(y+1)^{1/2} = \sin(x) + C \quad \Rightarrow \quad y = (\sin(x)+k)^2 - 1, \quad y = \sin^2(x) + 2\sin(x)$

7. $y' = \dfrac{3x^2+4x+2}{2y+1}, \quad y(0) = -1 \quad \Rightarrow \quad (2y+1)dy = (3x^2+4x+2)dx \quad \Rightarrow \quad y^2+y =$
$x^3+2x^2+2x+C, \quad y^2+y = x^3+2x^2+2x$

8. $y' = 2x\sin^2(y), \quad y(0) = \pi/4 \quad \Rightarrow \quad \csc^2(y)dy = 2xdx \quad \Rightarrow \quad -\cot(y) = x^2 + C, \quad y =$
$\cot^{-1}(1-x^2)$

9. $\sqrt{1-y^2}dx = \sqrt{1-x^2}dy, \quad y(0) = \dfrac{\sqrt{3}}{2} \quad \Rightarrow \quad \dfrac{dy}{\sqrt{1-y^2}} = \dfrac{dx}{\sqrt{1-x^2}} \quad \Rightarrow \quad \sin^{-1}(y) =$
$\sin^{-1}(x) + C \quad \Rightarrow \quad y = \sin(\sin^{-1}(x) + C \quad \Rightarrow \quad y = \sin(\sin^{-1}(x) - \pi/3)$

10. $\dfrac{dx}{dt} = 4(x^2+1), \quad x(\pi/4) = 1 \quad \Rightarrow \quad \dfrac{dx}{(x^2+1)} = 4dt \quad \Rightarrow \quad \tan^{-1}(x) = 4t + C$
$\Rightarrow \quad x = \tan(4t+C) \quad \Rightarrow \quad x = \tan(4t - 3\pi/4)$

11. $x^2y' = y - xy, \quad y(-1) = -1 \quad \Rightarrow \quad \dfrac{dy}{y} = \dfrac{1-x}{x^2} \quad \Rightarrow \quad \ln(|y|) = (-x^{-1} - \ln(|x|)) + C$
$\Rightarrow \quad -y = e^{-x^{-1}-\ln(|x|)+C} = \dfrac{ke^{x^{-1}}}{-x} \quad \Rightarrow \quad y = \dfrac{e^{-(x^{-1}+1)}}{x}$

2.3 First Order Linear Equations

The first order, linear, non-homogeneous differential equation has the form

$$\frac{dy}{dx} + p(x)y = f(x). \tag{7}$$

This equation is not (in general) separable. To solve Equation (7), we multiply by an *integrating factor.*

The integrating factor for this first order linear equation is

$$\mu(x) = ce^{\int p(x)\,dx}.$$

So to solve the equation (7) we multiply by the integrating factor and integrate. Notice that μ satisfies $\dfrac{d\mu}{dx} = p(x)\mu$ so multiplying the equation by μ we have

$$\mu\frac{dy}{dx} + p(x)\mu y = \mu f(x)$$

or

$$\mu\frac{dy}{dx} + \frac{d\mu}{dx}y = \mu f(x).$$

But by the product rule the left hand side is the exact derivative $[\mu y]'$ so we have

$$[\mu y]' = \mu f(x)$$

and integrating both sides gives

$$\mu y = \int \mu(x) f(x)\,dx + C,$$

or finally

$$y = \frac{1}{\mu}\int \mu(x) f(x)\,dx + \frac{C}{\mu}.$$

Remark 2.2. First Order, Linear Homogeneous Differential Equations. We have just shown that the first order, linear, homogeneous differential equation,

$$\frac{dy}{dx} + p(x)y = 0,$$

has the general solution

$$y = Ce^{-\int p(x)\,dx}. \tag{8}$$

The solution obtained above is exactly $\dfrac{C}{\mu}$. In other words the solution of the non-homogeneous problem always contains the solution of the homogeneous problem.

A very big point here is that the *general solution* of a non-homogeneous first order linear problem is the sum of two terms, i.e., the general solution is $y = y_p + y_c$ where y_p is called a *particular solution* and it satisfies $y' + p(x)y = f(x)$, and the general solution of the *homogeneous problem*, y_c, that satisfies $y' + p(x)y = 0$. In the book y_c is called the *complementary solution*.

Let us show that y_p satisfies the non-homogeneous problem, where

$$y_p = e^{-\int p\,dx}\left(\int e^{\int p\,dx} f(x)\,dx\right).$$

We have by the product rule and the chain rule

$$\begin{aligned} y_p' &= -pe^{-\int p\,dx}\left(\int e^{\int p\,dx} f(x)\,dx\right) + e^{-\int p\,dx}\left(e^{\int p\,dx} f(x)\right) \\ &= -py_p + f(x), \end{aligned}$$

and therefore

$$y_p' + y_p = f(x).$$

Example 2.5. Find the general solution of

$$y' + \frac{1}{x}y = 4x^2.$$

First we find the integrating factor.

$$\mu(x) = \exp\left(\int \frac{1}{x}\,dx\right) = e^{\ln x} = x$$

We multiply by the integrating factor

$$\frac{dy}{dx}(xy) = 4x^3$$

and integrate

$$xy = x^4 + C$$

Dividing by the integrating factor we have

$$y = x^3 + \frac{C}{x}.$$

The particular and complementary solutions are

$$y_p = x^3 \qquad \text{and} \qquad y_c = \frac{C}{x}.$$

Note that the general solution to the differential equation must contain an arbitrary constant.

Example 2.6. Solve the IVP $y' + 3x^2 y = 6x^2, \quad y(0) = 1 \Rightarrow \quad \mu = e^{\int 3x^2\,dx} = e^{x^3}$. Thus we have

$$\left[e^{x^3} y\right]' = 6x^2 e^{x^3}$$

which implies

$$e^{x^3} y = 6\int x^2 e^{x^3}\,dx + C = 2e^{x^3} + C.$$

So we have $e^{x^3} y = 2e^{x^3} + C$ and we can apply the IC $y(0) = 1$ to find C. Namely,

$$1 = 2 + C \quad \Rightarrow \quad C = -1$$

Therefore

$$y = 2 - e^{-x^3}.$$

Example 2.7. Solve the iVP $xy' - y = x^2 \sin(x)$ with $y(\pi) = 0$. First we must write the equation as $y' - \frac{1}{x} y = x\sin(x)$ from which we find $\mu = e^{-\int dx/x} = e^{-\ln(x)} = x^{-1}$. So we find

$$\left[x^{-1} y\right]' = \sin(x)$$

which implies

$$x^{-1} y = \int \sin(x)\,dx + C = -\cos(x) + C$$

and applying the IC $y(\pi) = 0$ we have

$$0 = 1 + C \quad \rightarrow \quad C = -1$$

so the unique solution is

$$y = -x(\cos(x) + 1).$$

Remark 2.3 (Integration by Parts Formula)**.** The next problem requires the use of integration by parts. You can do integration by parts any way you know how but I will show the way I do it which is the way I will do it in class. First let me derive the integration by parts formula from the product rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x) \quad \Rightarrow \quad f'(x)g(x) = [f(x)g(x)]' - f(x)g'(x)$$

Next we integrate both sides and use the fundamental theorem of calculus (which gives $\int [f(x)g(x)]'\,dx = f(x)g(x)$) to obtain the integration by parts formula for indefinite integrals

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx \tag{9}$$

For a definite integral the above formula becomes

$$\int_a^b f'(x)g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x)g'(x)\,dx.$$

The way you use it is to write one of the given functions as a derivative and then apply the above formula. Here are some examples:

1. Evaluate $\int x\ln(x)\,dx$

$$\begin{aligned}\int x\ln(x)\,dx &= \int \left(\frac{x^2}{2}\right)' \ln(x)\,dx \\ &= \frac{x^2}{2}\ln(x) - \int \frac{x^2}{2}\frac{1}{x}dx \\ &= \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x dx \\ &= \frac{x^2}{2}\ln(x) - \frac{1}{4}x^2 + C\end{aligned}$$

2. Evaluate $\displaystyle\int xe^x\,dx$

$$\begin{aligned}\int xe^x\,dx &= \int x\left(e^x\right)'\,dx\\ &= xe^x - \int e^x\,dx\\ &= xe^x - e^x + C\end{aligned}$$

3. Evaluate $\displaystyle\int x\cos(x)\,dx$

$$\begin{aligned}\int x\cos(x)\,dx &= \int x\left(\sin(x)\right)'\,dx\\ &= x\sin(x) - \int \sin(x)\,dx\\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

Example 2.8. $(x+1)y' + y = \ln(x)$ with $y(1) = 1$. The equation must be written as $y' + \dfrac{1}{(x+1)}y = \dfrac{\ln(x)}{(x+1)}$ from which we find $\mu = e^{\int dx/(x+1)} = e^{\ln(x+1)} = (x+1)$. So we have

$$[(x+1)y]' = \ln(x)$$

which implies (notice to integrate $\ln(x)$ we need integration by parts)

$$\int \ln(x)\,dx = \int (x)'\ln(x)\,dx = x\ln(x) - \int x\left(\frac{1}{x}\right)\,dx = x\ln(x) - x.$$

So we have

$$(x+1)y = \int \ln(x)\,dx + C = (x\ln(x) - x) + C.$$

At this point we evaluate the constant C using $y(1) = 1$

$$(1+1)\times 1 = (0-1) + C \quad\Rightarrow\quad C = 3.$$

and therefore

$$y = \frac{(x\ln(x) - x + 3)}{(x+1)}.$$

In the next two examples we consider problems with a right hand side defined in pieces.

Example 2.9. $y' + 2y = \begin{cases} 2, & 0 \le x < 3 \\ 0, & x \ge 3 \end{cases}$ with $y(0) = 0$. We need to solve two separate problems in a certain order. First we solve $y' + 2y = 2$ with $y(0) = 0$ on the interval $0 \le x < 3$. For this problem the integrating factor is $\mu = e^{2x}$ so we have $[e^{2x}y]' = 2e^{2x}$ which, after integration, implies $e^{2x}y = e^{2x} + C$. The initial condition gives $0 = 1 + C$ so $C = -1$. This gives $y = 1 - e^{-2x}$ for all $0 \le x < 3$. Now the continuity of the solution then requires that

$$y(3) = \lim_{x\to 3} y(x) = \left(1 - e^{-6}\right).$$

This now becomes the initial condition for the second problem:

$$y' + 2y = 0 \quad \text{with} \quad y(3) = \left(1 - e^{-6}\right) \quad \text{for} \quad x \ge 3.$$

Solving this problem we get (notice we have the same integrating factor) $[e^{2x}y]' = 0$ so $e^{2x}y = C$ and the initial condition implies $C = e^6\left(1 - e^{-6}\right) = \left(e^6 - 1\right)$. Finally then we get

$$y = \left(e^6 - 1\right)e^{-2x} \qquad \text{for } x \ge 3.$$

Putting these solutions together we have

$$y(x) = \begin{cases} \left(1 - e^{-2x}\right), & 0 \le x < 3 \\ \left(e^6 - 1\right)e^{-2x}, & x \ge 3 \end{cases}$$

Example 2.10. $y' + 2xy = \begin{cases} 2x, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$ with $y(0) = 2$. We need to solve two separate problems in a certain order. First we solve $y' + 2xy = 2x$ with $y(0) = 2$ on the interval $0 \le x < 1$. For this problem the integrating factor is $\mu = e^{x^2}$ so we have $[e^{x^2}y]' = 2xe^{x^2}$

which, after integration, implies $e^{x^2} y = e^{x^2} + C$. The initial condition gives $2 = 1 + C$ so $C = 1$. This gives $y = 1 + e^{-x^2}$ for all $0 \le x < 1$. Now the continuity of the solution then requires that

$$y(1) = \lim_{x\to 1} y(x) = \left(1 + e^{-1}\right).$$

Then we need to solve

$$y' + 2xy = 0 \quad \text{with} \quad y(1) = \left(1 + e^{-1}\right) \quad \text{for} \ \ x \ge 1.$$

Solving this problem we get (notice we have the same integrating factor) $[e^{x^2} y]' = 0$ so $e^{x^2} y = C$ and the initial condition implies $C = e^1 \left(1 + e^{-1}\right) = \left(e^6 + 1\right)$. Finally then we get

$$y = (e + 1)\, e^{-x^2} \qquad \text{for } x \ge 1.$$

Putting these solutions together we have

$$y(x) = \begin{cases} \left(1 + e^{-x^2}\right), & 0 \le x < 1 \\ (e + 1)\, e^{-x^2}, & x \ge 1 \end{cases}$$

2.4 Exact Equations

Any first order ordinary differential equation of the first degree can be written (in infinitely many ways) in differential form,

$$M(x, y)\, dx + N(x, y)\, dy = 0.$$

We know from Calculus III that if $F(x, y)$ is a function satisfying

$$dF = M\, dx + N\, dy = 0,$$

then this equation is called *exact* and an (implicit) general solution of the differential equation is

$$F(x, y) = c,$$

where c is an arbitrary constant. Since the differential of a function, $F(x, y)$, is

$$dF \equiv \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy,$$

M and N are the partial derivatives of F:

$$M(x, y) = \frac{\partial F}{\partial x}, \qquad N(x, y) = \frac{\partial F}{\partial y}.$$

A necessary and sufficient condition for exactness. Consider the ODE written in differential form,

$$M\, dx + N dy = 0.$$

A necessary and sufficient condition for exactness is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

What this means is that if the equation is exact then we are guaranteed there exists a function $F(x, y)$ so that

$$M = \frac{\partial F}{\partial x} \quad \text{and} \quad N = \frac{\partial F}{\partial y} \tag{10}$$

and therefore a general solution to the problem can be written as $F(x, y) = 0$.

More importantly (10) provides a method for finding $F(x, y)$. Namely, integrating the first equation of we see that

$$F(x, y) = \int M(\xi, y)\, d\xi + f(y),$$

for some $f(y)$.

If we differentiate this equation with respect to y and use the the second equation in (10)

$$\frac{\partial F}{\partial y} = N(x, y)$$

we arrive at an equation

$$\int M_y(x,y)\ dx + f'(y) = \frac{\partial F}{\partial y} = N(x,y).$$

It can be shown that, after simplifying, we will arrive at an equation only in y for which we can find $f(y)$ by integration.

Let us consider a simple example

Example 2.11.

$$xdx + ydy = 0.$$

Here $M = x$ and $N = y$ so

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x,y)$ so that $F_x = M = x$ and $F_y = N = y$.

Integrating $F_x = M$ with respect to x we have

$$F(x,y) = \int x\, dx = \frac{x^2}{2} + h(y).$$

Differentiating with respect to y and using $F_y = N$ we have

$$y = N = F_y = h'(y) \quad \Rightarrow \quad h'(y) = y.$$

Now, integrating we obtain

$$h(y) = \frac{y^2}{2}$$

and we find

$$F(x,y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

The implicit general solution is

$$\frac{x^2}{2} + \frac{y^2}{2} = c$$

which, by renaming the constant c can be written as

$$x^2 + y^2 = c.$$

Example 2.12.

$$(2x - 1)dx + (3y + 7)dy = 0.$$

Here $M = (2x - 1)$ and $N = (3y + 7)$ so

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x, y)$ so that $F_x = M = 2x - 1$ and $F_y = N = 3y + 7$.

