Sppring 2020. M3350.

Online Zoom Notes

(Suplements to Dr. Gilliam's notes)

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Last update: May 5, 2020.

Contents

CHAPTER 4

The Laplace Transform

4.1. Definition of the Laplace Transform

Summary. Formulas (2), (3), and table on page 215.

Definition:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Linear property:

$$\mathcal{L}\{\alpha f + \beta g\}(s) = \alpha\mathcal{L}\{f\}(s) + \beta\mathcal{L}\{g\}(s).$$

Factorials:

$$0! = 1, \quad 1! = 1, \quad 2! = 1(2) = 2, \quad 3! = 1(2)(3) = 6, \ldots$$

$$n! = (n-1)!n = 1(2)\ldots(n-1)(n).$$

Example:

$$\mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2!}\cdot\frac{2!}{s^3}\right\} = \frac{3}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{3}{2}t^2.$$

4.2. The Inverse Laplace Transform and Transforms of Derivatives

Summary. Theorem 4.2.1, formula (1), Theorem 4.2.2, also Remark on pages 223 and 224.

4.2.1. Inverse Laplace transforms.

Partial fractions: Prof. Gilliam's notes.

EXAMPLE 4.1.

$$F(s) = \frac{4s}{s^2+2s-3} = \frac{4s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}.$$

Then

$$\mathcal{L}^{-1}\{F\} = Ae^{-3t} + Be^t.$$

Find A, B.

(a) Direct method:

$$\frac{4s}{(s+3)(s-1)} = \frac{A(s-1)+B(s+3)}{(s+3)(s-1)}.$$

Then

$$4s = A(s-1)+B(s+3).$$

Either equate coefficients or select values.

For the latter method:

$s=-3$: $4(-3) = A(-3-1)$, hence $A = 3$.

$s=1$: $4(1) = B(1+3)$, hence $B=1$.

(b) Cover-up:

$$\frac{4s}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1}.$$

For A, take $s=-3$:

$$A = \frac{4(-3)}{(-3-1)} = 3$$

For B, take $s=1$,

$$B = \frac{4(1)}{(1+3)} = 1.$$

Example 4.2.

$$\frac{5}{s^3+5s} = \frac{5}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5} = \frac{A(s^2+5)+(Bs+C)s}{s(s^2+5)}.$$

Then

$$5 = A(s^2+5)+(Bs+C)s.$$

Take $s=0$: $A=1$.

Take $s=1$: $5 = 6A+B+C$, $B+C=-1$.

Take $s=-1$: $5=6A-(C-B)$, $B-C=-1$.

Hence $C=0$, $B=-1$.

Example 4.3.

$$F(s) = \frac{2s}{(s-2)(s-3)(s-6)} = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6}.$$

Then

$$\mathcal{L}^{-1}\{F\} = Ae^{2t} + Be^{3t} + Ce^{6t}.$$

Cover-up method:

$$A = \frac{2(2)}{(2-3)(2-6)} = \frac{4}{(-1)(-4)} = 1,$$

$$B = \frac{2(3)}{(3-2)(3-6)} = \frac{6}{-3} = -2,$$

$$C = \frac{2(6)}{(6-2)(6-3)} = \frac{12}{(4)(3)} = 1.$$

EXAMPLE 4.4.

$$F(s) = \frac{2s-4}{s(s+1)(s^2+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}.$$

Then

$$\mathcal{L}^{-1}\{F\} = A + Be^{-t} + C\cos(t) + D\sin(t).$$

Find A, B, C, D. We have

$$2s - 4 = A(s+1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s+1).$$

Take $s = 0$: $-4 = A$.

Take $s = -1$: $-6 = B(-1)(2) = -2B$, then $B = 3$.

Take $s = 1$:

$$-2 = -4(2)(2) + 3(1)(2) + (C+D)(1)(2) = -16 + 6 + 2(C+D) = -10 + 2(C+D),$$

then

$$C + D = 4.$$

Take $s = 2$:

$$0 = -4(3)(5) + 3(2)(5) + (2C+D)(2)(3) = -60 + 30 + 6(2C+D) = -30 + 6(2C+D).$$

Then

$$2C + D = 5.$$

We obtain $C = 1$, $D = 3$.

Another way, after A, B.

For s^3: $0 = A + B + C = -4 + 3 + C$. Then $C = 1$.

For s: $2 = A + B + D = -4 + 3 + D$. Then $D = 3$.

