Formulas Cullen Zill AEM Chapters 3

I. (Linear, Homogeneous, Constant Coefficients)

$\boxed{ay'' + by' + cy = 0}$ $\Rightarrow$ try $\boxed{y = e^{rx}}$ Characteristic Equation $\boxed{ar^2 + br + c = 0}$ has roots $r_1,\ r_2$. Discriminant: $\Delta = b^2 - 4ac$. Three Cases:

1. $\Delta > 0$ Real distinct roots $r_1 \neq r_2$ $\Rightarrow$ (general solution) $\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$
2. $\Delta = 0$ Real double root $r = r_1 = r_2$ $\Rightarrow$ (general solution) $\boxed{y = c_1 e^{rx} + c_2 x e^{rx}}$
3. $\Delta < 0$ Complet roots $r = \alpha \pm i\beta$ $\Rightarrow$ (general solution) $\boxed{y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)}$

II. (Euler Equation, $x > 0$) $\boxed{ax^2 y'' + bxy' + cy = 0}$ $\Rightarrow$ try $\boxed{y = x^r}$

Characteristic Equation $\boxed{ar^2 + (b-a)r + c = 0}$ has roots $r_1,\ r_2$.

Three Cases:

1. Real distinct roots $r_1 \neq r_2$ $\Rightarrow$ (general solution) $\boxed{y = c_1 x^{r_1} + c_2 x^{r_2}}$
2. Real double root $r = r_1 = r_2$ $\Rightarrow$ (general solution) $\boxed{y = c_1 x^r + c_2 \ln(x) x^r}$
3. Complet roots $r = \alpha \pm i\beta$ $\Rightarrow$ (general solution) $\boxed{y = c_1 x^\alpha \cos(\beta \ln(x)) + c_2 x^\alpha \sin(\beta \ln(x))}$

III. (Nonhomogeneous Linear) $\boxed{y'' + P(x)y' + Q(x)y = R(x)}$ General solution: $\boxed{y = y_c + y_p}$

y_c is the general solution of $y'' + P(x)y' + Q(x)y = 0$	and	y_p is any particular solution of $y'' + P(x)y' + Q(x)y = R(x)$

There are two methods:

A. (Undetermined Coefficients) Guess the form of y_p from $R(x)$. This method requires that P and Q to be constants and R is a sum of terms of the form x^k, $x^k e^{\alpha x}$, $x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$.

B. (Variation of Parameters) Look for a particular solution in the form $y_p = uy_1 + vy_2$. This approach leads to

$$\boxed{y_p(x) = -y_1(x)\int^x \frac{y_2(s)R(s)}{W(s)}\,ds + y_2(x)\int^x \frac{y_1(s)R(s)}{W(s)}\,ds, \qquad W(s) = \det\begin{bmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{bmatrix}}$$

IV. (Reduction of Order) Suppose that y_1 is a solution of $y'' + p(x)y' + q(x)y = 0$. A second solution can be found in the form $\boxed{y_2 = y_1 \displaystyle\int \frac{1}{y_1^2(x)} \exp\left(-\int^x p(x)\,ds\right)\,dx}$.

A homogeneous linear differential equation with constant real coefficients of order n has the form

$$\boxed{y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = 0. \qquad (*)}$$

We can introduce the notation $D = \dfrac{d}{dx}$ and write the above equation as

$$P(D)y \equiv \left(D^n + a_{n-1}D^{(n-1)} + \cdots + a_0\right) y = 0.$$

By the fundamental theorem of algebra we can factor $P(D)$ as

$$\boxed{(D-r_1)^{m_1} \cdots (D-r_k)^{m_k} (D^2 - 2\alpha_1 D + \alpha_1^2 + \beta_1^2)^{p_1} \cdots (D^2 - 2\alpha_\ell D + \alpha_\ell^2 + \beta_\ell^2)^{p_\ell},}$$

where $\displaystyle\sum_{j=1}^{k} m_j + 2\sum_{j=1}^{\ell} p_j = n.$

There are two types of factors $(D-r)^k$ and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^k$:

$$\boxed{\text{The general solution of } (D-r)^k y = 0 \text{ is } y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{rx}}$$

$$\boxed{\begin{array}{l}\text{The general solution of } (D^2 - 2\alpha D + \alpha^2 + \beta^2)^k y = 0 \text{ is} \\ y = \left(c_1 + c_2 x + \cdots + c_k x^{(k-1)}\right) e^{\alpha x}\cos(\beta x) + \left(d_1 + d_2 x + \cdots + d_k x^{(k-1)}\right) e^{\alpha x}\sin(\beta x).\end{array}}$$

The general solution contains one such term for each term in the factorization.

