Chapter 4 Laplace Transforms

4 Introduction

Reading assignment: In this chapter we will cover Sections 4.1 – 4.5.

4.1 Definition and the Laplace transform of simple functions

Given f, a function of time, with value $f(t)$ at time t, the Laplace transform of f which is denoted by $\mathcal{L}(f)$ (or F) is defined by

$$\mathcal{L}(f)(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad s > 0. \tag{1}$$

Example 4.1. 1. The Laplace transform is linear: $\mathcal{L}(\alpha f(t)+\beta g(t)) = \alpha\mathcal{L}(f(t))+\beta\mathcal{L}(g(t))$

$$\int_0^\infty e^{-st}\,(\alpha f(t) + \beta g(t))\, dt = \alpha \int_0^\infty e^{-st}\, f(t)\, dt + \beta \int_0^\infty e^{-st}\, g(t)\, dt.$$

2. $f(t) = 1 \quad \Rightarrow$

$$F(s) = \int_0^\infty e^{-st}\, 1\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$$

3. $f(t) = e^{at} \quad \Rightarrow \quad$ for $s > a$

$$F(s) = \int_0^\infty e^{-st}\, e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = -\frac{1}{(s-a)} e^{-(s-a)t}\Big|_0^\infty = \frac{1}{(s-a)}$$

4. For a positive integer n, $f(t) = t^n \quad \Rightarrow$

$$\begin{aligned} F(s) = \mathcal{L}(t^n) &= \int_0^\infty e^{-st}\, t^n\, dt = \int_0^\infty \left(\frac{-1}{s} e^{-st}\right)' t^n\, dt \\ &= \left(\frac{-1}{s} e^{-st}\right) t^n \Big|_0^\infty - \left(\frac{-n}{s}\right) \int_0^\infty e^{-st} t^{n-1}\, dt \\ &= \left(\frac{n}{s}\right) \int_0^\infty e^{-st}\, t^{n-1}\, dt = \left(\frac{n}{s}\right) \mathcal{L}(t^{n-1}) \end{aligned}$$

So we find that $\mathcal{L}(t^n) = \left(\frac{n}{s}\right)\mathcal{L}(t^{n-1})$. We can now use this formula over and over to successively reduce the power of t to obtain

$$\mathcal{L}(t^n) = \left(\frac{n}{s}\right)\mathcal{L}(t^{n-1}) = \left(\frac{n(n-1)}{s^2}\right)\mathcal{L}(t^{n-2}) = \cdots = \left(\frac{n!}{s^n}\right)\mathcal{L}(1) = \left(\frac{n!}{s^{n+1}}\right).$$

5. $f(t) = \cos(at) \quad \Rightarrow$ To compute the Laplace transform we will use the Euler formula described in the notes for Chapter 3.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{2}$$

which implies that

$$\cos(\theta) = \frac{e^{i\theta} + e^{i\theta}}{2}.$$

Also, using $i^2 = -1$ we can write

$$(s + ib)(s - ib) = s^2 - (ib)^2 = s^2 + b^2.$$

Combing the above we can write

$$\begin{aligned}
\mathcal{L}(\cos(bt)) =& \mathcal{L}\left(\frac{e^{ibt} + e^{ibt}}{2}\right) \\
=& \frac{1}{2}\left(\frac{1}{s - ib} + \frac{1}{s + ib}\right) \\
=& \frac{1}{2}\left(\frac{(s + ib)}{(s^2 + b^2)} + \frac{(s - ib)}{(s^2 + b^2)}\right) \\
=& \frac{1}{2}\left(\frac{(s + ib) + (s - ib)}{(s^2 + b^2)}\right) = \frac{s}{(s^2 + b^2)}.
\end{aligned}$$

So we arrive at

$$\mathcal{L}(\cos(bt) = \frac{s}{(s^2 + b^2)}.$$

6. Given a function $f(t)$ we can find $\mathcal{L}(f'(t))$ by applying integration by parts as follows

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\, dt = f(t)e^{-st}\Big|_0^\infty - (-s)\int_0^\infty e^{-st} f(t)\, dt = s\mathcal{L}(f(t)) - f(0)$$

or

$$\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - f(0)$$

7. Given a function $f(t)$ find $\mathcal{L}(f''(t))$ can be easily computed by using the previous formula

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s\big(s\mathcal{L}(f(t)) - f(0)\big) - f'(0) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0).$$

So we have

$$\mathcal{L}(f''(t)) = s^2\mathcal{L}(f(t)) - sf(0) - f'(0).$$

8. To compute the Laplace transform of $f(t) = \sin(bt)$ we will use two of the previous formulas.

$$\begin{aligned}
\mathcal{L}(\sin(bt)) =& \frac{-1}{b}\mathcal{L}(\cos(bt)') \\
=& \frac{-1}{b}\left[s\mathcal{L}(\cos(bt)) - \cos(0)\right] \\
=& \frac{-1}{b}\left[s\left(\frac{s}{(s^2+b^2)}\right) - 1\right] \\
=& \frac{-1}{b}\left[\frac{s^2}{(s^2+b^2)} - 1\right] \\
=& \frac{-1}{b}\left[\frac{s^2-(s^2+b^2)}{(s^2+b^2)}\right] \\
=& \frac{b}{(s^2+b^2)}
\end{aligned}$$

Therefore

$$\mathcal{L}(\sin(bt) = \frac{b}{(s^2+b^2)}.$$

Let us consider a few examples of finding Laplace transforms.

Example 4.2. 1. $\mathcal{L}(2t^4) = \dfrac{2 \times 4!}{s^5} = \dfrac{48}{s^5}.$

2. $\mathcal{L}(t^2+6t-3) = \mathcal{L}(t^2) + 6\mathcal{L}(t) - 3\mathcal{L}(1) = \dfrac{2}{s^3} + \dfrac{6}{s^2} - \dfrac{3}{s}$.

3. $\mathcal{L}(2\cos(3t) + 3\sin(2t) - 3e^{-7t}) = 2\mathcal{L}(\cos(3t)) + 3\mathcal{L}(\sin(2t)) - 6\mathcal{L}(e^{-7t}) = \dfrac{2s}{s^2+9} + \dfrac{6}{s^2+4} - \dfrac{6}{(s+7)}$.

4. $\mathcal{L}(2e^{-t} + 6e^{3t}) = \dfrac{2}{(s+1)} + \dfrac{6}{(s-3)}$.

Example 4.3. Find the Laplace transform of $f(t) = (1+e^{2t})^2$. To do this we first note that $f(t) = 1 + 2e^{2t} + e^{4t}$ so we have

$$\mathcal{L}(f(t)) = \mathcal{L}(1 + 2e^{2t} + e^{4t}) = \frac{1}{s} + \frac{2}{(s-2)} + \frac{1}{(s-4)}.$$

Example 4.4. Find the Laplace transform of $f(t) = (\cos(t) + \sin(t))^2$. To do this we first note that

$$f(t) = 1 + 2\sin(t)\cos(t) = 1 + \sin(2t)$$

so we have

$$\mathcal{L}(f(t)) = \mathcal{L}(1 + \sin(2t)) = \frac{1}{s} + \frac{2}{(s^2+4)}.$$

In order to do the next example we need one of the addition formulas from trig

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Example 4.5. Find the Laplace transform of $f(t) = \sin(t + \pi/2)$.

$$f(t) = \sin(t)\cos(\pi/2) + \sin(\pi/2)\cos(t) = \cos(t).$$

Therefore

$$\mathcal{L}(f(t)) = \mathcal{L}(\cos(t)) = \frac{s}{(s^2+1)}.$$

4.2 The Inverse Laplace Transform

Given a function $f(t)$ the operation of taking the Laplace transform is denoted by $\mathcal{L}(f(t)) = F(s)$ and the inverse process is denoted by $\mathcal{L}^{-1}(F(s)) = f(t)$. The process of computing the Laplace transform of a function turns out to be more challenging than most students like. It involves lots of algebra and using a table of Laplace transforms backwards. For example, if we were asked to find $\mathcal{L}^{-1}(3/s^3)$ we would write

$$\mathcal{L}^{-1}(3/s^3) = \frac{3}{2}\mathcal{L}^{-1}(2/s^3) = \frac{3}{2}t^2$$

since we know that $\mathcal{L}(t^2) = 2/s^3$ and we can adjust the constants to work out. Most generally this process will require the use of the method of partial fractions.

Partial Fractions

These notes are concerned with decomposing rational functions

$$\frac{P(s)}{Q(s)} = \frac{a_M s^M + a_{M-1}s^{M-1} + \cdots + a_1 s + a_0}{s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}$$

Note: We can (without loss of generality) assume that the coefficient of s^N in the denominator is 1. Also in our intended applications we will always have $M < N$.