4.2.2. Laplace transform of derivatives.

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - 1 \cdot f(0),$$

$$\mathcal{L}\{f''\}(s) = s^2\mathcal{L}\{f\}(s) - s \cdot f(0) - 1 \cdot f'(0),$$

$$\mathcal{L}\{f'''\}(s) = s^3\mathcal{L}\{f\}(s) - s^2 \cdot f(0) - s \cdot f'(0) - 1 \cdot f''(0),$$

$$\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\}(s) - s^{n-1} \cdot f(0) - s^{n-2} \cdot f'(0) - \ldots - s \cdot f^{(n-2)}(0) - 1 \cdot f^{(n-1)}(0).$$

Application to solving ODE.

$$Ay'' + By' + Cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Setp 1. Let $Y(s) = \mathcal{L}\{y\}(s)$, $F(s) = \mathcal{L}\{f\}(s)$.

Applying $\mathcal{L}$ to the ODE gives

$$\mathcal{L}\{Ay'' + By' + Cy\} = \mathcal{L}\{f(t)\},$$

$$A\mathcal{L}\{y''\} + B\mathcal{L}\{y'\} + C\mathcal{L}\{y\}\} = F(s),$$

$$A(s^2Y - sy(0) - y'(0)) + B(sY - y(0)) + CY = F(s),$$

$$(As^2 + Bs + C)Y - A(sy_0 + y_1) - By_0 = F(s).$$

Setp 2. Solve for $Y(s)$.

Setp 3. Solution of the ODE is $y(t) = \mathcal{L}^{-1}\{Y\}$. Calculatte $y(t)$.

4.3. The Shift/Translation Theorems

Summary. Theorem 4.3.1, formula (1), Definition 4.3.1, Theorem 4.3.2, formula (14), (15), (16).

Let

$$\mathcal{L}\{f(t)\}(s) = F(s).$$

4.3.1. Shift in s-variable. Then

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a),$$

$$\mathcal{L}^{-1}\{F(s-a)\}(t) = e^{at}f(t).$$

EXAMPLE 4.5.

$$\mathcal{L}\left\{t^3\right\}(s) = \frac{3!}{s^4}.$$

Then

$$\mathcal{L}\left\{e^{5t}t^3\right\}(s) = \mathcal{L}\left\{t^3\right\}(s-5) = \frac{3!}{(s-5)^4},$$

$$\mathcal{L}\left\{e^{-2t}t^3\right\}(s) = \mathcal{L}\left\{t^3\right\}(s+2) = \frac{3!}{(s+2)^4}.$$

EXAMPLE 4.6.

$$\mathcal{L}\left\{\cos(2t)\right\}(s) = \frac{s}{s^2+4}.$$

Then

$$\mathcal{L}\left\{e^{3t}\cos(2t)\right\}(s) = \mathcal{L}\left\{\cos(2t)\right\}(s-3) = \frac{(s-3)}{(s-3)^2+4},$$

$$\mathcal{L}\left\{e^{-4t}\cos(2t)\right\}(s) = \mathcal{L}\left\{\cos(2t)\right\}(s+4) = \frac{(s+4)}{(s+4)^2+4}.$$

EXAMPLE 4.7.

$$\mathcal{L}\left\{\sin(5t)\right\}(s) = \frac{5}{s^2+25}.$$

Then

$$\mathcal{L}\left\{e^{t}\sin(5t)\right\}(s) = \mathcal{L}\left\{\sin(5t)\right\}(s-1) = \frac{5}{(s-1)^2+25},$$

$$\mathcal{L}\left\{e^{-7t}\sin(5t)\right\}(s) = \mathcal{L}\left\{\sin(5t)\right\}(s+7) = \frac{5}{(s+7)^2+25}.$$

EXAMPLE 4.8.

$$s^2+2s+5 = s^2+2s+1^2-1^2+5 = (s+1)^2+4.$$

$$s^2+6s+34 = s^2+6s+3^2-3^2+34 = (s+3)^2+25.$$

$$F(s) = \frac{2s+5}{(s+3)^2+5^2} = \frac{2[(s+3)-3]+5}{(s+3)^2+5^2} = \frac{2(s+3)-6+5}{(s+3)^2+5^2} = \frac{2(s+3)-1}{(s+3)^2+5^2}$$

$$= \frac{2(s+3)}{(s+3)^2+5^2} - \frac{1}{5}\cdot\frac{5}{(s+3)^2+5^2}.$$

Then

$$\mathcal{L}^{-1}\left\{F\right\}(t) = 2e^{-3t}\cos(5t) - \frac{1}{5}e^{-3t}\sin(5t).$$

4.3.2. Shift in t-variable. Unit step/Heaviside function $u(t-a)$. Then

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t)\}(s),$$

$$\mathcal{L}\{f(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{f(t+a)\}(s),$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\}(t) = f(t-a)u(t-a).$$

Note

$$\mathcal{L}\{u(t-a)\}(s) = e^{-as}\mathcal{L}\{1\}(s) = \frac{e^{-as}}{s}.$$

EXAMPLE 4.9. $f(t) = 2 + u(t-3)(t-2) + u(t-4)(e^t - t)$.