Integrating $F_x = M = (2x - 1)$ with respect to x we have

$$F(x, y) = \int (2x - 1)\, dx = x^2 - x + h(y).$$

Differentiating with respect to y and using $F_y = N$ we have

$$(3y + 7) = N = F_y = h'(y) \quad \Rightarrow \quad h'(y) = 3y + 7.$$

Now, integrating we obtain

$$h(y) = \frac{3y^2}{2} + 7y$$

and we find

$$F(x, y) = x^2 - x + \frac{3y^2}{2} + 7y.$$

The implicit general solution is

$$x^2 - x + \frac{3y^2}{2} + 7y = c.$$

Example 2.13.

$$(2xy^2 - 3)dx + (2x^2y + 4)dy = 0.$$

Here $M = (2xy^2 - 3)$ and $N = (2x^2y + 4)$ so

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x, y)$ so that $F_x = M = (2xy^2 - 3)$ and $F_y = N = (2x^2y + 4)$.

Integrating $F_x = M = (2xy^2 - 3)$ with respect to x we have

$$F(x, y) = \int (2xy^2 - 3)\, dx = x^2y^2 - 3x + h(y).$$

Differentiating with respect to y and using $F_y = N$ we have

$$(2x^2y + 4) = N = F_y = 2x^2y + h'(y) \quad \Rightarrow \quad h'(y) = 4.$$

Now, integrating we obtain

$$h(y) = 4y$$

and we find

$$F(x, y) = x^2y^2 - 3x + 4y.$$

The implicit general solution is

$$x^2y^2 - 3x + 4y = c.$$

Example 2.14.

$$(x^2 - y^2)dx + (y^2 + 2xy)dy = 0.$$

Here $M = (x^2 - y^2)$ and $N = (y^2 + 2xy)$ so

$$\frac{\partial M}{\partial y} = -2y \neq 2y = \frac{\partial N}{\partial x}$$

so the equation is not exact.

Example 2.15.

$$(2x - y^3 + y^2 \sin(x))dx - (3xy^2 + 2y \cos(x))dy = 0.$$

Here $M = (2x - y^3 + y^2\sin(x))$ and $N = -(3xy^2 + 2y\cos(x))$ so

$$\frac{\partial M}{\partial y} = -3y^2 + 2y\sin(x) = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x,y)$ so that $F_x = M = (2x - y^3 + y^2\sin(x))$ and $F_y = N = -(3xy^2 + 2y\cos(x))$.

Integrating $F_x = M = (2x - y^3 + y^2\sin(x))$ with respect to x we have

$$F(x,y) = \int (2x - y^3 + y^2\sin(x))\, dx = x^2 - xy^3 - y^2\cos(x) + h(y).$$

Differentiating with respect to y and using $F_y = N$ we have

$$-(3xy^2 + 2y\cos(x)) = N = F_y = -3xy^2 - 2y\cos(x) + h'(y) \quad \Rightarrow \quad h'(y) = 0.$$

Now, integrating we obtain

$$h(y) = 0$$

and we find

$$F(x,y) = x^2 - xy^3 - y^2\cos(x).$$

The implicit general solution is

$$x^2 - xy^3 - y^2\cos(x) = c.$$

Example 2.16.

$$(y\ln(y) - e^{-xy})dx + (y^{-1} + x\ln(y))dy = 0.$$

Here $M = (y\ln(y) - e^{-xy})$ and $N = (y^{-1} + x\ln(y))$ so

$$\frac{\partial M}{\partial y} = \ln(y) + 1 + xe^{-xy} \neq \ln(y) = \frac{\partial N}{\partial x}$$

so the equation is not exact.

Example 2.17.

$$(2xe^x - y + 6x^2)dx - xdy = 0.$$

Here $M = (2xe^x - y + 6x^2)$ and $N = -x$ so

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x, y)$ so that $F_x = M = (2xe^x - y + 6x^2)$ and $F_y = N = -x$.

Integrating $F_y = N = -x$ with respect to y we have

$$F(x, y) = \int (-x)\, dy = -xy + h(x).$$

Differentiating with respect to x and using $F_x = (2xe^x - y + 6x^2)$ we have

$$(2xe^x - y + 6x^2) = M = F_x = -y + h'(x) \quad \Rightarrow \quad h'(x) = (2xe^x + 6x^2).$$

Now, integrating we obtain

$$h(y) = \int (2xe^x + 6x^2)\, dx$$

The first integral on the right requires integration by parts

$$\int xe^x\, dx = \int x\,(e^x)'\, dx = xe^x - \int e^x\, dx = xe^x - e^x$$

So we have

$$h(y) = 2xe^x - 2e^x + 2x^3$$

and we find

$$F(x, y) = -xy + 2xe^x - 2e^x + 2x^3.$$

The implicit general solution is

$$-xy + 2xe^x - 2e^x + 2x^3 = c.$$

Example 2.18.

$$(2x - y^3 + y^2\sin(x))dx - (3xy^2 + 2y\cos(x))dy = 0.$$

Here $M = (2x - y^3 + y^2\sin(x))$ and $N = -(3xy^2 + 2y\cos(x))$ so

$$\frac{\partial M}{\partial y} = -3y^2 + 2y\sin(x) = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x, y)$ so that $F_x = M = (2x - y^3 + y^2\sin(x))$ and $F_y = N = -(3xy^2 + 2y\cos(x))$.

Integrating $F_x = M = (2x - y^3 + y^2\sin(x))$ with respect to x we have

$$F(x, y) = \int (2x - y^3 + y^2\sin(x))\,dx = x^2 - xy^3 - y^2\cos(x) + h(y).$$

Differentiating with respect to y and using $F_y = N$ we have

$$-(3xy^2 + 2y\cos(x)) = N = F_y = -3xy^2 - 2y\cos(x) + h'(y) \quad \Rightarrow \quad h'(y) = 0.$$

Now, integrating we obtain

$$h(y) = 0$$

and we find

$$F(x, y) = x^2 - xy^3 - y^2\cos(x).$$

The implicit general solution is

$$x^2 - xy^3 - y^2\cos(x) = c.$$

Example 2.19.

$$(\tan(x) - \sin(x)\sin(y))dx + (\cos(x)\cos(y))dy = 0.$$

Here $M = (\tan(x) - \sin(x)\sin(y))$ and $N = (\cos(x)\cos(y))$ so

$$\frac{\partial M}{\partial y} = -\sin(x)\cos(y) = \frac{\partial N}{\partial x}$$

so the equation is exact and we know there exists a function $F(x, y)$ so that $F_x = M = (\tan(x) - \sin(x)\sin(y))$ and $F_y = N = (\cos(x)\cos(y))$.