By the fundamental theorem of algebra we know that the denominator factors into a product of powers of linear and quadratic terms where the quadratic terms correspond to complex roots. Namely, it can be written in the form

$$(s-r_1)^{m_1}\ \cdots\ (s-r_k)^{m_k}\,(s^2-2\alpha_1 s+\alpha_1^2+\beta_1^2)^{p_1}\ \cdots\ (s^2-2\alpha_\ell s+\alpha_\ell^2+\beta_\ell^2)^{p_\ell},$$

where $\displaystyle\sum_{j=1}^{k} m_j + 2\sum_{j=1}^{\ell} p_j = n.$

The process referred to as ***Partial Fractions*** is a method to reduce a complex rational function into a sum of much simpler terms of the form

$$\frac{c}{(s-r)^j} \quad \text{or} \quad \frac{cs+d}{(s^2-2\alpha s+\alpha^2+\beta^2)^j}$$

The most important point is to learn how to deal with certain types of terms that can appear. But first there is a special case that arises and is worth a special attention. This is the case of non-repeated linear terms.

I. Nonrepeated Linear Factors

If $Q(s) = (s - r_1)(s - r_2) \cdots (s - r_n)$ and $r_i \neq r_j$ for $i \neq j$

$$\frac{P(s)}{Q(s)} = \frac{A_1}{(s - r_1)} + \frac{A_2}{(s - r_2)} + \cdots + \frac{A_n}{(s - r_n)}$$

II. Repeated Linear Factors

If $Q(s)$ contains a factor of the form $(s-r)^m$ then you must have the following terms

$$\frac{A_1}{(s - r)} + \frac{A_2}{(s - r)^2} + \cdots + \frac{A_m}{(s - r)^m}$$

III. A Nonrepeated Quadratic Factor

If $Q(s)$ contains a factor of the form $(s^2 - 2\alpha s + \alpha^2 + \beta^2) = (s - \alpha)^2 + \beta^2$ then you must have the following term

$$\frac{A_1 s + B_1}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)}$$

IV. Repeated Quadratic Factors

If $Q(s)$ contains a factor of the form $(s^2 - 2\alpha s + \alpha^2 + \beta^2)^m$ then you must have the following terms

$$\frac{A_1 s + B_1}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)} + \frac{A_2 s + B_2}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)^2} + \cdots + \frac{A_m s + B_m}{(s^2 - 2\alpha s + \alpha^2 + \beta^2)^m}$$

Lets consider some examples of computing inverse Laplace transforms:

Example 4.6. 1. Find $\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = \frac{1}{2!}\mathcal{L}^{-1}\left(\frac{2!}{s^3}\right) = \frac{1}{2}t^2$.

2. Find $\mathcal{L}^{-1}\left(\frac{1-s}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2} - \frac{1}{s}\right) = t - 1$.

3. Find $\mathcal{L}^{-1}\left(\frac{(3s+7)}{s^2+16}\right) = 3\mathcal{L}^{-1}\left(\frac{s}{s^2+4^2} + \frac{7}{4}\frac{4}{s^2+4^2}\right) = 3\cos(4t) + \frac{7}{4}\sin(4t)$.

Example 4.7. Find $\mathcal{L}^{-1}\left(\frac{(s+1)^3}{s^4}\right)$. We need to first expand the numerator to get

$$\frac{(s+1)^3}{s^4} = \frac{s^3+3s^2+3s+1}{s^4} = \frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}.$$

So we have

$$\mathcal{L}^{-1}\left(\frac{(s+1)^3}{s^4}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{3}{s^2} + \frac{3}{s^3} + \frac{1}{s^4}\right) = 1 + 3t + \frac{3}{2}t^2 + \frac{1}{6}t^3.$$

Example 4.8. Find $\mathcal{L}^{-1}\left(\frac{1}{(4s+1)}\right)$. We have

$$\mathcal{L}^{-1}\left(\frac{1}{(4s+1)}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{(s+1/4)}\right) = \frac{1}{4}e^{-t/4}.$$

Example 4.9. Find $\mathcal{L}^{-1}\left(\frac{2s-6}{(s^2+9)}\right)$. For this example we have

$$\mathcal{L}^{-1}\left(\frac{2s-6}{(s^2+9)}\right) = 2\mathcal{L}^{-1}\left(\frac{s}{(s^2+9)}\right) - 2\mathcal{L}^{-1}\left(\frac{3}{(s^2+9)}\right) = 2\cos(3t) - 2\sin(3t).$$

Example 4.10. Find $\mathcal{L}^{-1}\left(\frac{4s}{(s^2+2s-3)}\right)$. For this example we first note that $(s^2+2s-3) = (s+3)(s-1)$ so we have

$$\mathcal{L}^{-1}\left(\frac{4s}{(s^2+2s-3)}\right) = \mathcal{L}^{-1}\left(\frac{4s}{(s+3)(s-1)}\right).$$

In order to carry out this inverse Laplace transform we must use partial fractions. Since the denominator consists of non-repeated terms we can use the first box (in the formulas for partial fractions) to write

$$\frac{4s}{(s^2+2s-3)} = \frac{A}{(s+3)} + \frac{B}{(s-1)}.$$

To find A and B we can use the “cover-up” method to find that $A = 3$ and $B = 1$ so that

$$\mathcal{L}^{-1}\left(\frac{4s}{(s^2+2s-3)}\right) = \mathcal{L}^{-1}\left(\frac{3}{(s+3)}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)}\right) = 3e^{-3t} + e^t.$$

Example 4.11. Find $\mathcal{L}^{-1}\left(\frac{2s}{(s-2)(s-3)(s-6)}\right)$ for this we need to do partial fractions. Namely we have

$$\frac{2s}{(s-2)(s-3)(s-6)} = \frac{A}{(s-2)} + \frac{B}{(s-3)} + \frac{C}{(s-6)} = \frac{1)}{(s-2)} + \frac{-2}{(s-3)} + \frac{1}{(s-6)}.$$

So $f(t) = e^{2t} - 2e^{3t} + e^{6t}$.

Example 4.12. Find $\mathcal{L}^{-1}\left(\frac{5}{s^3+5s}\right)$ for this we need to do partial fractions. Namely we have

$$\frac{5}{s^3+5s} = \frac{A}{s} + \frac{Bs+C}{s^2+5} = \frac{1}{s} + \frac{-s}{s^2+5}.$$

So $f(t) = 1 - \cos(\sqrt{5}t)$.

Example 4.13. Find $\mathcal{L}^{-1}\left(\frac{(2s-4)}{s(s+1)(s^2+1)}\right)$ for this we need to do partial fractions. Namely we have

$$\begin{aligned}\frac{(2s-4)}{s(s+1)(s^2+1)} &= \frac{A}{s} + \frac{B}{(s+1)} + \frac{Cs+D}{(s^2+1)} \\ &= \frac{A(s+1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s+1)}{s(s+1)(s^2+1)} \\ &= \frac{(A+B+C)s^3 + (A+C+D)s^2 + (A+B+D)s + (A)}{s(s+1)(s^2+1)}\end{aligned}$$

$$\begin{aligned}A+B+C &= 0 \\ A+C+D &= 0 \\ A+B+D &= 2 \\ A &= -4\end{aligned}$$

Using $A = -4$ these equations become

$$\begin{aligned}B+C &= 4 \\ C+D &= 4 \\ B+D &= 6\end{aligned}$$

Taking -1 times the first equation added to the third gives

$$\begin{aligned} C + D &= 4 \\ -C + D &= 2 \end{aligned}$$

and adding these together gives $2D = 6$ so that $D = 3$. But then we have $C = 4 - D = 1$. So we have $A = -4$, $C = 1$ and $D = 3$ so we can finally find B from the first equation as $B = -A - C = 4 - 1 = 3$

Thus we have

$$\frac{-4}{s} + \frac{3}{(s+1)} + \frac{s+3}{(s^2+1)}$$

so $f(t) = -4 + 3e^{-t} + \cos(t) + 3\sin(t)$.

Now lets solve an initial value problem using the formula for the Laplace transform of a derivative. In these examples we will use the shorthand notation $Y = \mathcal{L}(y)$.

Example 4.14. Find the solution of $y' - y = 1$ with $y(0) = 0$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$\mathcal{L}(y') - \mathcal{L}(y) = \mathcal{L}(1) \;\;\Rightarrow\;\; (sY - 0) - Y = \frac{1}{s} \;\;\Rightarrow\;\; Y = \frac{1}{s(s-1)}$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions.