We have

$$\mathcal{L}\left\{2\right\} = \frac{2}{s},$$

$$\mathcal{L}\left\{u(t-3)(t-2)\right\}(s) = e^{-3s}\mathcal{L}\left\{(t+3)-2\right\}(s) = e^{-3s}\mathcal{L}\left\{t+1\right\}(s) = e^{-3s}\left(\frac{1}{s^2}+\frac{1}{s}\right),$$

$$\begin{aligned}\mathcal{L}\left\{u(t-4)(e^t-t)\right\}(s) &= e^{-4s}\mathcal{L}\left\{e^{t+4}-(t+4)\right\}(s) = e^{-4s}\Big(e^4\mathcal{L}\left\{e^t\right\} - \mathcal{L}\left\{t\right\} - 4\mathcal{L}\left\{1\right\}\Big)\\ &= e^{-4s}\left(\frac{e^4}{s-1} - \frac{1}{s^2} - \frac{4}{s}\right).\end{aligned}$$

Hence,

$$\mathcal{L}\left\{f\right\}(s) = \frac{2}{s} + e^{-3s}\left(\frac{1}{s^2}+\frac{1}{s}\right) + e^{-4s}\left(\frac{e^4}{s-1}+\frac{1}{s^2}-\frac{4}{s}\right).$$

EXAMPLE 4.10.

$$F(s) = \frac{e^{-2s}}{s^2+9}.$$

Let

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t).$$

Then

$$\mathcal{L}^{-1}\left\{F(s)\right\}(t) = u(t-2)g(t-2) = u(t-2)\frac{1}{3}\sin(3(t-2)) = \frac{1}{3}u(t-2)\sin(3t-6).$$

EXAMPLE 4.11 (Example 4.35 in Prof. Gilliam's notes).

$$Y = \frac{s^2+27}{s^2(s-3)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{(s-3)^2} = \frac{As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2}{s^2(s-3)^2},$$

then

$$s^2 + 27 = As(s-3)^2 + B(s-3)^2 + Cs^2(s-3) + Ds^2.$$

Take $s = 0$:

$$27 = 9B, \quad B = 3.$$

Take $s = 3$:

$$9 + 27 = 9D, \quad D = 4.$$

Take $s = 1$:

$$28 = 4A + 4B - 2C + D = 4A + 12 - 2C + 4, \quad 2A - C = 6.$$

Take $s = -1$:

$$28 = -16A + 16B - 4C + D = -16A + 48 - 4C + 4, \quad 4A + C = 6.$$

Hence $A = 2$, $C = -2$. Thus,

$$Y = \frac{2}{s} + \frac{3}{s^2} - \frac{2}{s-3} + \frac{4}{(s-3)^2}.$$

EXAMPLE 4.12 (Example 4.36 in Prof. Gilliam's notes).

$$Y = \frac{2}{s(s^2-2s+2)} = \frac{2}{s((s-1)^2+1)} = \frac{A}{s} + \frac{B(s-1)+C}{(s-1)^2+1} = \frac{A[(s-1)^2+1] + s[B(s-1)+C]}{s((s-1)^2+1)},$$

then

$$2 = A[(s-1)^2+1] + s[B(s-1)+C].$$

Take $s = 0$: $2 = 2A$, then $A = 1$.

Take $s = 1$:

$$2 = A + C = 1 + C, \quad C = 1.$$

Take $s = 2$:

$$2 = 2A + 2(B+C), \quad 1 = A + B + C = 2 + B, \quad B = -1.$$

Then

$$Y = \frac{1}{s} - \frac{(s-1)}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}.$$

Solution

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = 1 - e^t\cos(t) + e^t\sin(t).$$

4.4. Additional Operational Properties

4.4.1. Derivative of the Laplace transform.

$$\frac{d}{ds}\mathcal{L}\{f(t)\}(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty (-t)e^{-st}f(t)dt = -\mathcal{L}\{tf(t)\}(s).$$

Thus,

$$\frac{d}{ds}\mathcal{L}\{f(t)\}(s) = -\mathcal{L}\{tf(t)\}(s).$$

Then

$$\frac{d^2}{ds^2}\mathcal{L}\{f(t)\}(s) = \frac{d}{ds}\left(\frac{d}{ds}\mathcal{L}\{f(t)\}(s)\right) = -\frac{d}{ds}\left(\mathcal{L}\{tf(t)\}(s)\right)$$
$$= -\left(-\mathcal{L}\{t\cdot tf(t)\}(s)\right) = 1\cdot\mathcal{L}\left\{t^2f(t)\right\}(s).$$