Integrating $F_y = N = (\cos(x)\cos(y))$ with respect to y we have

$$F(x, y) = \int (\cos(x)\cos(y))\, dy = \cos(x)\sin(y) + h(x).$$

Differentiating with respect to x and using $F_x = -\sin(x)\sin(y)$ we have

$$(\tan(x) - \sin(x)\sin(y)) = M = F_x = -\sin(x)\cos(y) + h'(x) \quad \Rightarrow \quad h'(x) = \tan(x).$$

Now, integrating we obtain

$$h(y) = \int \tan(x)\, dx$$

The first integral on the right requires a special substitution. Writing $\tan(x) = \sin(x)/\cos(x)$ we set $u = \cos(x)$ which implies $du = -\sin(x)\, dx$ and the integral becomes

$$\int \tan(x)\, dx = -\int \frac{du}{u} = -\ln(|u|) = -\ln(|\cos(x)|)$$

So we have

$$h(y) = -\ln(|\cos(x)|)$$

and we find

$$F(x, y) = \cos(x)\sin(y) - \ln(|\cos(x)|).$$

The implicit general solution is

$$\cos(x)\sin(y) - \ln(|\cos(x)|) = c.$$

Integrating Factors

Consider a differential equation written in differential form

$$\widetilde{M}\,dx + \widetilde{N}\,dy = 0.$$

When this equation is not exact it can sometimes be made exact using an integrating factor. The idea is this, it may be that $\widetilde{M}_y \neq \widetilde{N}_x$ but we can look for a function μ to multiply times the equation so that the resulting equation

$$\mu\widetilde{M}\,dx + \mu\widetilde{N}\,dy = 0$$

is exact, i.e., Defining $M = \mu\widetilde{M}$ and $N = \mu\widetilde{N}$ we obtain an exact equation $M\,dx + N\,dy = 0$.

The problem of finding such an integrating factor can be very difficult and we will only investigate two possible scenarios. We may look for an integrating factor that is a function of x alone, i.e., $\mu = \mu(x)$. To test the new equation for exactness we would seek $\mu(x)$ so that

$$(\mu(x)\widetilde{M})_y - (\mu(x)\widetilde{N})_x = \mu(x)\widetilde{M}_y - \mu(x)\widetilde{N}_x - \mu'(x)\widetilde{N} = 0$$

This can be rewritten as

$$\mu'(x) - \left(\frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{N}}\right)\mu(x) = 0.$$

Assuming that

$$\left(\frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{N}}\right) \equiv f(x)$$

is a function of x alone then denoting it by $f(x)$ we obtain a first order linear equation

$$\mu'(x) - f(x)\mu(x) = 0,$$

with solution

$$\mu(x) = C\exp\left(\int f(x)\,dx\right).$$

Thus we obtain the following

$$\text{If} \quad f(x) = \left(\frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{N}} \right) \quad \Rightarrow \quad \mu(x) = \exp\left(\int f(x)\, dx \right) \tag{11}$$

In a similar way we could ask if there is an integrating factor as a function of y alone. To test the new equation for exactness we would seek $\mu(y)$ so that

$$(\mu(y)\widetilde{M} - (\mu(y)\widetilde{N})_x = \mu'(y)\widetilde{M} + \mu(y)\widetilde{M})_y - \mu(y)\widetilde{N}_x = 0$$

This can be rewritten as

$$\mu'(y) + \left(\frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} \right) \mu(y) = 0.$$

Assuming that

$$\left(\frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} \right)$$

is a function of y alone then denoting it by $g(y)$ we obtain a first order linear equation

$$\mu'(y) + g(y)\mu(y) = 0,$$

with solution

$$\mu(y) = C \exp\left(- \int g(y)\, dy \right).$$

Thus we obtain the following

$$\text{If} \quad g(y) = \left(\frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} \right) \quad \Rightarrow \quad \mu(y) = \exp\left(- \int g(y)\, dy \right) \tag{12}$$

Example 2.20. Consider $y\,dx - x\,dy = 0$. We have $\widetilde{M} = y$ and $\widetilde{N} = -x$ and

$$\widetilde{M}_y - \widetilde{N}_x = 1 + 1 = 2.$$

If we divide by $\widetilde{N}$ we obtain a function of x alone

$$f(x) = \frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{N}} = \frac{2}{-x}.$$

We obtain

$$\mu(x) = e^{\int f(x)\,dx} = e^{\int -2dx/x} = e^{-2\ln(x)} = x^{-2}.$$

Multiplying the equation through by $\mu(x)$ we have

$$yx^{-2}\,dx - x^{-1}\,dy = 0$$

so $M = yx^{-2}$ and $N = -x^{-1}$ and this equation is exact since

$$M_y = x^{-2} = N_x.$$

If we divide by $\widetilde{M}$ we obtain a function of y alone

$$g(y) = \frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} = \frac{2}{y}.$$

We obtain

$$\mu(y) = e^{-\int g(y)\,dy} = e^{\int -2dy/y} = e^{-2\ln(y)} = y^{-2}.$$

Multiplying the equation through by μ we have

$$y^{-1}\,dx - xy^{-2}\,dy = 0$$

so $M = y^{-1}$ and $N = -xy^{-2}$ and this equation is exact since

$$M_y = -y^{-2} = N_x.$$

Example 2.21. Consider $(2y^2+3x)\,dx+2xy\,dy = 0$. We have $\widetilde{M} = (2y^2+3x)$ and $\widetilde{N} = 2xy$ and

$$\widetilde{M}_y - \widetilde{N}_x = 4y - 2y = 2y.$$

If we divide by $\widetilde{N}$ we obtain a function of x alone

$$f(x) = \frac{\widetilde{M_y} - \widetilde{N}_x}{\widetilde{N}} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = \frac{1}{x}.$$

We obtain

$$\mu = e^{\int f(x)\, dx} = e^{\int dx/x} = e^{\ln(x)} = x.$$

Multiplying the equation through by μ we have

$$(2xy^2 + 3x^2)\, dx + 2x^2 y\, dy = 0$$

so $M = (2xy^2 + 3x^2)$ and $N = 2x^2 y$ and this equation is exact since

$$M_y = 4xy = N_x.$$

Example 2.22. Consider $(2x + yx^{-1})\, dx + (xy - 1)\, dy = 0$. We have $\widetilde{M} = (2x + yx^{-1})$ and $\widetilde{N} = (xy - 1)$ and

$$\widetilde{M_y} - \widetilde{N}_x = x^{-1} - y.$$

If we divide by $\widetilde{N}$ we obtain a function of x alone

$$f(x) = \frac{\widetilde{M_y} - \widetilde{N}_x}{\widetilde{N}} = \frac{x^{-1} - y}{xy - 1} = -\frac{1}{x}\,\frac{xy - 1}{xy - 1} = -\frac{1}{x}.$$

We obtain

$$\mu = e^{\int f(x)\, dx} = e^{-\int dx/x} = e^{-\ln(x)} = x^{-1}.$$

Multiplying the equation through by μ we have

$$(2 + yx^{-2})\, dx + (y - x^{-1})\, dy = 0$$

so $M = (2 + yx^{-2})$ and $N = (y - x^{-1})$ and this equation is exact since

$$M_y = x^{-2} = N_x.$$

Example 2.23. Consider $(y^2+2xy)\,dx - x^2, dy = 0$. We have $\widetilde{M} = (y^2+2xy)$ and $\widetilde{N} = -x^2$ and

$$\widetilde{M}_y - \widetilde{N}_x = (2y+2x) - (-2x) = 2y + 4x.$$

If we divide by $\widetilde{M}$ we obtain a function of y alone

$$g(y) = \frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} = \frac{2y+4x}{(y^2+2xy)} = \frac{2}{y}\,\frac{(y+2x)}{(y+2x)} = \frac{2}{y}.$$