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s(s-1)}\right) = \mathcal{L}^{-1}\left(\frac{-1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)}\right) = -1 + e^t.$$

Example 4.15. Find the solution of $y'' - y = 1$ with $y(0) = 0$, $y'(0) = 1$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$s^2Y - 0s - 1 - Y = \frac{1}{s} \;\Rightarrow\; (s^2-1)Y = \frac{1}{s} + 1 = \frac{s+1}{s} \;\Rightarrow\; Y = \frac{s+1}{s(s^2-1)} = \frac{1}{s(s-1)}$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions.

$$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s(s-1)}\right) = \mathcal{L}^{-1}\left(\frac{-1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)}\right) = -1 + e^t.$$

Example 4.16. Find the solution of $y'' + 5y' + 4y = 0$ with $y(0) = 3$ and $y'(0) = 0$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$\mathcal{L}(y'') + 5\mathcal{L}(y') + 4\mathcal{L}(y) = 0 \;\;\Rightarrow\;\; (s^2Y - 3s) + 5(sY - 3) + 4Y = 0$$

Solving for Y we get

$$Y = \frac{3(s+5)}{(s^2 + 5s + 4)} \;\;\Rightarrow\;\; Y = \frac{3(s+5)}{(s+4)(s+1)}$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions.

$$\begin{aligned} y = \mathcal{L}^{-1}(Y) &= \mathcal{L}^{-1}\left(\frac{3(s+5)}{(s+4)(s-1)}\right) \\ &= \mathcal{L}^{-1}\left(\frac{-1}{(s+4)}\right) + \mathcal{L}^{-1}\left(\frac{4}{(s+1)}\right) \\ &= -e^{-4t} + 4e^{-t}. \end{aligned}$$

Example 4.17. Find the solution of $y'' + 5y' + 6y = 6$ with $y(0) = 1$ and $y'(0) = 1$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$\mathcal{L}(y'') + 5\mathcal{L}(y') + 6\mathcal{L}(y) = \mathcal{L}(6) \;\;\Rightarrow\;\; (s^2Y - s - 1) + 5(sY - 1) + 6Y = \frac{6}{s}$$

Solving for Y we get

$$Y = \frac{s^2 + 6s + 6}{s(s^2 + 5s + 6)} \;\;\Rightarrow\;\; Y = \frac{s^2 + 6s + 6}{s(s+2)(s+3)}$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions.

$$Y = \frac{s^2 + 6s + 6}{s(s+2)(s+3)} = \frac{1}{s} + \frac{1}{(s+2)} - \frac{1}{(s+3)}$$

$$y = \mathcal{L}^{-1}(Y) = 1 + e^{-2t} - e^{-3t}.$$

Example 4.18. Find the solution of $y'' + y = \sqrt{2}\sin(\sqrt{2}t)$ with $y(0) = 10$ and $y'(0) = 0$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sqrt{2}\sin(\sqrt{2}t)) \;\Rightarrow\; (s^2Y - 10s) + Y = \frac{2}{s^2+2}$$

$$(s^2+1)Y = \frac{2}{s^2+2} + 10s = \frac{10s^3+20s+2}{(s^2+2)} \;\Rightarrow\; Y = \frac{10s^3+20s+2}{(s^2+2)(s^2+1)}.$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions (this time I give the answer and let you do the work)

$$Y = \frac{10s^3+20s+2}{(s^2+2)(s^2+1)} = \frac{10s+2}{(s^2+1)} - \frac{2}{(s^2+2)}$$

$$y = 10\cos(t) + 2\sin(t) - \sqrt{2}\sin(\sqrt{2}t).$$

4.3 The Shift Theorems

Two of the most important and useful results we need to discuss are the first and second shift (or Translation) theorems.

Theorem 4.1 (First Shift Theorem)**.** *If $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(e^{at}f(t)) = F(s-a)$ for any real number a.*

Proof.

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$$

□

Lets consider some examples.

Example 4.19.

$$\mathcal{L}(t^2e^{3t}) = \mathcal{L}(t^2)\big|_{s-3} = \frac{2}{(s-3)^3}$$

Example 4.20.

$$\mathcal{L}(e^{-t}\sin(2t)) = \mathcal{L}(\sin(2t))\big|_{s+1} = \frac{2}{(s+1)^2+4}$$

Example 4.21.

$$\mathcal{L}(e^{3t}(t+1)^2) = \mathcal{L}(t^2+2t+1)\big|_{s-3} = \frac{2}{(s-3)^3} + \frac{2}{(s-3)^2} + \frac{1}{(s-3)}$$

Example 4.22.

$$\begin{aligned}\mathcal{L}(t(e^t+e^{2t})^2) &= \mathcal{L}(t(e^{2t}+2e^{3t}+e^{4t}))\\ &= \mathcal{L}(te^{2t}) + 2\mathcal{L}(te^{3t}) + \mathcal{L}(te^{4t}))\\ &= \frac{1}{(s-2)^2} + 2\frac{1}{(s-3)^2} + \frac{1}{(s-4)^2}\end{aligned}$$

Next we consider some examples of inverse Laplace transforms using the shift theorems.

Example 4.23. Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+2s+5}\right)$. We note that the denominator does not factor since the discriminant is $(2)^2-4(1)(5)<0$ so we know the result will involve sines and cosines. We proceed by completing the square in the denominator.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+2^2}\right) = \frac{1}{2}e^{-t}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}e^{-t}\sin(2t).$$

Example 4.24. Find $\mathcal{L}^{-1}\left(\dfrac{2s+5}{s^2+6s+34}\right)$. Once again we note that the denominator does not factor since the discriminant is $(6)^2-4(1)(34)<0$ so we know the result will involve sines and cosines. We proceed by completing the square in the denominator.

$$\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+6s+34}\right) = \mathcal{L}^{-1}\left(\frac{2s+5}{(s+3)^2+5^2}\right).$$

So we see that the denominator has a shifted s value but the numerator does not. To fix this we write

$$2s+5 = 2[(s+3)-3]+5 = 2(s+3)-1.$$

So we have

$$\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+6s+34}\right) = \mathcal{L}^{-1}\left(\frac{2(s+3)-1}{(s+3)^2+5^2}\right)$$

$$
\begin{aligned}
&= 2\mathcal{L}^{-1}\left(\frac{(s+3)}{(s+3)^2+5^2}\right) - \frac{1}{5}\mathcal{L}^{-1}\left(\frac{5}{(s+3)^2+5^2}\right) \\
&= 2e^{-3t}\mathcal{L}^{-1}\left(\frac{s}{s^2+5^2}\right) - \frac{1}{5}e^{-3t}\mathcal{L}^{-1}\left(\frac{5}{s^2+5^2}\right) \\
&2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t).
\end{aligned}
$$

The Second Shift (Translation) Theorem

Consider finding the following Laplace transform

$$
f(t) = \begin{cases} e^{-2t} & \text{for } 0 \leq t < 1 \\ 0 & \text{for } t \geq 1 \end{cases}.
$$

The only way to do this up to this point is to apply the definition

$$
\begin{aligned}
\mathcal{L}(f) &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st} e^{-2t}\, dt \\
&= \int_0^1 e^{(-(s+2)t)}\, dt = \left.\frac{e^{(-(s+2)t)}}{-(s+2)}\right|_0^1 \\
&= \frac{1 - e^{-(s+2)}}{(s+2)}.
\end{aligned}
$$

We will ow introduce one of the most important tools in this block of material that gives a much simpler way to do the above problem. Namely we will introduce the second shift theorem. In order to present the second shift theorem we first need to discuss the unit step function (or Heaviside function)

$$
u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}
$$

or the shifted unit step function

$$
u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a. \end{cases}
$$

With this we can state the second shift theorem

Theorem 4.2 (Second Shift Theorem)**.** *If* $\mathcal{L}(f(t)) = F(s)$ *then*

$$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s) \quad \textit{for any} \quad a > 0.$$

Proof.

$$\begin{aligned}
\mathcal{L}(u(t-a)f(t-a)) &= \int_0^\infty e^{-st}u(t-a)f(t-a)\,dt = \int_a^\infty e^{-st}f(t-a)\,dt \\
(\text{ set } \tau = t-a) \qquad &= \int_0^\infty e^{-s(\tau+a)}f(\tau)\,d\tau = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau \\
&= e^{-as}\int_0^\infty e^{-st}\,f(\tau)\,dt = e^{-as}F(s).
\end{aligned}$$

□

A very useful modification of the second shift theorem shows how to take the Laplace transform of a unit step function times a function that is not shifted.

Theorem 4.3 (Modified Second Shift Theorem)**.**

$$\mathcal{L}(u(t-a)g(t)) = e^{-as}\mathcal{L}(g(t+a)) \quad \textit{for any} \quad a > 0.$$

Proof. Let $f(t) = g(t+a)$ then we have $f(t-a) = g(t)$ and we can use the second shift theorem to write

$$\mathcal{L}(u(t-a)g(t)) = \mathcal{L}(u(t-a)f(t-a)) = e^{-as}\mathcal{L}(f(t)) = e^{-as}\mathcal{L}(g(t+a)).$$

□

One of the most important things you need to come to grips with is the use of the unit step function to express a function defined in pieces. Let me give a very generic example.

Example 4.25. Suppose you want to take the Laplace transform of a general function given by

$$f(t) = \begin{cases} f_1(t), & 0 \leq t < t_1 \\ f_2(t), & t_1 \leq t < t_2 \\ f_3(t), & t \geq t_2 \end{cases}.$$

The first step is to express this function using the unit step function as follows

$$f(t) = f_1(t) + u(t - t_1)(f_2(t) - f_1(t)) + u(t - t_2)(f_3(t) - f_2(t)).$$

Then you could proceed to compute the Laplace transform by applying Theorem 4.3.