Therefore,

$$\frac{d^2}{ds^2}\mathcal{L}\{f(t)\}(s) = \mathcal{L}\left\{t^2f(t)\right\}(s).$$

Repeating this gives

$$\frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s) = (-1)^n\mathcal{L}\{t^nf(t)\}(s),$$
$$\mathcal{L}\{t^nf(t)\}(s) = (-1)^n\frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s).$$

4.4.2. Laplace transform of integrals.

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{1}{s}\mathcal{L}\{f(t)\}(s).$$

Later, we can use convolution:

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \mathcal{L}\{1*f\} = \mathcal{L}\{1\}\mathcal{L}\{f\} = \frac{1}{s}\mathcal{L}\{f(t)\}(s).$$

Convolutions.

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

We have

$$(f*g)(t) = (g*f)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Also,

$$(f * g) * h = f * (g * h).$$

Let $F(s) = \mathcal{L}\{f\}$ and $G(s) = \mathcal{L}\{g\}$. Then

$$\begin{aligned} \mathcal{L}\{f * g\} &= F \cdot G, \\ \mathcal{L}^{-1}\{F \cdot G\} &= f * g. \end{aligned}$$

EXAMPLE 4.13.

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{14}{(s+2)(s-5)}\right\} &= 14\mathcal{L}^{-1}\left\{\frac{1}{s+2}\cdot\frac{1}{s-5}\right\} \\ &= 14\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = 14e^{-2t} * e^{5t} \\ &= 14\int_0^t e^{-2\tau}e^{5(t-\tau)}d\tau = 14e^{5t}\int_0^t e^{-7\tau}d\tau \\ &= \frac{14e^{5t}}{-7}(e^{-7t}-1) = -2(e^{-2t}-e^{5t}). \end{aligned}$$

Trigonometric identities.

$$\begin{aligned} \cos(a)\cos(b) &= \frac{1}{2}\Big(\cos(a+b)+\cos(a-b)\Big), \\ \sin(a)\sin(b) &= \frac{1}{2}\Big(\cos(a-b)-\cos(a+b)\Big), \\ \sin(a)\cos(b) &= \frac{1}{2}\Big(\sin(a+b)+\sin(a-b)\Big). \end{aligned}$$

Example 4.14 (Example 4.48 in Prof. Gilliam's notes).

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\cdot\frac{1}{s^2+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = (\cos t) * (\sin t) \\
&= \int_0^t \cos\tau \sin(t-\tau)d\tau = \frac{1}{2}\int_0^t (\sin t + \sin(t-2\tau))d\tau \\
&= \frac{1}{2}t\sin t - \frac{1}{2}\int_0^t \sin(2\tau - t)d\tau \\
&= \frac{1}{2}t\sin t + \frac{1}{2}\cdot\frac{1}{2}\cos(2\tau - t)\Big|_{\tau=0}^{\tau=t} \\
&= \frac{1}{2}t\sin t + \frac{1}{4}(\cos t - \cos(-t)) \\
&= \frac{1}{2}t\sin t.
\end{aligned}$$

Example 4.15.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)^2}\right\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\cdot\frac{2}{s^2+4}\right\} \\
&= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{4}\sin(2t) * \sin(2t) \\
&= \frac{1}{4}\int_0^t \sin(2\tau)\sin(2(t-\tau)d\tau = \frac{1}{4}\cdot\frac{1}{2}\int_0^t (\cos(4\tau - 2t) - \cos(2t))d\tau \\
&= \frac{1}{8}\left\{\frac{1}{4}\sin(4\tau - 2t)\Big|_{\tau=0}^{\tau=t} - t\cos(2t)\right\} \\
&= \frac{1}{8}\left\{\frac{1}{4}[\sin(2t) - \sin(-2t)] - t\cos(2t)\right\} \\
&= \frac{1}{8}\left\{\frac{1}{2}\sin(2t) - t\cos(2t)\right\}.
\end{aligned}$$

Example 4.16.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+3)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\cdot\frac{1}{s+3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} \\
&= \frac{1}{2}t^2 * e^{-3t} = \frac{1}{2}\int_0^t \tau^2 e^{-3(t-\tau)}d\tau
\end{aligned}$$