We can also argue as before and seek solutions of $(*)$ in the form $y = e^{rx}$ to get a characteristic polynomial

$$r^n + a_{n-1}r^{(n-1)} + \cdots + a_0 = 0.$$

In either case we find that the general solution consists of a sum of n expressions $\{y_j\}_{j=1}^n$ where each like x^k, $x^k e^{rx}$, $x^k e^{\alpha x}\cos(\beta x)$ or $x^k e^{\alpha x}\sin(\beta x)$. The y_j are linearly independent and the general solution is $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$. In the case $n = 3$ the Wronskian of three functions y_1, y_2, y_3 is

$$W = W(y_1, y_2, y_3) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}.$$

If the functions are solutions of a linear homogeneous ODE then the functions are linearly independent on an interval I if and only if the wronskian is not zero at a single $x \in I$ (and therefore for all $x \in I$).

Nonhomogeneous Linear Equation with Constant Coefficients

Our goal is to find the general solution of $\boxed{ay'' + by' + cy = f(x) \ (*)}$

The general solution of $(*)$ is obtained as $\boxed{y = y_c + y_p}$ where

1. y_c is the general solution of the homogeneous problem, i.e. $y_c = c_1 y_1 + c_2 y_2$ where y_1, y_2 are two linearly independent solutions of $ay'' + by' + cy = 0$.
2. y_p is (any) particular solution of the nonhomogeneous problem $(*)$.

The main problem then is to find y_p.

Remarks on the *Method of Undetermined Coefficients*

Remark:

1. The general solution of the homogeneous problem is given as a sum of numbers times terms of the form
$$p(x), \quad p(x)e^{ax}, \quad p(x)e^{\alpha x}\cos(\beta x), \quad p(x)e^{\alpha x}\sin(\beta x) \tag{1}$$
where $p(x)$ is a polynomial in x. No other types of solutions are possible!
2. This remains true for the nonhomogeneous problem $(*)$ provided the right hand side $f(x)$ is also given as a sum of terms of the form (1).
3. The main thing is to find y_p and here we consider the case of $f(x)$ in the form (1).

We consider $p(x) = Cx^m + \cdots$ is a polynomial of degree m.

$$ay'' + by' + cy = p(x)e^{r_0 x} \quad \Rightarrow \quad y_p = x^s(A_m x^m + \cdots + A_1 x + A_0)e^{r_0 x}$$

1. $s = 0$ if r_0 is not a root of the characteristic polynomial $\boxed{ar^2 + br + c = 0 \quad (\dagger)}$.
2. $s = 1$ if r_0 is a simple root of $(\dagger)$.
3. $s = 2$ if r_0 is a double root of $(\dagger)$.

N.B. The above case includes the case $r_0 = 0$ in which case the right side is $p(x)$.

$$ay'' + by' + cy = \begin{cases} p(x)e^{\alpha x}\cos(\beta x) \\ \text{or} \\ p(x)e^{\alpha x}\sin(\beta x) \end{cases} \Rightarrow \quad y_p = \begin{array}{l} x^s(A_m x^m + \cdots + A_1 x + A_0)e^{\alpha x}\cos(\beta x) \\ +x^s(B_m x^m + \cdots + B_1 x + B_0)e^{\alpha x}\sin(\beta x) \end{array}$$

1. $s = 0$ if $r_0 = \alpha + i\beta$ is not a root of $(\dagger)$.
2. $s = 1$ if $r_0 = \alpha + i\beta$ is a simple root of $(\dagger)$.