Small-Sample Confidence Intervals and Prediction Intervals

MATH 3342
Section 7.3

Example

- A sample of 100 soda cans, from a population with soda volume being Normally distributed having $s = 0.20$ oz, produced a sample mean equal to 12.09 oz.
  - With this large of a sample we would construct a CI as in section 7.2.
- What if our sample only contained 10 soda cans?

Assumptions

- The data is from a SRS.
- Observations from the population are either from:
  - A Normal distribution with **unknown** mean μ and **unknown** standard deviation σ.

    OR
  - A symmetric, single-peaked distribution with **unknown** mean μ and **unknown** standard deviation σ.
    - This assumption results in approximations

Small-Sample Distribution

- Under these assumptions, for small or moderate *n*,

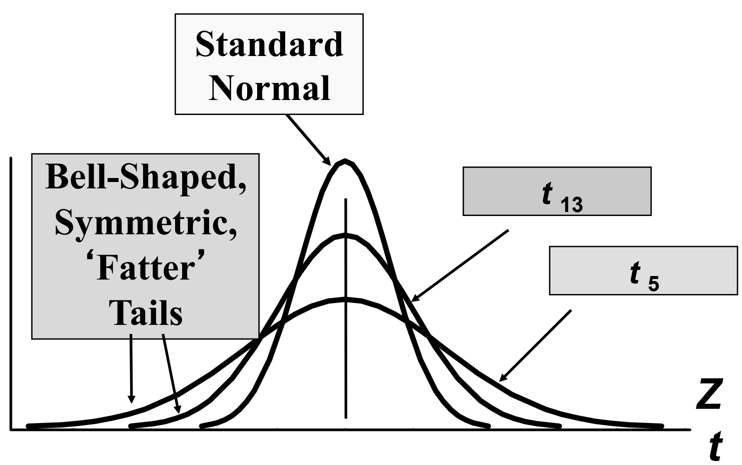
$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

- Does NOT have a Normal distribution.
- It has a **t distribution** with *n – 1 degrees of freedom*.
  - Also called Student's t distribution

Student's t Distributions

- The t-distribution family has the following properties:
  - Bell-shaped and symmetric
  - Greater area in the tails than in the tails of the Normal
- The t-distribution approaches the normal distribution as the degrees of freedom increase.
- Let t_v denote the t distribution with v df.
- Critical values are provided in Appendix Table A.5

Convergence to the Normal Distribution



One-Sample t Confidence Intervals

$$\overline{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}}$$

where:

- $t_{\alpha/2,n-1}$ = Critical value for $t_{n\text{-}1}$
- $\overline{x}$ = Sample mean
- s = Sample standard deviation
- *n* = Sample size

Example

- A sample of 10 soda cans, from a population with soda volume being Normally distributed having $s = 0.20$ oz, produced a sample mean equal to 12.09 oz.

$$\overline{x} \pm t_{\alpha/2,9} \frac{s}{\sqrt{n}} \longrightarrow 12.09 \pm 2.262 \frac{0.20}{\sqrt{10}}$$

$$12.09 \pm 0.1431$$

11.957 ounces —— ***t*** **- CI** —— **12.233 ounces**

11.966 ounces —— **Z - CI** —— **12.214 ounces**

One-Sample t Confidence Bounds

- An upper confidence bound for μ is:

$$\bar{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}$$

- A lower confidence bound for μ is:

$$\bar{x} - t_{\alpha,n-1}\frac{s}{\sqrt{n}}$$

Example

- Suppose you randomly select 8 participants who took an IQ test and record their scores.

126	92	80	71
118	107	122	73

$$\bar{x} = 98.6 \quad s = 22.5$$

- If you were to pick a 9^{th} person at random, what would you predict that their IQ is?

Prediction Intervals

Suppose we observe $X_1 = x_1, \ldots, X_n = x_n$ from a Normal distribution.

A point predictor for X_{n+1} is $\bar{X}$ with prediction error $\bar{X} - X_{n+1}$.

$E(\bar{X} - X_{n+1}) = E(\bar{X}) - E(X_{n+1}) = \mu - \mu = 0$

X_{n+1} is independent of $\bar{X}$, so

$$V(\bar{X} - X_{n+1}) = V(\bar{X}) + V(X_{n+1}) = \frac{\sigma^2}{n} - \sigma^2 = \sigma^2\left(1 + \frac{1}{n}\right)$$

Prediction Intervals

$$Z = \frac{\bar{X} - X_{n+1} - 0}{\sqrt{\sigma^2\left(1 + \frac{1}{n}\right)}}$$ has a standard Normal distribution.

$$T = \frac{\bar{X} - X_{n+1} - 0}{S\sqrt{1 + \frac{1}{n}}}$$ has a t_{n-1} distribution.

Prediction Intervals

- A **prediction interval** (PI) for a single observation to be selected from a Normal population distribution is:

$$\bar{x} \pm t_{\alpha/2,n-1} \cdot s\sqrt{1+\frac{1}{n}}$$

- The **prediction level** is 100(1 – α)%
- Lower and upper prediction bounds can be found similarly to the confidence bounds.

Example

- Suppose you randomly select 8 participants who took an IQ test and record their scores.

126	92	80	71
118	107	122	73

$$\bar{x} = 98.6 \quad s = 22.5$$

- Calculate a 95% PI for a 9th randomly selected participant.

$$\bar{x} \pm t_{\alpha/2,n-1} \cdot s\sqrt{1+\frac{1}{n}} \longrightarrow 98.6 \pm 2.365 \cdot 22.5\sqrt{1+\frac{1}{8}}$$

$$98.6 \pm 56.4$$