We obtain

$$\mu = e^{-\int g(y)\,dy} = e^{-2\int dy/y} = e^{-2\ln(y)} = y^{-2}.$$

Multiplying the equation through by μ we have

$$(1+2xy^{-1})\,dx - (x^2y^{-2})\,dy = 0$$

so $M = (1+2xy^{-1})$ and $N = -(x^2y^{-2})$ and this equation is exact since

$$M_y = -2xy^{-2} = N_x.$$

Example 2.24. Consider $(xy)\,dx + (2x^2+3y^2-20), dy = 0$. We have $\widetilde{M} = (xy)$ and $\widetilde{N} = (2x^2+3y^2-20)$ and

$$\widetilde{M}_y - \widetilde{N}_x = (x) - (4x) = -3x.$$

If we divide by $\widetilde{M}$ we obtain a function of y alone

$$g(y) = \frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} = \frac{-3x}{(xy)} = \frac{-3}{y}.$$

We obtain

$$\mu = e^{-\int g(y)\,dy} = e^{3\int dy/y} = e^{3\ln(y)} = y^3.$$

Multiplying the equation through by μ we have

$$(xy^4)\,dx + (2x^2y^3 + 3y^5 - 20y^3)\,dy = 0$$

so $M = (xy^4)$ and $N = (2x^2y^3 + 3y^5 - 20y^3)$ and this equation is exact since

$$M_y = 4xy^3 = N_x.$$

Example 2.25. Consider $(x + y)\sin(y)\,dx + (x\sin(y) + \cos(y)), dy = 0$. We have $\widetilde{M} = (x + y)\sin(y)$ and $\widetilde{N} = (x\sin(y) + \cos(y))$ and

$$\widetilde{M}_y - \widetilde{N}_x = (\sin(y) + (x + y)\cos(y)) - \sin(y) = (x + y)\cos(y).$$

If we divide by $\widetilde{M}$ we obtain a function of y alone

$$g(y) = \frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{M}} = \frac{(x + y)\cos(y)}{(x + y)\sin(y)} = \frac{\cos(y)}{\sin(y)}.$$

We obtain

$$\mu = e^{-\int g(y)\,dy} = e^{-\int \cos(y)/\sin(y)\,dy} = e^{-\int du/u} = e^{-\ln(u)} = u^{-1} = \frac{1}{\sin(y)}.$$

Multiplying the equation through by μ we have

$$(x + y)\,dx + (x + \cot(y))\,dy = 0$$

so $M = (x + y)$ and $N = (x + \cot(y))$ and this equation is exact since

$$M_y = 1 = N_x.$$

Example 2.26. Solve the IVP $(3e^x y + 4)\,dx + e^x\,dy = 0$ with $y(0) = 0$. First let us find a general solution.

We have $\widetilde{M} = (3e^x y + 4)$ and $\widetilde{N} = e^x$ and

$$\widetilde{M}_y - \widetilde{N}_x = 3e^x - e^x = 2e^x.$$

If we divide by $\widetilde{N}$ we obtain a function of x alone

$$f(x) = \frac{\widetilde{M}_y - \widetilde{N}_x}{\widetilde{N}} = \frac{2e^x}{e^x} = 2.$$

We obtain

$$\mu = e^{\int f(x)\,dx} = e^{\int 2dx} = e^{2x}.$$

Multiplying the equation through by μ we have

$$(3e^{3x}y + 4e^{2x})\,dx + e^{3x}\,dy = 0$$

so $M = (3e^{3x}y + 4e^{2x})$ and $N = e^{3x}$ and this equation is exact since

$$M_y = 3e^{3x} = N_x.$$

So now we need to find the general solution of the exact equation $(3e^{3x}y + 4e^{2x})\,dx + e^{3x}\,dy = 0$ $F_x = (3e^{3x}y + 4e^{2x})$ and $F_y = e^{3x}$ Integrating F_y with respect to y gives

$$F = \int F_y\,dy = \int e^{3x}\,dy = ye^{3x} + h(x).$$

Differentiating this with respect to x and using $F_x = (3e^{3x}y + 4e^{2x})$ we have

$$(3e^{3x}y + 4e^{2x}) = F_x = 3ye^{3x} + h'(x)$$

so

$$h'(x) = 4e^{2x}, \quad \Rightarrow \quad h(x) = \int 4e^{2x}\,dx = 2e^{2x}.$$

Therefore a general solution is

$$ye^{3x} + 2e^{2x} = C.$$

We now apply the initial condition $y(0) = 0$ to obtain $C = 2$ and we have $ye^{3x} + 2e^{2x} = 2$ which is an implicit solution. Finally then we can obtain an explicit solution by solving for

y

$$y = 2e^{-x} - 2e^{-3x}.$$

Example 2.27. Find an integrating factor in the form $\mu = x^n y^m$ for the equation

$$(2y^2 - 6xy)\,dx + (3xy - 4x^2)\,dy = 0 \tag{13}$$

So we have $\widetilde{M} = (2y^2 - 6xy)$, $\widetilde{N} = (3xy - 4x^2)$ and we seek an integrating factor μ. This means we need $(\mu\widetilde{M})_y = (\mu\widetilde{N})_x$ or

$$\frac{\partial[x^n y^m(2y^2 - 6xy)]}{\partial y} = \frac{\partial[x^n y^m(3xy - 4x^2)]}{\partial x}$$

Simplifying this a bit by collecting the x and y terms we obtain

$$\frac{\partial(2x^n y^{m+2} - 6x^{n+1}y^{m+1})}{\partial y} = \frac{\partial(3x^{n+1}y^{m+1} - 4x^{n+2}y^m)}{\partial x}$$

or

$$2(m+2)x^n y^{m+1} - 6(m+1)x^{n+1}y^m = 3(n+1)x^n y^{m+1} - 4(n+2)x^{n+1}y^m.$$

Now on each side we have powers of the form $x^n y^{m+1}$ and $x^{n+1}y^m$ and the corresponding coefficients of these terms on each side must be the same so that implies

$$2(m+2) = 3(n+1) \qquad \text{and} \qquad -6(m+1) = -4(n+2)$$

or

$$\begin{aligned} 3n - 2m &= 1 \\ 2n - 3m &= -1 \end{aligned}$$

To solve this system system of equations you could multiply the first equation by $-2/3$ and add it to the second equation (so that the n's cancel) to get

$$\begin{aligned} 3n - 2m &= 1 \\ -5/3m &= -5/3 \end{aligned}$$

From the second equation we get $m = 1$ and then the first equation becomes $3n - 2 = 1$ which implies $n = 1$ so we obtain an integrating factor $\mu = xy$.

Lets check to make sure it works. Multiplying the differential equation (13) by μ we get

$$(2xy^3 - 6x^2y^2)\, dx + (3x^2y^2 - 4x^3y)\, dy = 0.$$

So $M = (2xy^3 - 6x^2y^2)$ and $N = (3x^2y^2 - 4x^3y)$ and we have

$$M_y = (6xy^2 - 12x^2y) \quad \text{and} \quad N_x = (6xy^2 - 12x^2y)$$

and since $M_y = N_x$ the equation is exact.