Using integration by parts

alternate sign	derivative in τ	anti-derivative in τ
$+$	τ^2	$e^{-3(t-\tau)}$
$-$	2τ	$\frac{1}{3}e^{-3(t-\tau)}$
$+$	2	$\frac{1}{9}e^{-3(t-\tau)}$
$-$	0	$\frac{1}{27}e^{-3(t-\tau)}$

Then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+3)}\right\} &= \frac{1}{2}\left\{\tau^2\frac{1}{3}e^{-3(t-\tau)}\Big|_{\tau=0}^{\tau=t} - 2\tau\frac{1}{9}e^{-3(t-\tau)}\Big|_{\tau=0}^{\tau=t} + \frac{2}{27}e^{-3(t-\tau)}\Big|_{\tau=0}^{\tau=t} - 0\right\}\\ &= \frac{1}{2}\left\{\frac{t^2}{3} - \frac{2t}{9} + \frac{2}{27}(1-e^{-3t})\right\}.\end{aligned}$$

Example 4.17.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{2}{s^2+4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}\\ &= \frac{1}{2}t * \sin(2t) = \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)d\tau\end{aligned}$$

Using integration by parts

alternate sign	derivative in τ	anti-derivative in τ
$+$	$t-\tau$	$\sin(2\tau)$
$-$	-1	$-\frac{1}{2}\cos(2\tau)$
$+$	0	$-\frac{1}{4}\sin(2\tau)$

Then

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} &= \frac{1}{2}\left\{-\frac{1}{2}(t-\tau)\cos(2\tau)\Big|_{\tau=0}^{\tau=t} - \frac{1}{4}\sin(2\tau)\Big|_{\tau=0}^{\tau=t} + 0\right\}\\ &= \frac{1}{2}\left\{\frac{1}{2}t - \frac{1}{4}(\sin(2t)-0)\right\}.\end{aligned}$$

Volterra Integral Equations.

Example 4.18. Solve for $y(t)$:

$$y(t) = -t + \int_0^t y(t-\tau)\sin(\tau)d\tau.$$

Write

$$y(t) = -t + y(t) * \sin(t).$$

Take Laplace transform of the equation:

$$Y = \mathcal{L}\{y(t)\} = -\mathcal{L}\{t\} + \mathcal{L}\{y(t) * \sin(t)\} = -\frac{1}{s^2} + \mathcal{L}\{y(t)\}\mathcal{L}\{\sin(t)\} = -\frac{1}{s^2} + Y \cdot \frac{1}{s^2+1}.$$

Solve for $Y(s)$. Then solution is $y(t) = \mathcal{L}^{-1}\{Y\}(t)$.

EXAMPLE 4.19. Solve for $y(t)$:

$$y(t) + \int_0^t (t-\tau)y(\tau)d\tau = t.$$

Write

$$y(t) + g(t) * y(t) = t,$$

where $g(t-\tau) = t-\tau$. Then $g(t) = t$. Thus,

$$y(t) + t * y(t) = t,$$

Take Laplace transform of the equation:

$$\mathcal{L}\{y(t)\} + \mathcal{L}\{t * y(t)\} = \mathcal{L}\{t\},$$
$$Y + \mathcal{L}\{t\}\mathcal{L}\{y(t)\} = \frac{1}{s^2},$$
$$Y + \frac{1}{s^2} \cdot Y = \frac{1}{s^2}.$$

Solve for $Y(s)$. Then solution is $y(t) = \mathcal{L}^{-1}\{Y\}(t)$.

4.4.3. Laplace transform of periodic functions. Suppose f is T-periodic, that is $f(t+T) = f(t)$ for all $t \geq 0$.

Define function f_T by

$$f_T(t) = \begin{cases} f(t), & \text{if } 0 \leq t < T, \\ 0, & \text{if } t > T. \end{cases}$$

Then

$$f_T(t) = f(t) - u(t-T)f(t).$$

We have

$$\mathcal{L}\{f_T(t)\} = \mathcal{L}\{f(t)\} - e^{-Ts}\mathcal{L}\{f(t+T)\} = \mathcal{L}\{f(t)\} - e^{-Ts}\mathcal{L}\{f(t)\}.$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1-e^{-Ts}}. \tag{4.1}$$

Note that $\mathcal{L}\{f_T(t)\}(s) = \int_0^T e^{-st} f(t)dt$, then

$$\mathcal{L}\{f(t)\}(s) = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-Ts}}. \tag{4.2}$$

However, we will calculate $\mathcal{L}\{f_T(t)\}$ in (4.1) by different ways, rather than use the the integral in (4.2). (Read also another proof of (4.2) in the textbook or Prof. Gilliam's notes.)