Here is an example

$$f(t) = \begin{cases} 2, & 0 \leq t < 3 \\ t, & 3 \leq t < 4 \\ e^t, & t \geq 4 \end{cases}.$$

The first step is to express this function using the unit step function as follows

$$f(t) = 2 + u(t - 3)(t - 2) + u(t - 4)(e^t - t).$$

So we need to find

$$\mathcal{L}(f(t)) = \mathcal{L}(2) + \mathcal{L}(u(t - 3)(t - 2)) + \mathcal{L}(u(t - 4)(e^t - t)).$$

Let us do these transforms one at a time

1. $\mathcal{L}(2) = \dfrac{2}{s}$
2. For the second term we use Theorem 4.3 with $g(t) = t - 2$ and $a = 3$ which gives $g(t + 3) = t + 1$ so we have

$$\mathcal{L}(u(t - 3)(t - 2)) = e^{-3s}\mathcal{L}(t + 1) = e^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

3. For the third term we could use Theorem 4.3 but it is easier to break it up into two

problems. We have

$$\mathcal{L}(u(t-4)(e^t - t)) = \mathcal{L}(u(t-4)e^t) - \mathcal{L}(u(t-4)t).$$

For the second term we use Theorem 4.3 to get

$$-\mathcal{L}(u(t-4)t) = -e^{-4s}\mathcal{L}(t+4) = -e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

where we have used $g(t) = t$ and $g(t+4) = t+4$.

For the remaining term we apply the first shift theorem (Theorem 4.1) as follows.

$$\mathcal{L}(u(t-4)e^t) = \mathcal{L}(u(t-4))\big|_{s-1} = \frac{e^{-4(s-1)}}{(s-1)}.$$

Finally then we have

$$\mathcal{L}(f) = \frac{2}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + \frac{e^{-4(s-1)}}{(s-1)}$$

Example 4.26. Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}.$$

First we write $f(t) = t - u(t-1)t$ and then compute

$$\begin{aligned}\mathcal{L}(f(t)) &= \mathcal{L}(t - u(t-1)t) = \mathcal{L}(t) - \mathcal{L}(tu(t-1)) = \frac{1}{s^2} - e^{-s}\mathcal{L}((t+1)) \\ &= \frac{1}{s^2} - e^{-s}\big[\mathcal{L}(t) + \mathcal{L}(1)\big] = \frac{1}{s^2} - e^{-s}\left[\frac{1}{s^2} + \frac{1}{s}\right]\end{aligned}$$

Example 4.27. Find the Laplace transform of

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ \sin(t), & t \ge \pi \end{cases}.$$

First we write $f(t) = u(t - \pi)\sin(t)$ and then compute

$$\begin{aligned}\mathcal{L}(f(t)) = \mathcal{L}(u(t-\pi)\sin(t)) = e^{-\pi s}\mathcal{L}(\sin(t+\pi))\\ = e^{-\pi s}\mathcal{L}(-\sin(t)) = -e^{-\pi s}\frac{1}{(s^2+1)}\end{aligned}$$

where we have used the addition formula

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

to get $\sin(t+\pi) = \sin(t)\cos(\pi) + \sin(\pi)\cos(t) = -\sin(t)$.

Example 4.28. Find the Laplace transform of

$$f(t) = \begin{cases} \sin(2t), & 0 \le t < \pi \\ \sin(t), & t \ge \pi \end{cases}.$$

First we write $f(t) = \sin(2t) + u(t-\pi)(\sin(t) - \sin(2t))$ and then for the first term we compute

$$\mathcal{L}(\sin(2t)) = \frac{2}{s^2+4},$$

and for the second term we apply Theorem 4.3. Namely we set

$g(t) = (\sin(t) - \sin(2t))$ and we need $g(t+\pi) = (\sin(t+\pi) - \sin(2(t+\pi)))$

Applying the addition formula

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$$

we get

$$\sin(t+\pi) = \sin(t)\cos(\pi) + \sin(\pi)\cos(t) = -\sin(t)$$

and

$$\sin(2(t+\pi)) = \sin(2t+2\pi) = \sin(2t)\cos(2\pi) + \sin(2\pi)\cos(2t) = \sin(2t)$$

So we have

$$\mathcal{L}(u(t-\pi)(\sin(t)-\sin(2t))) = -e^{-\pi s}\mathcal{L}(\sin(t)+\sin(2t)) = -e^{-\pi s}\left(\frac{1}{s^2+1}+\frac{2}{s^2+4}\right).$$

Finally then the answer is

$$f(t) = \frac{2}{s^2+4} - e^{-\pi s}\left(\frac{1}{s^2+1}+\frac{2}{s^2+4}\right).$$

Example 4.29. Find the Laplace transform of

$$f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & t \ge 3 \end{cases}.$$

First we write $f(t) = 2 - 4u(t-3)$ and then compute

$$\mathcal{L}(f(t)) = \frac{2}{s} - 4\mathcal{L}(u(t-3)) = \frac{2}{s} - 4\frac{e^{-3s}}{s}.$$

Example 4.30. Find the Laplace transform of $f(t) = u(t-1)e^{2t}t$. Use the first shift theorem to get rid of the e^{2t} and then apply the second shift theorem

$$\begin{aligned}\mathcal{L}(f(t)) &= \mathcal{L}(u(t-1)e^{2t}t) = \mathcal{L}(u(t-1)t)\big|_{(s-2)} \\ &= \left[e^{-s}\mathcal{L}((t+1))\right]\big|_{(s-2)} = e^{-(s-2)}\left[\frac{1}{(s-2)^2}+\frac{1}{(s-2)}\right].\end{aligned}$$

Example 4.31. Find $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right)$. Since this function of s contains an exponential e^{-as} we know the inverse Laplace transform must involve the second shift theorem. Namely we have

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = \mathcal{L}^{-1}\left(e^{-s}\,\frac{1}{s^2+s}\right)$$

So by the second shift theorem backwards we have

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = u(t-1)\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right)\Bigg|_{(t-1)}.$$

In order to continue we need to use partial fractions to simplify matters

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{(s+1)}\right) = 1 - e^{-t}.$$

Combining these results we have

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = u(t-1)\left[1 - e^{-(t-1)}\right].$$

Example 4.32. Find $\mathcal{L}^{-1}\left(\frac{s\ e^{-5s}}{s^2+4}\right)$. Since this function of s contains an exponential e^{-as} we know the inverse Laplace transform must involve the second shift theorem. Namely we have

$$\mathcal{L}^{-1}\left(\frac{s\ e^{-5s}}{s^2+4}\right) = \mathcal{L}^{-1}\left(e^{-s}\ \frac{s}{s^2+4}\right) = u(t-5)\cos(2(t-5)).$$

Now lets solve an initial value problem.

Example 4.33. Solve $y'' - y = 2u(t-1)$ and $y(0) = 4$, $y'(0) = 2$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$\mathcal{L}(y'') - \mathcal{L}(y) = \frac{2\ e^{-s}}{s} \quad \Rightarrow \quad (s^2Y - 4s - 4) - Y = \frac{2\ e^{-s}}{s}.$$

Solving for Y we get

$$Y = \frac{2e^{-s}}{s(s^2-1)} + \frac{(4s+2)}{(s^2-1)}$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions. But since partial fractions only applies to rational functions we need to do two separate partial fractions. Namely

$$\frac{2}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s+1)}.$$

We easily find $A = -2$, $B = 1$ and $C = 1$. We also need to do

$$\frac{(4s+2)}{(s-1)(s+1)} = \frac{D}{(s-1)} + \frac{E}{(s+1)}$$

Once again this is very easy and we get $D = 3$ and $E = 1$. So we have

$$
\begin{aligned}
y = \mathcal{L}^{-1}(Y) &= \mathcal{L}^{-1}\left(\frac{2\ e^{-s}}{s(s-1)(s+1)} + \frac{(4s+2)}{(s^2-1)}\right) \\
&= \mathcal{L}^{-1}\left(e^{-s}\ \frac{-2}{s}\right) + \mathcal{L}^{-1}\left(e^{-s}\ \frac{1}{(s-1)}\right) + \mathcal{L}^{-1}\left(e^{-s}\ \frac{1}{(s+1)}\right) \\
&\quad + \frac{3}{(s-1)} + \frac{1}{(s+1)} \\
&= u(t-1)\left[e^{-(t-1)} + e^{-(t-1)} - 2\right] + 3e^t + e^{-t}.
\end{aligned}
$$

Example 4.34. Solve $y'' + 4y = -4u(t-1)$ and $y(0) = 4$, $y'(0) = 2$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$
\mathcal{L}(y'') + 4\mathcal{L}(y) = \frac{-4\ e^{-s}}{s} \quad \Rightarrow \quad (s^2 Y - 4s - 2) + 4Y = \frac{-4\ e^{-s}}{s}.
$$

Solving for Y we get

$$
Y = \frac{-4\ e^{-s}}{s(s^2+4)} + \frac{4s+2}{s^2+4}.
$$

So to find y we need to take the inverse Laplace transform which requires we do partial fractions on the terms multiplying e^{-s}. Namely

$$
\frac{-4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{(s^2+4)}.
$$

We easily find $A = -1$, $B = 1$ and $C = 0$ so we have

$$
\begin{aligned}
y = \mathcal{L}^{-1}(Y) &= \mathcal{L}^{-1}\left(\frac{-4\ e^{-s}}{s(s^2+4)} + + \frac{4s+2}{s^2+4}\right) \\
&= \mathcal{L}^{-1}\left(e^{-s}\left[\frac{-1}{s} + \frac{s}{(s^2+4)}\right] + \frac{4s+2}{s^2+4}\right) \\
&= u(t-1)\left[-1 + \cos(2(t-1))\right] + 4\cos(2t) + \sin(2t).
\end{aligned}
$$

