# Basic Properties of Confidence Intervals

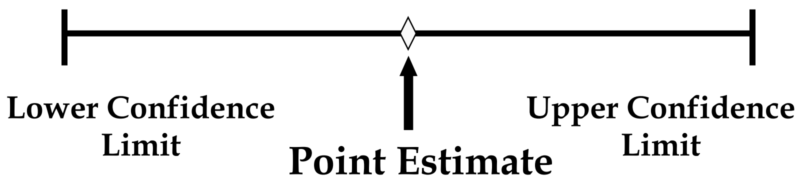
MATH 3342
Section 7.1

## Example

- A sample of 100 soda cans, from a population with soda volume being Normally distributed having $\sigma = 0.20$ oz, produced a sample mean equal to 12.09 oz.

- What is a likely range of values for the population mean $\mu$ of soda volume?

# Confidence Intervals




# Confidence Intervals

- Provides a range of plausible values for a parameter.
  - Developed from a sample
  - For a given confidence level
- A **confidence level C** is a measure of the degree of reliability of the interval.
  - Gives the probability that the interval will capture the true parameter in repeated samples.
  - C is usually 90% or higher.

# Assumptions for This Section

Suppose μ is the parameter of interest:

1. The data is from a SRS. There are no non-sampling errors.
2. The variable of interest is exactly Normal distributed.
3. We don't know the population mean μ, but we **do** know the population standard deviation σ.
   - Not necessarily a practical assumption.

# Construction of a 95% CI

- We build our interval starting from the formula for a z-score and a probability statement about a standard normal RV.

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \qquad P(-1.96 < Z < 1.96) = 0.95$$

# Construction of a 95% CI

$$
\begin{aligned}
0.95 &= P(-1.96 < Z < 1.96) \\
&= P\left(-1.96 < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < 1.96\right) \\
&= P\left(-1.96 \cdot \sigma/\sqrt{n} < \bar{X}-\mu < 1.96 \cdot \sigma/\sqrt{n}\right) \\
&= P\left(-\bar{X} - 1.96 \cdot \sigma/\sqrt{n} < -\mu < -\bar{X} + 1.96 \cdot \sigma/\sqrt{n}\right) \\
&= P\left(\bar{X} - 1.96 \cdot \sigma/\sqrt{n} < \mu < \bar{X} + 1.96 \cdot \sigma/\sqrt{n}\right)
\end{aligned}
$$

# Construction of a 95% CI

$\left(\bar{X} - 1.96 \cdot \sigma/\sqrt{n}, \bar{X} + 1.96 \cdot \sigma/\sqrt{n}\right)$ is a Random Interval.

The width is not random, only the *center* is.

*Interpretation*: The probability that this random interval includes the true value of $\mu$ is 95%.

# 95% CI for $\mu$

Suppose we observe $X_1 = x_1, \ldots, X_n = x_n$ and compute $\bar{x}$ and use it in place of $\bar{X}$, then:

$\left(\bar{x} - 1.96 \cdot \sigma/\sqrt{n}, \bar{x} + 1.96 \cdot \sigma/\sqrt{n}\right)$ is a 95% CI for $\mu$.

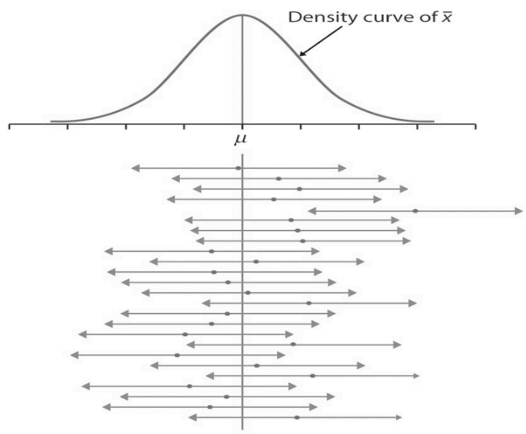
Alternatively,

$\bar{x} - 1.96 \cdot \sigma/\sqrt{n} < \mu < \bar{x} + 1.96 \cdot \sigma/\sqrt{n}$ with 95% confidence.

# Interpretation

- When we substitute in the observed sample mean, the interval is no longer random!
- Therefore:
  - The probability statement is not about any one observed interval
  - Rather, a statement about the **long-term relative frequency** if we repeatedly sample and compute intervals in this manner.

# Interpretation




# Example

A sample of 100 soda cans, from a population with soda volume being Normally distributed having σ = 0.20 oz, produced a sample mean equal to 12.09 oz. A 95% confidence interval would be:

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$

$$12.09 \pm 1.96 \frac{0.20}{\sqrt{100}}$$

$$12.09 \pm 0.039$$

**12.051 ounces** ——————————— **12.129 ounces**

# 100(1- $\alpha$ )% Confidence Intervals

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

where: $z_{\alpha/2}$ = Standard Normal Critical Value

$\sigma$ = Population standard deviation

n = Sample size

# Common Critical Values

| Confidence Level | 90% | 95% | 99% |
|---|---|---|---|
| $z_{\alpha/2}$ | 1.645 | 1.960 | 2.576 |

# Example

A sample of 100 soda cans, from a population with soda volume being Normally distributed having σ = 0.20 oz, produced a sample mean equal to 12.09 oz. A **99%** confidence interval would be:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$12.09 \pm 2.576 \frac{0.20}{\sqrt{100}}$$

$$12.09 \pm 0.052$$

**12.051 ounces** ——— **12.129 ounces**

**12.038 ounces** ——— **12.142 ounces**

# How do we reduce width?

- Lower the confidence level. This results in a lower critical value $z_{\alpha/2}$.
- Increase the sample size $n$. This reduces the variability of the sampling distribution.

# Example: Impact of *n*

If instead of sample of 100 cans, suppose a sample of 400 cans, from a population with σ = 0.20, produced a sample mean equal to 12.09. A 95% confidence interval would be:

$$\bar{x} \pm z \frac{\sigma}{\sqrt{n}}$$

$$12.09 \pm 1.96 \frac{0.20}{\sqrt{400}}$$

$$12.09 \pm 0.0196$$

**12.0704 ounces** ——— **n=400** ——— **12.1096 ounces**

**12.051 ounces** ——— **n=100** ——— **12.129 ounces**

# The Minimum *n* for a Desired Width

$$n = \left(2 z_{\alpha/2} \frac{\sigma}{w}\right)^2$$

where:

$z_{\alpha/2}$ = Critical value

$w$ = Desired width

$\sigma$ = Population standard deviation

# Always round up for *n* !!!

- Suppose the calculation says that 58.2 observations are needed.
- 58 observations would produce a slightly wider interval than is wanted.
- 59 observations would produce a slightly narrower interval than desired.
- Estimate would still then be within the desired margin of error with the desired level of confidence

# Example

The manager of the Georgia Timber Mill wishes to construct a 90% confidence interval within 0.50 inches in estimating the mean diameter of logs. Assume a population standard deviation of 4.8 inches.

$$n = \left(2 \cdot 1.645 \frac{4.8}{1}\right)^2 = 249.38 \boxed{\approx} 250$$