EXAMPLE 4.20. $f(t) = t$ for $0 < t < 2$ and has period $T = 2$. Let

$$f_T(t) = \begin{cases} t, & \text{if } 0 \le t < 2, \\ 0, & \text{if } t > 2. \end{cases}$$

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-2s}}.$$

Now calculate $\mathcal{L}\{f_T(t)\}(s)$. We have

$$f_T(t) = t - tu(t-2).$$

Then

$$\mathcal{L}\{f_T(t)\}(s) = \frac{1}{s^2} - e^{-2s}\mathcal{L}\{t+2\}(s) = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = \frac{\frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)}{1 - e^{-2s}}.$$

EXAMPLE 4.21. Function $f(t)$ has period $T = 2$, and

$$f(t) = \begin{cases} 1, & \text{if } 0 \le t < 1, \\ -1, & \text{if } 1 < t < 2. \end{cases}$$

Let

$$f_T(t) = \begin{cases} f(t), & \text{if } 0 \le t < 2, \\ 0, & \text{if } t > 2 \end{cases} = \begin{cases} 1, & \text{if } 0 \le t < 1, \\ -1, & \text{if } 1 < t < 2, \\ 0, & \text{if } t > 2. \end{cases}$$

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-2s}}.$$

Now calculate $\mathcal{L}\{f_T(t)\}(s)$. We have

$$f_T(t) = 1 - 2u(t-1) + 1 \cdot u(t-2).$$

Then

$$\mathcal{L}\{f_T(t)\}(s) = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}.$$

Therefore,

$$\mathcal{L}\{f(t)\}(s) = \frac{\dfrac{1}{s} - 2\dfrac{e^{-s}}{s} + \dfrac{e^{-2s}}{s}}{1 - e^{-2s}}.$$

Example 4.22. Function $f(t)$ has period $T = 2\pi$, and

$$f(t) = \begin{cases} \sin(t), & \text{if } 0 \le t < \pi, \\ 0, & \text{if } \pi < t < 2\pi. \end{cases}$$

Let

$$f_T(t) = \begin{cases} f(t), & \text{if } 0 \le t < 2\pi, \\ 0, & \text{if } t > 2\pi \end{cases} = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi, \\ 0, & \text{if } \pi < t < 2\pi, \\ 0, & \text{if } t > 2\pi \end{cases} = \begin{cases} \sin(t) & \text{if } 0 \le t < \pi, \\ 0, & \text{if } t > \pi. \end{cases}$$

Then

$$\mathcal{L}\{f(t)\}(s) = \frac{\mathcal{L}\{f_T(t)\}(s)}{1 - e^{-2\pi s}}.$$

Now calculate $\mathcal{L}\{f_T(t)\}(s)$. We have

$$f_T(t) = \sin(t) - \sin(t)u(t-\pi).$$

Then

$$\begin{aligned} \mathcal{L}\{f_T(t)\}(s) &= \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} = \frac{1}{s^2+1} + e^{-\pi s}\mathcal{L}\{\sin(t)\} \\ &= \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} = \frac{1+e^{-\pi s}}{s^2+1}. \end{aligned}$$

Therefore,

$$\mathcal{L}\{f\}(s) = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})} = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})(1+e^{-\pi s})} = \frac{1}{(s^2+1)(1-e^{-\pi s})}.$$

4.5. The Dirac Delta Function

DEFINITION 4.5.1. *The Dirac delta function* $\delta(t)$ *is chracterized by the following two properties.*

(i)

$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0, \\ \infty, & \text{if } t = 0. \end{cases}$$

(ii)

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0),$$

for any function f *which is continuous on an open interval containing* 0.

In fact, $\delta(t)$ is not a function, but rather a *generalized function*.

Let $a \in \mathbb{R}$. Shifting in t by a, we have $\delta(t-a)$. Then, similarly,

(i)

$$\delta(t-a) = \begin{cases} 0, & \text{if } t \neq a, \\ \infty, & \text{if } t = a. \end{cases}$$

(ii)

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a),$$

for any function f which is continuous on an open interval containing a.

For $a \geq 0$,

$$\mathcal{L}\left\{\delta(t-a)\right\}(s) = \int_0^{\infty} e^{-st}\delta(t-a)dt = e^{-as}.$$

When $a = 0$, we have

$$\mathcal{L}\left\{\delta(t)\right\}(s) = 1.$$

In general,

$$\mathcal{L}\left\{f(t)\delta(t-a)\right\}(s) = \int_0^{\infty} e^{-st}f(t)\delta(t-a)dt = f(a)e^{-as}. \tag{4.3}$$

EXAMPLE 4.23. Find $\mathcal{L}\left\{e^{-2t}\delta(t-3)\right\}(s)$.