Example 4.35. Solve $y'' - 6y' + 9y = 27t$ and $y(0) = 0$, $y'(0) = 1$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$
\mathcal{L}(y'') - 6\mathcal{L}(y') + 9\mathcal{L}(y) = \frac{27}{s^2} \quad \Rightarrow \quad (s^2 Y - 1) - 6(sY) + 9Y = \frac{27}{s^2}.
$$

Solving for Y we get something that requires a partial fraction with repeated terms

$$Y = \frac{s^2+27}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-3)} + \frac{D}{(s-3)^2}$$

After a bit of tedious work (which I suggest you do for practice) we get

$$\frac{2}{s} + \frac{3}{s^2} + \frac{-2}{(s-3)} + \frac{4}{(s-3)^2}$$

$$y = \mathcal{L}^{-1}(Y) = 2 + 3t - 2e^{3t} + 4te^{3t}.$$

Example 4.36. Solve $y'' - y' = 2e^t\cos(t)$ and $y(0) = 0$, $y'(0) = 0$. First we take the Laplace transform of both sides (using $Y = \mathcal{L}(y)$) to get

$$\mathcal{L}(y'') - \mathcal{L}(y') = \frac{2(s-1)}{(s-1)^2+1} \quad \Rightarrow \quad (s^2 - s)Y = \frac{2(s-1)}{(s^2-2s+2)}.$$

Solving for Y we get something that requires a partial fraction. Namely

$$\begin{aligned} Y &= \frac{2(s-1)}{s(s-1)(s^2-2s+2)} = \frac{2}{s(s^2-2s+2)} \\ &= \frac{A}{s} + \frac{Bs+C}{s^2-2s+2} \\ &= \frac{A(s^2-2s+2) + s(Bs+C)}{s(s^2-2s+2)} \\ &= \frac{(A+B)s^2 + (-2A+C)s + 2A}{s(s^2-2s+2)} \end{aligned}$$

From this we see that $2A = 2$ so that $A = 1$. Then $-2A + C = 0$ so $C = 2$. And finally $A + B = 0$ so $B = -1$ and we have

$$Y = \frac{1}{s} - \frac{s-2}{s^2-2s+2}.$$

Next we need to find the inverse Laplace transform. The first term is very easy but for the second we must use the First Shift Theorem (Theorem 4.1). First we use completing the square to write the problem as

$$\mathcal{L}^{-1}\left(\frac{s-2}{s^2-2s+2}\right) = \mathcal{L}^{-1}\left(\frac{(s-1)-1}{(s-1)^2+1}\right)$$

Notice that

$$\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2+1}\right) = e^t\cos(t)$$

and

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+1}\right) = e^t\sin(t)$$

So after applying Theorem 4.1 we obtain

$$y = 1 - e^t\cos(t) - e^t\sin(t).$$

Example 4.37. Next consider an example of computing an inverse Laplace transform using Theorem 4.1. Find

$$\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+6s+34}\right)$$

To do this we will apply the First Shift Theorem (Theorem 4.1) after completing the square in the denominator and using the adding zero trick in the numerator. Namely we have

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+6s+34}\right) &= \mathcal{L}^{-1}\left(\frac{2s+5}{(s+3)^2+5^2}\right)\\
&= \mathcal{L}^{-1}\left(\frac{2[(s+3)-3]+5}{(s+3)^2+5^2}\right)\\
&= \mathcal{L}^{-1}\left(\frac{2(s+3)-1}{(s+3)^2+5^2}\right)\\
&= \mathcal{L}^{-1}\left(\frac{2(s+3)}{(s+3)^2+5^2}\right) - \frac{1}{5}\mathcal{L}^{-1}\left(\frac{5}{(s+3)^2+5^2}\right)\\
&= 2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t).
\end{aligned}$$

4.4 Additional Operational Properties

In this section we cover several very useful theorems for Laplace transforms.

Theorem 4.4. *If* $\mathcal{L}(f(t)) = F(s)$ *then* $\mathcal{L}(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n}F(s)$.

Proof.

$$\mathcal{L}(t^n f(t)) = \int_0^\infty e^{-st}\, t^n f(t)\,dt = \int_0^\infty te^{-st}[t^{(n-1)}f(t)]\,dt = -\frac{d}{ds}\mathcal{L}(t^{(n-1)} f(t)).$$

Repeating this calculation n times we arrive at

$$\begin{aligned}\mathcal{L}(t^n f(t)) &= -\frac{d}{ds}\mathcal{L}(t^{(n-1)} f(t)) \\ &= (-1)^2\frac{d^2}{ds^2}\mathcal{L}(t^{(n-2)} f(t)) \\ &\vdots \\ &= (-1)^n\frac{d^n}{ds^n}\mathcal{L}(t^{(0)} f(t)) = (-1)^n\frac{d^n}{ds^n}F(s).\end{aligned}$$

□

Example 4.38. Find the Laplace transform of $f(t) = t\sin(2t)$. Applying Theorem 4.4 we have

$$\mathcal{L}(t\sin(2t)) = -\frac{d}{ds}\left(\frac{2}{(s^2+4)}\right) = -\left(\frac{-4s}{(s^2+4)^2}\right) = \frac{4s}{(s^2+4)^2}.$$

Example 4.39. Find the Laplace transform of $f(t) = te^{2t}\cos(t)$. In this case we first apply the first shift theorem to write

$$L(f(t)) = L(te^{2t}\cos(t)) = L(t\cos(t))\big|_{(s-2)}.$$

So we next need to compute $L(t\cos(t))$ and for this we apply Theorem 4.4 to obtain

$$\mathcal{L}(t\cos(t)) = -\frac{d}{ds}\left(\frac{s}{(s^2+1)}\right) = -\left(\frac{(s^2+1)-s(2s)}{(s^2+1)^2}\right) = \frac{-s^2+1}{(s^2+1)^2}.$$

Finally then we arrive at

$$L(f(t)) = \frac{-(s-2)^2+1}{((s-2)^2+1)^2}.$$

Example 4.40. Find the inverse Laplace transform $f(t)$ of $F(s) = \ln\left(\frac{s+2}{s-5}\right)$. We use Theorem 4.4 as follows:

$$\frac{dF}{ds} = \frac{d}{ds}\left(\ln(s+2) - \ln(s-5)\right) = \frac{1}{s+2} - \frac{1}{s-5}$$

so we have

$$\mathcal{L}(tf(t)) = -\frac{dF}{ds} = -\left(\frac{1}{s+2} - \frac{1}{s-5}\right) \quad\Rightarrow\quad tf(t) = e^{5t} - e^{-2t}$$

or

$$f(t) = \frac{e^{5t} - e^{-2t}}{t}.$$

Example 4.41. Find the inverse Laplace transform $f(t)$ of $F(s) = \ln\left(\frac{s+2}{s-5}\right)$. We use Theorem 4.4 as follows:

$$\frac{dF}{ds} = \frac{d}{ds}\left(\ln(s+2) - \ln(s-5)\right) = \frac{1}{s+2} - \frac{1}{s-5}$$

so we have

$$\mathcal{L}(tf(t)) = -\frac{dF}{ds} = -\left(\frac{1}{s+2} - \frac{1}{s-5}\right) \quad \Rightarrow \quad tf(t) = e^{5t} - e^{-2t}$$

or

$$f(t) = \frac{e^{5t} - e^{-2t}}{t}.$$

Example 4.42. Find the inverse Laplace transform $f(t)$ of $F(s) = \tan^{-1}\left(\frac{1}{s}\right)$. We use the above theorem as follows:

$$\frac{dF}{ds} = \frac{d}{ds}\left(\tan^{-1}\left(\frac{1}{s}\right)\right) = \frac{-1}{s^2+1}$$

so we have

$$\mathcal{L}(tf(t)) = -\frac{dF}{ds} = \left(\frac{1}{s^2+1}\right) \quad \Rightarrow \quad tf(t) = \sin(t)$$

or

$$f(t) = \frac{\sin(t)}{t}.$$

Theorem 4.5. *If* $\mathcal{L}(f(t)) = F(s)$ *then* $\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}F(s)$.

Proof. Let $g(t) = \int_0^t f(\tau)\,d\tau$ so that $g'(t) = f(t)$.

$$\mathcal{L}(f(t)) = \mathcal{L}(g'(t)) = s\mathcal{L}(g(t)) - g(0) = s\mathcal{L}(g(t)).$$

Therefore, dividing by s we have

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \mathcal{L}(g(t)) = \frac{1}{s}\mathcal{L}(f(t)) = \frac{1}{s}F(s).$$

□

Example 4.43. Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{1}{s(s-5)}$. Using the previous result and the fact that $e^{5t} = \mathcal{L}^{-1}(1/(s-5))$ we have

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s(s-5)}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s}\,\frac{1}{(s-5)}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{s}\,\mathcal{L}(e^{5t})\right) = \int_0^t e^{5\tau}\,d\tau \\
&= \frac{1}{5}\left[e^{5\tau}\right]\Big|_0^t = \frac{1}{5}\left[e^{5t} - 1\right].
\end{aligned}$$

Convolutions and their Applications

Definition 4.1 (Convolution)**.** Given two functions $f(t)$ and $g(t)$ we define the *convolution product* by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau. \tag{3}$$

Theorem 4.6 (Two Important Properties)**.** *1.* $(f * g)(t) = (g * f)(t)$*, i.e., Convolution multiplication is commutative.*

2. If $F(s) = \mathcal{L}(f(t))$ *and* $G(s) = \mathcal{L}(g(t))$*, then* $\mathcal{L}(f * g)(s) = F(s)G(s)$*.*

Proof. 1. To show that $(f * g)(t) = (g * f)(t)$ we begin with the definition

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(\tau)g(t-\tau)\,d\tau \\
&\text{(Let } w = t - \tau \;\; \Rightarrow \;\; dw = -d\tau, \;\; \tau = t - w) \\
&= \int_t^0 f(t-w)g(w)\,(-dw) \\
&= \int_0^t f(t-w)g(w)\,dw = (g * f)(t).
\end{aligned}$$

2. If $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$, then we want to show that $\mathcal{L}(f * g)(s) = F(s)G(s)$.