2.5 Substitutions

2.5.1 Homogeneous Equations

Consider a first order equation

$$\frac{dy}{dx} = f(x, y).$$

We say that this equation is *Homogeneous* if the right hand side can be written in the form

$$\frac{dy}{dx} = F(y/x). \tag{14}$$

If this is possible then we can reduce the equation to a separable equation using the change of dependent variable

$$u = y/x.$$

Note that $u = y/x$ implies $y = xu$ which, in turn, implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Therefore the equation (14) can be written as

$$u + x\frac{du}{dx} = F(u)$$

which is separable. Namely, we can write

$$\frac{du}{F(u)-u} = \frac{dx}{x}$$

and we obtain an implicit solution by integration

$$\int \frac{du}{F(u)-u} = \int \frac{dx}{x} + C.$$

Example 2.28. Consider the problem

$$x\frac{dy}{dx} = \frac{y^2}{x} + y, \quad y(1) = 1.$$

Dividing by x we obtain

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right).$$

Proceeding as above, we set $u = y/x$ implies $y = xu$ which, in turn, implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Thus the equation becomes

$$u + x\frac{du}{dx} = u^2 + u.$$

Now we simplify by subtracting the u on both sides and separate the variables to obtain

$$\frac{du}{u^2} = \frac{dx}{x}.$$

Integrating both sides we have

$$-\frac{1}{u} = \ln|x| + C.$$

So we get

$$u = \frac{-1}{\ln|x| + C}$$

or

$$\frac{y}{x} = \frac{-1}{\ln|x| + C}$$

From the IC we have

$$1 = \frac{-1}{\ln|1| + C} = \frac{-1}{C} \quad \Rightarrow \quad C = -1.,$$

which gives

$$\frac{y}{x} = \frac{-1}{\ln|x| - 1}$$

and finally

$$y = \frac{x}{1 - \ln|x|}$$

Example 2.29. Consider the problem $(x - y)\,dx + x\,dy = 0$ which can be written as

$$x\frac{dy}{dx} = \frac{y - x}{x} = \frac{y}{x} - 1.$$

Set $u = y/x$ implies $y = xu$ which implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Thus the equation becomes

$$u + x\frac{du}{dx} = u - 1.$$

Now we simplify by subtracting the u on both sides and separate the variables to obtain

$$du = -\frac{dx}{x}.$$

Integrating both sides we have

$$u = -\ln|x| + C.$$

So we get

$$y = x(C - \ln|x|)$$

Example 2.30. Consider the problem

$$\frac{dy}{dx} = \frac{y - x}{y + x}.$$

Dividing the top and bottom by x we have

$$\frac{dy}{dx} = \frac{(y/x) - 1}{(y/x) + 1}.$$

Set $u = y/x$ implies $y = xu$ which implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Thus the equation becomes

$$u + x\frac{du}{dx} = \frac{u-1}{u+1}.$$

Now we subtract the u from both sides

$$x\frac{du}{dx} = \frac{u-1}{u+1} - u = \frac{(u-1) - u(u+1)}{u+1} = \frac{-(u^2+1)}{u+1}.$$

Separate variables to get

$$\frac{(u+1)du}{u^2+1} = -\frac{dx}{x}.$$

Integrating both sides we have

$$\int \frac{u\,du}{u^2+1} + \int \frac{du}{u^2+1} = -\ln|x| + C.$$

For the first integral use the substitution $w = u^2 + 1$ which implies $dw = 2udu$ and the integral becomes

$$\int \frac{u\,du}{u^2+1} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln(|w|) = \frac{1}{2}\ln(u^2+1) = \ln((y/x)^2+1)^{1/2}$$

and

$$\int \frac{du}{u^2+1} = \tan^{-1}(u) = \tan^{-1}(y/x).$$

Combining these results we have

$$\ln((y/x)^2+1)^{1/2} + \tan^{-1}(y/x) = -\ln|x| + C.$$

Example 2.31. Consider the similar problem

$$\frac{dy}{dx} = \frac{y+x}{y-x}.$$

Dividing the top and bottom by x we have

$$\frac{dy}{dx} = \frac{(y/x)+1}{(y/x)-1}.$$

Set $u = y/x$ implies $y = xu$ which implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Thus the equation becomes

$$u + x\frac{du}{dx} = \frac{u+1}{u-1}.$$

Now we subtract the u from both sides

$$x\frac{du}{dx} = \frac{u+1}{u-1} - u = \frac{(u+1) - u(u-1)}{u-1} = \frac{-(u^2-2u+1)}{u-1}.$$

Separate variables to get

$$\frac{(u-1)du}{u^2-2u+1} = -\frac{dx}{x}.$$

Integrating both sides we have

$$\int \frac{(u-1)\,du}{u^2-2u+1} = -\ln|x| + C.$$

For the integral on the left use the substitution $w = u^2-2u+1$ which implies $dw = 2(u-1)du$ and the integral becomes

$$\int \frac{(u-1)\,du}{u^2+1} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln(|w|) = \frac{1}{2}\ln(u^2-2u+1) = \ln((y/x)^2 - 2(y/x)+1)^{1/2}$$

Combining these results we have

$$\ln((y/x)^2 - 2(y/x) + 1)^{1/2} = -\ln|x| + C.$$

Example 2.32. Consider the IVP

$$\frac{dy}{dx} = \frac{x + ye^{y/x}}{xe^{y/x}}, \quad y(1) = 0.$$

Dividing the top and bottom by x we have

$$\frac{dy}{dx} = \frac{1 + (y/x)e^{y/x}}{e^{y/x}}.$$

Set $u = y/x$ implies $y = xu$ which implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Thus the equation becomes

$$u + x\frac{du}{dx} = \frac{1 + ue^u}{e^u} = e^{-u} + u.$$

Now we subtract the u from both sides

$$x\frac{du}{dx} = e^{-u}.$$

Separate variables to get

$$e^u\, du = \frac{dx}{x}.$$

Integrating both sides we have

$$e^u = \ln(|x|) + C.$$

Applying the IC we get

$$1 = 0 + C \quad \Rightarrow \quad C = 1.$$

Combining these results we have

$$e^{y/x} = \ln(|x|) + 1.$$

Take logarithm of both sides to get

$$\frac{y}{x} = \ln(\ln(|x|) + 1)$$

and finally

$$y = x \ln(\ln(|x|) + 1).$$

Example 2.33. Consider the IVP

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}, \quad y(1) = 1.$$

Dividing the bottom into the top

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{y}{x}\right) - \left(\frac{x}{y}\right)\right].$$

Set $u = y/x$ implies $y = xu$ which implies

$$\frac{dy}{dx} = u + x\frac{du}{dx}.$$

Thus the equation becomes

$$u + x\frac{du}{dx} = \frac{1}{2}\left[u - \frac{1}{u}\right].$$

Now we subtract the u from both sides

$$x\frac{du}{dx} = -\frac{1}{2}\left[u + \frac{1}{u}\right] = -\frac{1}{2}\left[\frac{u^2+1}{u}\right].$$

Separate variables to get

$$\frac{2u\,du}{u^2+1} = -\frac{dx}{x}.$$

Integrating both sides we have

$$\int \frac{2u\,du}{u^2+1} = -\ln(|x|) + C.$$

If we set $w = u^2 + 1$ then $dw = 2udu$ and the integral on the left becomes

$$\int \frac{2u\,du}{u^2+1} = \int \frac{dw}{w} = \ln(|w|)$$

so we have

$$\ln(u^2+1) = -\ln(|x|) + C$$

or

$$\ln((y/x)^2+1) = -\ln(|x|) + C$$

Applying the IC we get

$$\ln(2) = 0 + C \quad \Rightarrow \quad C = \ln(2).$$

Combining these results we have (notice $x = 1 > 0$ which implies $\ln(|x|) = \ln(x)$)

$$\ln((y/x)^2+1) = -\ln(|x|) + \ln(2) = \ln(2/x).$$

Exponentiate both sides to get

$$(y/x)^2 + 1 = 2/x$$

and the multiply by x^2,

$$y^2 + x^2 = 2x$$

move x^2 to the right and take the square root

$$y = \sqrt{2x - x^2}.$$

Notice we take the positive square root since (from the IC) $y > 0$.