Solution 1: Using translation in t,

$$\mathcal{L}\left\{e^{-2t}\delta(t-3)\right\}(s) = \mathcal{L}\left\{\delta(t-3)\right\}(s+2) = e^{-3(s+2)}.$$

Solution 2: By definition,

$$\mathcal{L}\left\{e^{-2t}\delta(t-3)\right\}(s) = \int_0^\infty e^{-st}e^{-2t}\delta(t-3)dt = e^{-3s}e^{-2(3)}.$$

Solution 3: By using (4.3),

$$\mathcal{L}\left\{e^{-2t}\delta(t-3)\right\}(s) = e^{-2(3)}e^{-3s}.$$

Example 4.24.

$$\mathcal{L}\left\{\delta(t-1)-\delta(t-3)\right\} = e^{-s}-e^{-3s},$$

$$\mathcal{L}\left\{t\delta(t-1)\right\} = 1\cdot e^{-s},$$

$$\mathcal{L}\left\{\delta(t-\pi)\sin t\right\} = (\sin\pi)e^{-\pi s},$$

or directly,

$$\mathcal{L}\left\{\delta(t-\pi)\sin t\right\} = \int_0^\infty e^{-st}\sin(t)\delta(t-\pi)dt = e^{-\pi s}(\sin\pi).$$

Example 4.25 (Example 4.62 in Prof. Gilliam's notes).

$$\begin{aligned} Y &= \frac{2s+5}{(s+2)(s+3)} + \frac{e^{-2(s+1)}}{(s+2)(s+3)} \\ &= \frac{1}{s+2} + \frac{1}{s+3} + e^{-2}e^{-2s}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\}. \end{aligned}$$

Solution is

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{Y\right\} \\ &= e^{-2t}+e^{-3t}+e^{-2}u(t-2)\left\{e^{-2(t-2)}-e^{-3(t-2)}\right\}. \end{aligned}$$

Remarks.

(a) Consider ODE:

$$ay''+by'+c=\delta(t), \quad y(0)=0=y'(0),$$

where a, b, c are constants.

Let $Y=\mathcal{L}\left\{y(t)\right\}(s)$. Then we have

$$as^2Y+bsY+cY=\mathcal{L}\left\{\delta(t)\right\}=1,$$

$$Y=\frac{1}{as^2+bs+c}.$$

Set

$$H(s) = \frac{1}{as^2 + bs + c}, \quad h(t) = \mathcal{L}^{-1}\{H\}.$$

(H is called the transfer function.)

Then $Y = H$ and

$$y(t) = \mathcal{L}^{-1}\{Y\} = h(t).$$

The solution $h(t)$ is called the reponse to the inpulse δ.

(b) Now, consider ODE:

$$ay'' + by' + c = f(t), \quad y(0) = 0 = y'(0),$$

where a, b, c are constants.

Let $Y = \mathcal{L}\{y(t)\}(s)$. Then we have

$$as^2Y + bsY + cY = \mathcal{L}\{f(t)\} = F(s),$$

$$Y = \frac{1}{as^2 + bs + c} \cdot F(s).$$

Set

$$H(s) = \frac{1}{as^2 + bs + c}, \quad h(t) = \mathcal{L}^{-1}\{H\}.$$

Then $Y = H \cdot F$ and

$$y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\{H \cdot F\} = h(t) * f(t).$$

The solution $y(t)$ is the convolution of the reponse $h(t)$ and the forcing function $f(t)$.

If initial data is $y(0)$ and $y'(0)$ are not specified, then solution

$$y_p = h(t) * f(t)$$

plays the role of a particular solution. Recall the general solution is

$$y = y_p + y_c.$$

(c) Finally, relation between unit step function and Dirc delta function is

$$u'(t - a) = \delta(t - a).$$

APPENDIX A

Some problems

Review p. 18. Find

$$f(t) = \mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s^3} - \frac{2s}{(s-1)^2+9}\right)\right\}.$$

Solution. Let

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}, \quad h(t) = \mathcal{L}^{-1}\left\{\frac{2s}{(s-1)^2+9}\right\}.$$

Then

$$f(t) = u(t-2)\Big(g(t-2) - h(t-2)\Big).$$

Calculations

$$g(t) = \frac{1}{2}t^2,$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{2(s-1)+2}{(s-1)^2+9}\right\} = 2e^t\cos(3t) + \frac{2}{3}e^t\sin(3t).$$

Then

$$f(t) = u(t-2)\Big(\frac{1}{2}(t-2)^2 - 2e^{t-2}\cos(3(t-2)) - \frac{2}{3}e^{t-2}\sin(3(t-2))\Big).$$