We have

$$
\begin{aligned}
F(s)G(s) &= \left(\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right)\left(\int_0^\infty e^{-sw}\, g(w)\,dw\right) \\
&= \int_0^\infty \left(\int_0^\infty e^{-s(\tau+w)} f(\tau)g(w)\,d\tau\right) dw = \int_0^\infty f(\tau)\ \left(\int_0^\infty e^{-s(\tau+w)} g(w)\,dw\right) d\tau \\
&\text{(Let } t=\tau+w \ \ \Rightarrow \ \ dt = dw \text{ and } w = t-\tau) \\
&= \int_0^\infty f(\tau)\ \left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right) d\tau \\
&\text{(Change the order of integration)} \\
&= \int_0^\infty e^{-st}\ \left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right) dt = \mathcal{L}((f*g)(t)).
\end{aligned}
$$

□

Remark 4.1. The second part of the theorem gives a very simple proof of an earlier result.

$$
\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \mathcal{L}((1*f)(t)) = \mathcal{L}(1)\mathcal{L}(f(t)) = \frac{1}{s}F(s).
$$

This result can be useful in finding inverse Laplace transforms since

$$
\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t f(\tau)\,d\tau.
$$

Example 4.44. Find the inverse Laplace transform of $\dfrac{1}{s(s^2+1)}$ and $\dfrac{1}{s^2(s^2+1)}$.

For the first problem we have

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\sin(t))\right) \\
&= \int_0^t \sin(\tau)\,d\tau = -\cos(\tau)\Big|_0^t \\
&= 1-\cos(t).
\end{aligned}
$$

And, using the above we can solve the second problem as follows

$$
\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\ \mathcal{L}(1-\cos(\tau))\right)
$$

$$
\begin{aligned}
&= \int_0^t (1 - \cos(\tau))\, d\tau = (\tau - \sin(\tau))\Big|_0^t \\
&= t - \sin(t).
\end{aligned}
$$

Example 4.45. Show that

$$
\mathcal{L}^{-1}\left(\frac{2k^3}{(s^2+k^2)^2}\right) = \sin(kt) - kt\cos(kt). \qquad (4)
$$

For this problem will use one of the addition formulas from trig:

$$
\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)
$$

which implies that

$$
\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right].
$$

By our main property of the convolution we have

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2k^3}{(s^2+k^2)^2}\right) &= 2k\,\mathcal{L}^{-1}\left(\frac{k}{(s^2+k^2)}\right)\left(\frac{k}{(s^2+k^2)}\right) \\
&= 2k\left(\sin(kt) * (\sin(kt)\right) = \int_0^t \sin(k\tau)\sin(k(t-\tau))\, d\tau \\
&= k\int_0^t \left[\cos(k(2\tau - t)) - \cos(kt)\right]\, d\tau \\
&= k\left[\frac{1}{k}\sin(kt) - t\cos(t)\right] = \sin(kt) - kt\cos(kt).
\end{aligned}
$$

Example 4.46. Let us consider an example of finding a convolution by two different methods. The first method will be to use the definition of convolution. Let $f(t) = t$ and $g(t) = e^t$ and find $h(t) = (f * g)(t)$. Recall the $f * g = g * f$ so we have

$$
\begin{aligned}
h(t) = (f * g)(t) &= \int_0^t (t - v)e^v\, dv \\
&= \int_0^t ve^{t-v}\, dv = e^t \int_0^t ve^{-v}\, dv \\
&= e^t\left[\int_0^t v\left(-e^{-v}\right)'\, dv\right]
\end{aligned}
$$

$$
\begin{aligned}
&= e^t \left[-ve^{-v}\Big|_0^t - \int_0^t \left(-e^{-v}\right)\, dv \right] \\
&= e^t \left[-te^{-t} + \int_0^t e^{-v}\, dv \right] \\
&= e^t \left[-te^{-t} - \left(e^{-t} - 1\right) \right] \\
&= -t - 1 + e^t
\end{aligned}
$$

Next we compute the same result using Laplace transforms. We have $h(t) = (f * g)(t)$ so that

$$
H(s) = F(s)G(s) = \mathcal{L}(t)\mathcal{L}(e^t) = \frac{1}{s^2(s-1)}
$$

We use partial fractions to find

$$
h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)}\right) = \mathcal{L}^{-1}\left(\frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)}\right)
$$

We easily find $A = -1, B = -1, C = 1$

$$
\begin{aligned}
&= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{-1}{s^2} + \frac{1}{(s-1)}\right) \\
&= -1 - t + e^t
\end{aligned}
$$

Example 4.47. Consider another example of finding a convolution. Let $f(t) = \sin(t)$ and $g(t) = \cos(t)$ and find $h(t) = (f * g)(t)$. One way to compute the resulting integral is to use a trig identity

$$
\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)
$$

which, upon adding the two equations (one for plus and one for minus) together, implies

$$
\sin(a)\cos(b) = \frac{1}{2}\big[\sin(a+b) + \sin(a-b)\big].
$$

So for this example we set $a = v$ and $b = t - v$ and

$$
\sin(v)\cos(t-v) = \frac{1}{2}\big[\sin(t) + \sin(2v - t)\big].
$$

So we have

$$
\begin{aligned}
h(t) = (f * g)(t) &= \int_0^t \sin(v)\cos(t-v)\,dv \\
&\int_0^t \frac{1}{2}\big[\sin(t) + \sin(2v-t)\big]\,dv \\
&= \frac{1}{2}\,t\sin(t) + \frac{1}{2}\int_0^t \sin(2v-t)\,dv \\
&\text{Let } u = 2v - t \quad \Rightarrow \quad du = 2dv \\
&= \frac{1}{2}\,t\sin(t) + \frac{1}{4}\int_{-t}^t \sin(u)\,du \\
&= \frac{1}{2}\,t\sin(t)
\end{aligned}
$$

since the second integral is 0.

If we try to compute the same result using Laplace transforms. We have $h(t) = (f * g)(t)$ so that

$$H(s) = F(s)G(s) = \mathcal{L}(\sin(t))\mathcal{L}(\cos(t)) = \frac{s}{(s^2+1)^2}$$

This is already in partial fractions form so partial fractions will not help. The only way out here is to use a formula in the book (Example 1, page 221, Cullen and Zill, "Advanced Engineering math, 4th Ed.)

$$\mathcal{L}(t\sin(kt)) = \frac{2ks}{(s^2+k^2)^2}.$$

In the present case we then have (with $k = 1$)

$$h(t) = \mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) = \frac{1}{2}\,t\sin(t).$$

Example 4.48. Let us consider an example solving an initial value problem.

$$y'' + y = 2\cos(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Applying Laplace transforms we get

$$(s^2+1)Y = \frac{2s}{s^2+1}$$

or

$$Y = \frac{2s}{(s^2+1)^2}$$

To solve the initial value problem we need to find the inverse Laplace transform

$$y = \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right).$$

This problem is related to the above Example 4.47. Indeed we have

$$\begin{aligned} y = \mathcal{L}^{-1}\left(\frac{2s}{(s^2+1)^2}\right) &= 2(\sin(t) * \cos(t)) \\ &= 2\int_0^t \sin(t-\tau)\cos(\tau)\,d\tau \\ &= \int_0^t [\sin(t) \ + \sin(t-2\tau)]\,d\tau \\ &= t\sin(t) \end{aligned}$$

where on the third we have used the addition formula

$$\sin(a)\cos(b) = \frac{1}{2}\big[\sin(a+b) + \sin(a-b)\big]$$

with $a = t - \tau$ and $b = \tau$ to write

$$2\sin(t-\tau)\cos(\tau) = \sin(t) + \sin(t-2\tau)$$

and on the last step we have

$$\int_0^t \sin(t-2\tau)\,d\tau = 0$$

as above.

Volterra Integral Equations

Laplace transform methods are particularly useful in solving Volterra Integral equation. An example of a Volterra equation is an equation in the form

$$y(t) = f(t) + \int_0^t k(t-\tau)\,y(\tau)\,d\tau.$$

Notice that the the integral term is a convolution, so if we apply the Laplace transform to this equation with $Y(s) = \mathcal{L}(y)$, $F(s) = \mathcal{L}(f)$ and $K(s) = \mathcal{L}(k)$ we have

$$Y(s) = F(s) + K(s)Y(s)$$

which implies that

$$Y(s) = \frac{F(s)}{1 - K(s)}.$$

So to find $y(t)$ we only need to take the inverse Laplace transform of the right hand side of the above equation.