2.5.2 Bernoulli equations

A Bernoulli equation is any equation that can be written in the form

$$y' + p(x)y = f(x)y^n, \qquad n \neq 0, 1. \tag{15}$$

Note that if $n = 0, 1$ then the equation is first order linear.

The Bernoulli equation can be reduced to a first order linear equation using the following substitution:

$$u = y^{1-n}.$$

This relation implies

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Multiplying the Bernoulli equation by $(1-n)y^{-n}$ we see that it can be written as

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)p(x)y^{1-n} = (1-n)f(x)y^n y^{-n}$$

which simplifies to

$$u' + (1-n)p(x)u = (1-n)f(x), \tag{16}$$

which is first order linear and can be solved using the methods discussed in the Section on first order linear problems.

Example 2.34. Consider the problem

$$y' + y = y^4.$$

This is a Bernoulli equation with $n = 4$ so $(1-n) = -3$ and we set $u = y^{-3}$. Then we obtain the linear equation

$$u' - 3u = -3.$$

To solve this problem we find the integrating factor

$$\mu = e^{-3\int dx} = e^{-3x}.$$

Multiplying by μ we obtain

$$\left(e^{-3x}u\right)' = -3e^{-3x}.$$

Next we integrate to obtain

$$\int \left(e^{-3x}u\right)'\,dx = \int -3e^{-3x}\,dx$$

which gives

$$e^{-3x}u = e^{-3x} + C$$

or

$$u(x) = 1 + Ce^{3x}.$$

Finally, converting back to y we have

$$y(x)^{-3} = 1 + Ce^{3x}.$$

Example 2.35. Consider the problem

$$xy' + y = y^{-2} \text{ with } y(1) = 2.$$

First we note the equation is not in the correct form and we must divide by x to get

$$y' + \frac{1}{x}y = x^{-1}y^{-2}.$$

This is a Bernoulli equation with $n = -2$ so $(1-n) = 3$ and we set $u = y^3$. Then we obtain the linear equation

$$u' + \frac{3}{x}u = \frac{3}{x}.$$

To solve this problem we find the integrating factor

$$\mu = e^{3\int dx/x} = e^{3\ln(x)} = x^3.$$

Multiplying by μ we obtain

$$\big(x^3u\big)' = 3x^2.$$

Next we integrate to obtain

$$\int \big(x^3u\big)\ dx = \int 3x^2, dx = x^3 + C$$

which gives

$$x^3u = x^3 + C$$

or

$$x^3y^3 = x^3 + C$$

Applying the IC we have

$$8 = 1 + C \quad \Rightarrow \quad C = 7$$

$$x^3y^3 = x^3 + 7$$

Dividing by x^3 and taking the cube root of both sides gives

$$y(x) = (1 + 7/x^3)^{1/3}.$$

Example 2.36. Consider the problem

$$xy' + y = y^2 \text{ with } y(1) = \frac{1}{2}.$$

First we note the equation is not in the correct form and we must divide by x to get

$$y' + \frac{1}{x}y = x^{-1}y^2.$$

This is a Bernoulli equation with $n = 2$ so $(1-n) = -1$ and we set $u = y^{-1}$. Then we obtain the linear equation

$$u' - \frac{1}{x}u = -\frac{1}{x}.$$

To solve this problem we find the integrating factor

$$\mu = e^{-\int dx/x} = e^{-\ln(x)} = x^{-1}.$$

Multiplying by μ we obtain

$$\left(x^{-1}u\right)' = -x^{-2}.$$

Next we integrate to obtain

$$\int \left(x^{-1}u\right)\, dx = -\int x^{-2}, dx = x^{-1} + C$$

which gives

$$x^{-1}u = x^{-1} + C$$

or

$$x^{-1}y^{-1} = x^{-1} + C$$

Applying the IC we have

$$2 = 1 + C \quad \Rightarrow \quad C = 1$$

$$x^{-1}y^{-1} = x^{-1} + 1$$

Multiply by x and raising both sides to the (-1) power we have

$$y(x) = \frac{1}{x+1}.$$

Example 2.37. Consider the problem

$$y' - \frac{3}{x}y = 9x^4y^{1/3} \text{ with } y(1) = 8.$$

This is a Bernoulli equation with $n = 1/3$ so $(1-n) = 2/3$ and we set $u = y^{2/3}$. Then we obtain the linear equation

$$u' - \frac{2}{x}u = 6x^4.$$

To solve this problem we find the integrating factor

$$\mu = e^{-2\int dx/x} = e^{-2\ln(x)} = x^{-2}.$$

Multiplying by μ we obtain

$$\left(x^{-2}u\right)' = 6x^2.$$

Next we integrate to obtain

$$\int \left(x^{-2}u\right)\, dx = 6\int x^2, dx = 2x^3 + C$$

which gives

$$x^{-2}u = 2x^3 + C$$

or

$$x^{-2}y^{2/3} = 2x^3 + C$$

Applying the IC we have

$$4 = 2 + C \quad \Rightarrow \quad C = 2$$

$$x^{-2}y^{2/3} = 2x^3 + 2$$

Multiply by x^2 and raising both sides to the $(3/2)$ power we have

$$y(x) = (2x^5 + 2x^2)^{3/2}.$$

Example 2.38. Consider the problem

$$y' + \frac{4}{x}y = 16x^{-4}y^{-3/4} \text{ with } y(1) = 1.$$

This is a Bernoulli equation with $n = -3/4$ so $(1-n) = 7/4$ and we set $u = y^{7/4}$. Then we obtain the linear equation

$$u' + \frac{7}{x}u = 28x^{-4}.$$

To solve this problem we find the integrating factor

$$\mu = e^{7\int dx/x} = e^{7\ln(x)} = x^7.$$

Multiplying by μ we obtain

$$\left(x^7 u\right)' = 28x^3.$$

Next we integrate to obtain

$$\int \left(x^7 u\right)\, dx = 28 \int x^3\, dx = 7x^3 + C$$

which gives

$$x^7 u = 7x^3 + C$$

or

$$x^7 y^{7/4} = 7x^3 + C$$

Applying the IC we have

$$1 = 7 + C \quad \Rightarrow \quad C = -6$$

$$x^7 y^{7/4} = 7x^3 - 6$$

Divide by x^7 and raising both sides to the $(4/7)$ power we have

$$y(x) = \left(\frac{7x^3 - 6}{x^7}\right)^{4/7}.$$

Example 2.39. Consider the problem

$$y' + 2xy = 2xy^2 \text{ with } y(0) = -1.$$

This is a Bernoulli equation with $n = 2$ so $(1-n) = -1$ and we set $u = y^{-1}$. Then we obtain the linear equation

$$u' - 2xu = -2x.$$

To solve this problem we find the integrating factor

$$\mu = e^{-2\int x dx} = e^{-x^2}.$$

Multiplying by μ we obtain

$$\left(e^{-x^2}u\right)' = -2xe^{-x^2}.$$

Next we integrate to obtain

$$\int \left(e^{-x^2}u\right)\, dx = \int -2xe^{-x^2}\, dx = +C$$

We integrate the right hand side using a simple substitution $w = -x^2$ which implies $dw = -2x\, dx$

$$\int -2xe^{-x^2}\, dx = \int e^w\, dw = e^w = e^{-x^2}.$$

which gives

$$e^{-x^2}u = e^{-x^2} + C$$

or

$$e^{-x^2}y^{-1} = e^{-x^2} + C$$

Applying the IC we have

$$-1 = 1 + C \quad \Rightarrow \quad C = -2$$

$$e^{-x^2}y^{-1} = e^{-x^2} - 2$$

Multiply by e^{x^2} and raising both sides to the (-1) power we have

$$y(x) = \left(1 - 2e^{x^2}\right)^{-1}.$$

2.5.3 RHS in the form $f(ax + by + c)$

If for the equation $y' = f(x, y)$ right hand side can be written as function of a linear relation in x and y, i.e., it has the form $f(ax + by + c)$ then a simple substitution transforms the problem into a separable problem. Namely, if we set $v = ax + by + c$ then $v' = a + bf(v)$ which is separable.