HW 9, pb 13. Solve equation

$$y'' + 16y = \begin{cases} 16, & 0 \le t < 3, \\ 0, & 3 \le t, \end{cases} \qquad y(0) = 3, \quad y'(0) = 4.$$

Solution. Write equation as

$$y'' + 16y = 16 - 16u(t-3).$$

Denote $Y = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation gives

$$\begin{aligned}(s^2Y - 3s - 4) + 16Y &= \frac{16}{s} - 16\frac{e^{-3s}}{s},\\ (s^2+16)Y &= 3s + 4 + \frac{16}{s} - 16\frac{e^{-3s}}{s},\\ Y &= \frac{3s+4}{s^2+16} + \frac{16}{s(s^2+16)} - \frac{16e^{-3s}}{s(s^2+16)}\\ &= \frac{3s+4}{s^2+16} + G(s) - G(s)e^{-3s},\end{aligned}$$

where

$$G(s) = \frac{16}{s(s^2+16)}.$$

Let

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{16}{s(s^2+16)}\right\}.$$

Solution is

$$\begin{aligned}y(t) = \mathcal{L}^{-1}\{Y\} &= \mathcal{L}^{-1}\left\{\frac{3s+4}{s^2+16}\right\} + g(t) - u(t-3)g(t-3)\\ &= 3\cos(4t) + \sin(4t) + g(t) - u(t-3)g(t-3).\end{aligned}$$

Using partial fraction:

$$G(s) = \frac{16}{s(s^2+16)} = \frac{A}{s} + \frac{Bs+4C}{s^2+16} = \frac{A(s^2+16) + (Bs+4C)s}{s(s^2+16)}.$$

Then

$$16 = A(s^2+16) + (Bs+4C)s.$$

Take $s = 0$: $16 = 16A$. Then $A = 1$.

Take $s = 1$: $16 = 17A + B + 4C$, hence $B + 4C = -1$.

Take $s = -1$: $16 = 17A - (-B + 4C)$, hence $-B + 4C = 1$.

Solving for B, C gives $C = 0$, $B = -1$.

Thus,

$$g(t) = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+4C}{s^2+16}\right\} = A + B\cos(4t) + C\sin(4t) = 1 - \cos(4t).$$

Therefore, the solution is

$$\begin{aligned}y(t) &= 3\cos(4t) + \sin(4t) + 1 - \cos(4t) - u(t-3)(1 - \cos(4(t-3)))\\ &= 1 + 2\cos(4t) + \sin(4t) - u(t-3)(1 - \cos(4t - 12)).\end{aligned}$$

HW 10, pb 5. Solve equation

$$y(t) + 9\int_0^t (t-v)y(v)dv = -3t.$$

Solution. Write equation as

$$y(t) + 9t * y(t) = -3t.$$

Denote $Y = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the equation gives

$$\begin{aligned}
\mathcal{L}\{y(t)\} + 9\mathcal{L}\{t * y(t)\} &= -3\mathcal{L}\{t\}, \\
Y + 9\mathcal{L}\{t\}\mathcal{L}\{y\} &= -3\mathcal{L}\{t\}, \\
Y + \frac{9}{s^2}Y &= -\frac{3}{s^2}, \\
\frac{s^2+9}{s^2}Y &= -\frac{3}{s^2}, \\
Y &= -\frac{3}{s^2+9}.
\end{aligned}$$

Solution is

$$y(t) = \mathcal{L}^{-1}\{Y\} = -\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = -\sin(3t).$$

APPENDIX B

Corrections for Review Notes

These are corrections for typos/misprints in the Review Notes (posted on line.)

- Page 2 last line and page 3 first line: $3x^3$ should be $3x^3dx$.
- Page 8: ignore the example for part 5 (homogeneous equations). It is done in more details on page 11.
- Page 13: Part B, it should be written more explicitly as follows.

  When z is not a root:

$$y = (A_m t^m + A_{m-1} t^{m-1} + \ldots + A_1 t + A_0) e^{\alpha t} \cos(\beta t)$$
$$+ (B_m t^m + B_{m-1} t^{m-1} + \ldots + B_1 t + B_0) e^{\alpha t} \sin(\beta t).$$

  When z is a root:

$$y = t \cdot (A_m t^m + A_{m-1} t^{m-1} + \ldots + A_1 t + A_0) e^{\alpha t} \cos(\beta t)$$
$$+ t \cdot (B_m t^m + B_{m-1} t^{m-1} + \ldots + B_1 t + B_0) e^{\alpha t} \sin(\beta t).$$

- Page 19, last line: the first term on the right-hand side $\tau\frac{1}{3}$ should be $\tau^3\frac{1}{3}$.