Example 4.49. Consider the equation

$$y(t) = -t + \int_0^t y(t-\tau)\sin(\tau)\,d\tau.$$

Applying the Laplace transform we have

$$Y(s) = -\frac{1}{s^2} + \left(\frac{1}{s^2+1}\right)Y(s)$$

which implies

$$Y(s) = \frac{-(s^2+1)}{s^4} = -\frac{1}{s^2} - \frac{1}{s^4}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t) = -\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = -t - \frac{1}{6}t^3.$$

Example 4.50. Consider a mixed integro-differential equation

$$y'(t) = 1 - \int_0^t y(t-\tau)e^{-2\tau}\,d\tau, \quad y(0) = 1.$$

Applying the Laplace transform we have

$$sY(s) - 1 = \frac{1}{s} - \left(\frac{1}{s+2}\right)Y(s)$$

which implies

$$\left(s + \frac{1}{s+2}\right) Y(s) = \frac{1}{s} + 1$$

or

$$\left(\frac{s^2 + 2s + 1}{s+2}\right) Y = \frac{s+1}{s}.$$

Solving for Y and noting that $s^2 + 2s + 1 = (s+1)^2$ we arrive at

$$Y = \frac{(s+1)(s+2)}{s(s+1)^2} = \frac{(s+2)}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}.$$

So we have

$$Y(s) = \frac{2}{s} - \frac{1}{s+1}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t) = 2 - e^{-t}.$$

Example 4.51. Consider the equation

$$y(t) = 2t - \int_0^t y(t-\tau) e^{\tau} \, d\tau.$$

Applying the Laplace transform we have

$$Y(s) = \frac{2}{s^2} - \left(\frac{1}{s-1}\right) Y(s)$$

which implies

$$\left(1 + \frac{1}{s-1}\right) Y = \frac{2}{s^2}$$

or

$$\left(\frac{s}{s-1}\right) Y = \frac{2}{s^2}$$

or

$$Y(s) = \frac{2(s-1)}{s^3} = \frac{2}{s^2} - \frac{2}{s^3}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t) = 2t - t^2.$$

Example 4.52. Consider the equation

$$y(t) = 2t - \int_0^t y(t-\tau)e^{\tau}\, d\tau.$$

Applying the Laplace transform we have

$$Y(s) = \frac{2}{s^2} - \left(\frac{1}{s-1}\right) Y(s)$$

which implies

$$\left(1 + \frac{1}{s-1}\right) Y = \frac{2}{s^2}$$

or

$$\left(\frac{s}{s-1}\right) Y = \frac{2}{s^2}$$

or

$$Y(s) = \frac{2(s-1)}{s^3} = \frac{2}{s^2} - \frac{2}{s^3}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t) = 2t - t^2.$$

Example 4.53. Consider the equation

$$y(t) + \int_0^t (t-\tau)y(\tau)\, d\tau = t.$$

Applying the Laplace transform we have

$$Y(s) + \frac{1}{s^2}Y(s) = \frac{1}{s^2}$$

which implies

$$\left(1+\frac{1}{s^2}\right)Y(s)=\frac{1}{s^2}$$

or

$$\left(\frac{s^2+1}{s^2}\right)Y(s)=\frac{1}{s^2}$$

or

$$Y(s)=\frac{1}{s^2+1}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t)=\sin(t).$$

Example 4.54. Consider the equation

$$y(t)-\int_0^t y(\tau)\,d\tau=e^{2t}.$$

Applying the Laplace transform we have

$$Y(s)+\frac{1}{s}Y(s)=\frac{1}{s-2}$$

which implies

$$\left(1+\frac{1}{s}\right)Y(s)=\frac{1}{s-2}$$

or

$$\left(\frac{s+1}{s}\right)Y(s)=\frac{1}{s-2}$$

or

$$Y(s)=\frac{s}{(s+1)(s-2)}=\frac{-1}{s-1}+\frac{2}{s-2}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t)=2e^{2t}-e^{t}.$$

Example 4.55. Consider the equation

$$y'(t) = 1 - \int_0^t y(\tau)\,d\tau, \quad y(0) = 1.$$

Applying the Laplace transform we have

$$sY(s) - 1 = \frac{1}{s} - \frac{1}{s}Y(s)$$

which implies

$$\left(s + \frac{1}{s}\right)Y(s) = \frac{1}{s} + 1$$

or

$$\left(\frac{s^2+1}{s}\right)Y(s) = \frac{s+1}{s}$$

or

$$Y(s) = \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}.$$

Applying the inverse Laplace transform to both sides we get

$$y(t) = \cos(t) + \sin(t).$$

Periodic Functions

Theorem 4.7. *If $f(t)$ is a periodic function with period $T > 0$, i.e., $f(t+T) = f(t)$ for all $t > 0$ then*

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

Proof. Assume that $f(t)$ is a periodic function with period $T > 0$, i.e., $f(t+T) = f(t)$ for all $t > 0$. Then we have

$$\begin{aligned}\mathcal{L}(f(t)) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&(\text{ Set } t = \tau + T \ \Rightarrow dt = d\tau) \\
&= \int_0^T e^{-st} f(t)\, dt + \int_0^\infty e^{-s(\tau+T)} f(\tau+T)\, d\tau \\
&= \int_0^T e^{-st} f(t)\, dt + e^{-sT} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau \\
&= \int_0^T e^{-st} f(t)\, dt + e^{-sT} \mathcal{L}(f(t)).
\end{aligned}
$$

From this we obtain

$$\mathcal{L}(f(t)) = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}.$$

□

Example 4.56. Consider the function $f(t) = \begin{cases} 1, & \text{for } 0 < t < 1 \\ 0 & \text{for } 1 \le t < 2 \end{cases}$ and $f(t+2) = f(t)$ for all t. Then $f(t)$ is periodic with period $T = 2$ and we have

$$\mathcal{L}(f(t)) = \frac{\int_0^2 e^{-st}\, t\, dt}{1 - e^{-2s}}.$$

In order to evaluate the integral in the numerator we need to use integration by parts

$$
\begin{aligned}
\int_0^2 e^{-st} f(t)\, dt &= \int_0^1 e^{-st}\, dt \\
&= \left.\frac{e^{-st}}{-s}\right|_0^1 = \frac{(1 - e^{-s})}{s}
\end{aligned}
$$

So we have

$$\mathcal{L}(f(t)) = \frac{1 - e^{-s}}{s(1 - e^{-2s})} = \frac{1}{s(1 + e^{-s})}.$$

Example 4.57. Consider the function $f(t) = \begin{cases} 1, & \text{for } 0 < t < 1 \\ -1 & \text{for } 1 \le t < 2 \end{cases}$ and $f(t+2) = f(t)$ for all t. Then $f(t)$ is periodic with period $T = 2$ and we have

$$\mathcal{L}(f(t)) = \frac{\int_0^2 e^{-st}\, t\, dt}{1 - e^{-2s}}.$$

In order to evaluate the integral in the numerator we need to use integration by parts

$$\begin{aligned}\int_0^2 e^{-st} f(t)\,dt &= \int_0^1 e^{-st}\,dt - \int_1^2 e^{-st}\,dt \\ &= \frac{e^{-st}}{-s}\Big|_0^1 - \frac{e^{-st}}{-s}\Big|_1^2 = \frac{(1-s^{-s})}{s} + \frac{(e^{-s}-e^{-2s})}{s} \\ &= \frac{(1-e^{-s})^2}{s((1-e^{-2s})}\end{aligned}$$

So we have

$$\mathcal{L}(f(t)) = \frac{(1-e^{-s})^2}{s(1-e^{-2s})} = \frac{(1-e^{-s})}{s(1+e^{-s})}.$$

Example 4.58. Consider the function $f(t) = t$ for $0 < t < 2$ and $f(t+2) = f(t)$ for all t. Then $f(t)$ is periodic with period $T = 2$ and we have

$$\mathcal{L}(f(t)) = \frac{\int_0^2 e^{-st}\,t\,dt}{1-e^{-2s}}.$$

In order to evaluate the integral in the numerator we need to use integration by parts

$$\begin{aligned}\int_0^2 e^{-st}\,t\,dt &= \int_0^2 \left(\frac{e^{-st}}{-s}\right)' t\,dt \\ &= t\left(\frac{e^{-st}}{-s}\right)\Big|_0^2 + \frac{1}{s}\int_0^2 e^{-st}\,dt \\ &= -\frac{2e^{-2s}}{s} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\Big|_0^2 \\ &= -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} \\ &= \frac{1-2se^{-2s}-e^{-2s}}{s^2}.\end{aligned}$$