Example 2.40. Consider the problem

$$y' = e^{-(x+y)} - 1, \quad \text{with } \ y(0) = 0.$$

We set $v = x + y$ which implies $v' = 1 + y' = 1 + (e^{-v} - 1) = e^{-v}$. The equation $v' = e^{-v}$ is separable and separating the variables we have

$$e^v \frac{dv}{dx} = 1 \quad \Rightarrow \quad e^v \, dv = dx \quad \Rightarrow \quad e^v = x + C \Rightarrow \quad e^{x+y} = x + C.$$

Thus we have an implicit general solution $e^{x+y} = x + C$.

Next we apply the IC to get

$$1 = 0 + C \quad \Rightarrow \quad C = 1$$

so we have

$$e^{x+y} = x + 1.$$

Finally we take the natural log of both sides and subtract x from both sides to get

$$y(x) = \ln(x + y) - x.$$

Example 2.41. Consider the problem

$$y' = (x + y + 1)^2, \quad \text{with } \ y(1) = -2.$$

We set $v = x + y + 1$ which implies $v' = 1 + y' = 1 + v^2$ which is separable and separating the variables we have

$$\int \frac{dv}{v^2 + 1} = \int dx = x + C \quad \Rightarrow \quad \tan^{-1}(v) = x + C.$$

Thus we have an implicit general solution $\tan^{-1}(x+y+1) = x + C$.

Next we apply the IC to get

$$\tan^{-1}(0) = 1 + C \quad \Rightarrow \quad C = -1$$

so we have

$$\tan^{-1}(x+y+1) = x - 1.$$

Finally we take tan of both sides and subtract $x+1$ from both sides to get

$$y(x) = \tan(x-1) - (x+1).$$

Example 2.42. Consider the problem

$$y' = \tan^2(x+y), \quad \text{with} \;\; y(1) = -1.$$

We set $v = x + y$ which implies $v' = 1 + y' = 1 + \tan^2(v)$ which is separable and separating the variables we have

$$\int \frac{dv}{\tan^2(v)+1} = \int dx = x + C.$$

So the main problem is that we need to evaluate the integral

$$\int \frac{dv}{\tan^2(v)+1}$$

Here re recall a main trig identity $\sin^2(\theta) + \cos^2(\theta) = 1$ which, dividing by $\cos^2(\theta)$, gives $\tan^2(\theta) + 1 = \sec^2(\theta)$. So we have

$$\int \frac{dv}{\tan^2(v)+1} = \int \frac{dv}{\sec^2(v)} = \int \cos^2(v)\, dv$$

At this point we need another trig identity (the half angle formula)

$$\cos^2(v) = \frac{1+\cos(2v)}{2}$$

and our integral becomes

$$\int \frac{dv}{\tan^2(v)+1} = \frac{1}{2}\int (1+\cos(2v))\,dv = \frac{1}{2}\left(x + \frac{1}{2}\sin(2v)\right)$$

Thus we have an implicit general solution

$$\frac{1}{2}\left(v + \frac{1}{2}\sin(2v)\right) = x + C$$

or

$$\frac{1}{2}\left(x+y + \frac{1}{2}\sin(2(x+y))\right) = x + C$$

Next we apply the IC to get

$$\frac{1}{2}\left(0 + \frac{1}{2}\sin(0)\right) = 1 + C \quad \Rightarrow \quad C = -1$$

so we have

$$\frac{1}{2}\left(x+y + \frac{1}{2}\sin(2(x+y))\right) = x - 1$$

Solving for y would not be easy so we stop with a little algebraic simplifying

$$\left(x+y + \frac{1}{2}\sin(2(x+y))\right) = 2x - 2$$

or

$$\sin(2(x+y)) = 2(x-y-2).$$

Example 2.43. Consider the problem

$$y' = \cos(x+y), \quad \text{with } \; y(0) = \pi/2.$$

We set $v = x + y$ which implies $v' = 1 + y' = 1 + \cos(v)$ which is separable and separating the variables we have

$$\int \frac{dv}{1+\cos(v)} = \int dx = x + C.$$

So the main problem is that we need to evaluate the integral

$$\int \frac{dv}{1+\cos(v)}$$

Here re recall a main trig identity $\sin^2(\theta)+\cos^2(\theta)=1$ which implies $\sin^2(\theta)=1-\cos^2(\theta)$. So we multiply the numerator and denominator by $1-\cos(v)$ and we have

$$\int \frac{dv}{1+\cos(v)} = \int \frac{(1-\cos(v))}{(1-\cos(v))}\frac{dv}{1+\cos(v)} = \int \frac{(1-\cos(v))\,dv}{\sin^2(v)}$$
$$= \int (\csc^2(v) - \csc(v)\cot(v))\,dv = -\cot(v)+\csc(v)$$

Thus we have an implicit general solution

$$\csc(x+y)-\cot(x+y)=x+C$$

Next we apply the IC to get

$$\csc(\pi/2)-\cot(\pi/2)=0+C$$

We have $\csc(\pi/2)=1$ and $\cot(\pi/2)=0$ so $C=1$ and we have the implicit solution

$$\csc(x+y)-\cot(x+y)=x+1.$$

Example 2.44. Consider the problem

$$y'=\tan(x+y)-1.$$

We set $v=x+y$ which implies $v'=1+y'=1+\tan(v)-1=\tan(v)$ which is separable and separating the variables we have

$$\int \cot(v)\, dv = \int dx = x + C.$$

So the main problem is that we need to evaluate the integral

$$\int \frac{\cos(v)\, dv}{\sin(v)}$$

We use the simple substitution $w = \sin(v)$ which implies $dw = \cos(v)dv$

$$\int \frac{\cos(v)\, dv}{\sin(v)} = \int \frac{dw}{w} = \ln(w).$$

Thus we have an implicit general solution

$$\ln(\sin(x + y)) = x + C$$

Example 2.45. Consider the problem

$$y' = \frac{y - x + 1}{y - x + 2}.$$

There are several choices here for v but we set $v = y - x + 2$ which implies $v' = y' - 1 = \frac{v - 1}{v} - 1 = -\frac{1}{v}$ which is separable and separating the variables we have

$$\int v\, dv = -\int dx = -x + C.$$

So the main problem is that we need to evaluate the integral

$$\int v\, dv = \frac{1}{2}v^2 = \frac{1}{2}(x - y + 2)^2$$

Thus we have an implicit general solution

$$\frac{1}{2}(x - y + 2)^2 = -x + C$$

If we had chose a different value for v, e.g., $v = y - x$ then we would have gotten $v' = y' - 1 = \frac{v+1}{v+2} - 1 = \frac{-1}{v+2}$ which is separable and separating the variables we have

$$\int (v+1)\,dv = -\int dx = -x + C.$$

So the main problem is that we need to evaluate the integral

$$\int (v+2)\,dv = \frac{1}{2}v^2 + 2v = \frac{1}{2}(x-y)^2 + 2(y-x)$$

Thus we have an implicit general solution

$$\frac{1}{2}(x-y)^2 + 2(y-x) = -x + C$$

The two different answers given above only differ by a constant.