So we have

$$\mathcal{L}(f(t)) = \frac{1-2se^{-2s}-e^{-2s}}{s^2(1-e^{-2s})}.$$

Example 4.59. Consider the function $f(t) = \begin{cases} \sin(t), & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi \le t < 2\pi \end{cases}$ and $f(t+2\pi) = f(t)$

for all t. Then $f(t)$ is periodic with period $T = 2\pi$ and we have

$$\mathcal{L}(f(t)) = \frac{\int_0^2 e^{-st}\, t\, dt}{1 - e^{-2s}}.$$

In order to evaluate the integral in the numerator we need to use integration by parts

$$\int_0^{2\pi} e^{-st}\, f(t)\, dt = \int_0^{\pi} e^{-st} \sin(t)\, dt$$

To do this integral we need to use integration by parts:

$$\begin{aligned}
\int_0^{\pi} e^{-st}\, \sin(t)\, dt &= \int_0^{\pi} \left(\frac{e^{-st}}{-s}\right)' \sin(t)\, dt \\
&= \left(\frac{e^{-st}}{-s}\right) \sin(t)\Big|_0^{\pi} + \frac{1}{s}\int_0^{\pi} e^{-st} \cos(t)\, dt \\
&= \frac{1}{s}\int_0^{\pi} \left(\frac{e^{-st}}{-s}\right)' \cos(t)\, dt \\
&= \frac{1}{s}\left[\left(\frac{e^{-st}}{-s}\right) \cos(t)\Big|_0^{\pi} - \int_0^{\pi} \left(\frac{e^{-st}}{-s}\right) (-\sin(t))\, dt\right] \\
&= \frac{1}{s^2}(1 + e^{-\pi s}) - \frac{1}{s^2}\int_0^{\pi} e^{-st}\, \sin(t)\, dt
\end{aligned}$$

So

$$\int_0^{\pi} e^{-st}\, \sin(t)\, dt = \frac{(1 + e^{-\pi s})}{s^2}.$$

So we have

$$\mathcal{L}(f(t)) = \frac{(1 + e^{-\pi s})}{s^2(1 - e^{-2\pi s})} = \frac{1}{s^2((1 - e^{-\pi s})}.$$

4.5 The Dirac Delta Function

In this section we cover applications of Laplace transforms involving the Dirac Delta Function. The first thing you should know is that the “Dirac Delta Function” is not a function. This will become clear from its defining property. What is it? It is a limit of functions in a special sense which we now describe. Let $a > 0$ be given and define

$$\delta_n(t-a) = \begin{cases} 0, & 0 < t < a \\ n, & a \le t < a + 1/n \\ 0, & t \ge a + 1/n \end{cases} .$$

Then we define the Dirac Delta Function supported at $t = a$, $\delta(t-a)$, applied to a continuous function $f(t)$ by the formula

$$\int \delta(t-a)\, f(t)\, dt = \lim_{n\to\infty} \int \delta_n(t-a)\, f(t)\, dt.$$

Theorem 4.8. *For any continuous function $f(t)$ we have*

$$\int \delta(t-a)\, f(t)\, dt = f(a).$$

Proof. We need to show that

$$\lim_{n\to\infty} \int \delta_n(t-a)\, f(t)\, dt = f(a).$$

To this end we first define

$$F(t) = \int_0^t f(\tau)\, d\tau.$$

Then, from the fundamental theorem of calculus we have $F'(t) = f(t)$.

We also have

$$\begin{aligned} \lim_{n\to\infty} \int \delta_n(t-a)\, f(t)\, dt &= \lim_{n\to\infty} n \int_a^{a+1/n} f(t)\, dt \\ &= \lim_{n\to\infty} \frac{F(a+1/n) - F(a)}{1/n} \\ &= F'(a) = f(a). \end{aligned}$$

□

Example 4.60. Consider a few integrals involving the delta function which are easily computed using the definition and property that $\delta(t-c)$ is zero everywhere except at c.

$$\int_a^b f(t)\delta(t-c)\,dt = \begin{cases} f(c), & \text{if } c \in [a,b] \\ 0, & \text{if } c \notin [a,b]. \end{cases}$$

1. $\displaystyle\int_{-\infty}^{\infty} (t^2-1)\delta(t)\,dt = -1$
2. $\displaystyle\int_{-\infty}^{\infty} \sin(3t)\delta(t-\pi/2)\,dt = \sin(3\pi/2)$
3. $\displaystyle\int_{0}^{\infty} e^{-2t}\delta(t-1)\,dt = e^{-2}$
4. $\displaystyle\int_{3}^{\infty} (t^2+2)\delta(t-2)\,dt = 0$ because $t=2$ in not in the interval $[3,\infty)$ and $\delta(t-2)$ is zero everywhere except at $t=2$.

Now we consider the Laplace transform of the delta function.

Theorem 4.9.

$$\mathcal{L}\left(\delta(t-a)\right) = e^{-as}.$$

Proof. We only need to apply the result of Theorem 4.8

$$\mathcal{L}(\delta(t-a)) = \int_0^t e^{-as}\delta(t-a)\,dt = e^{-as}.$$

□

Example 4.61. Consider the differential equation $y'' + y = \delta(t-\pi)$ with $y(0) = 0$ and $y'(0) = 0$. Applying the Laplace transform we have

$$(s^2Y - 0s - 0) + Y = e^{-\pi s} \quad \Rightarrow \quad Y = e^{-\pi s}\left(\frac{1}{s^2+1}\right).$$

Applying the inverse Laplace transform and using the second shift theorem we have

$$y(t) = u(t-\pi)\sin(t-\pi) = -u(t-\pi)\sin(t).$$

Example 4.62. Consider the differential equation $y'' + 5y' + 6y = e^{-t}\delta(t-2)$ with $y(0) = 2$ and $y'(0) = -5$. Applying the Laplace transform we have

$$(s^2Y - 2s + 5) + 5(sY - 2) + 6Y = e^{-2(s+1)} \quad \Rightarrow \quad (s^2 + 5s + 6)Y = (2s + 5) + e^{-2(s+1)}.$$

Note that $(s^2Y - 2s + 5) = (s+2)(s+3)$ so we have

$$Y = \frac{(2s+5)}{(s+2)(s+3)} + \frac{e^{-2(s+1)}}{(s+2)(s+3)}.$$

Next we apply partial fractions to each term to get

$$\frac{(2s+5)}{(s+2)(s+3)} = \frac{1}{(s+2)} + \frac{1}{(s+3)} \text{ and } \frac{1}{(s+2)(s+3)} = \frac{1}{(s+2)} - \frac{1}{(s+3)}.$$

Applying the inverse Laplace transform and using the second shift theorem we have

$$y(t) = \left(e^{-2t} + e^{-3t}\right) + u(t-2)\left(e^{-2(t-2)} - e^{-3(t-4/3)}\right).$$

Example 4.63. Consider the differential equation $y'' + y = \delta(t - \pi/2) + \delta(t - 3\pi/2)$ with $y(0) = 1$ and $y'(0) = 0$. Applying the Laplace transform we have

$$(s^2Y - s) + Y = e^{-\pi/2s} + e^{-3\pi/2s} \quad \Rightarrow \quad (s^2 + 1)Y = s + e^{-\pi/2s} + e^{-3\pi/2s}.$$

So we get

$$Y = \frac{s}{s^2+1} + e^{-\pi/2s}\left(\frac{1}{s^2+1}\right) + e^{-3\pi/2s}\left(\frac{1}{s^2+1}\right).$$

Applying the inverse Laplace transform and using the second shift theorem we have

$$y(t) = \cos(t) + u(t - \pi/2)\sin(t - \pi/2) + u(t - 3\pi/2)\sin(t - 3\pi/2).$$

Example 4.64. Consider the differential equation $y'' + 4y' + 5y = \delta(t - 2\pi)$ with $y(0) = 0$ and $y'(0) = 2$. Applying the Laplace transform we have

$$(s^2Y - 2) + 4sY + 5Y = e^{-2\pi s} \quad \Rightarrow \quad (s^2 + 4s + 5)Y = 2 + e^{-2\pi s}.$$

$$Y = \frac{2}{(s^2 + 4s + 5)} + \frac{e^{-2\pi s}}{(s^2 + 4s + 5)}.$$

We need to complete the square: $(s^2 + 4s + 5) = (s + 2)^2 + 1$ so we have

$$Y = \frac{2}{(s + 2)^2 + 1} + \frac{e^{-2\pi s}}{(s + 2)^2 + 1}.$$

Applying the inverse Laplace transform and using the first and second shift theorems we have

$$y(t) = 2e^{-2t} \sin(t) + u(t - 2\pi)e^{-2(t-2\pi)} \sin(t - 2\pi).$$

Table of Laplace Transforms

$f(t)$ for $t \geq 0$	$\widehat{f} = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$ $(n = 0, 1, \ldots)$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$ $(a > 0)$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
$f'(t)$	$s\mathcal{L}(f) - f(0)$
$f''(t)$	$s^2\mathcal{L}(f) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^n\mathcal{L}(f) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0)$
$t^n f(t)$	$(-1)^n \frac{d^n F}{ds^n}(s)$
$e^{at} f(t)$	$\mathcal{L}(f)(s-a)$
$u(t-a) = \begin{cases} 0 & t \leq a \\ 1 & t > a \end{cases}$	$\frac{e^{-as}}{s}$
$u(t-a)f(t-a)$	$e^{-as}\mathcal{L}(f)(s)$
$\delta(t-a)$	e^{-as}
$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau$	$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$