Cumulative Distribution Functions and Expected Values

MATH 3342
SECTION 4.2

The Cumulative Distribution Function (cdf)

- The **cumulative distribution function** F(*x*) for a continuous RV X is defined for every number *x* by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$

- For each *x*, F(*x*) is the area under the density curve to the left of *x*.

The Uniform Distribution

- Recall:
- A continuous RV X is said to have a **uniform distribution** over the interval [A, B] if the pdf is:

$$f(x;A,B)=\left\{\begin{array}{ll}\dfrac{1}{B-A} & A\le x\le B\\ 0 & otherwise\end{array}\right\}$$

The Uniform cdf

- The cdf of the uniform distribution is obtained as follows:

$$F(x)=\int_{-\infty}^{x} f(y)dy=\int_{A}^{x}\frac{1}{B-A}dy$$

$$=\frac{1}{B-A}\cdot[y]_A^x$$

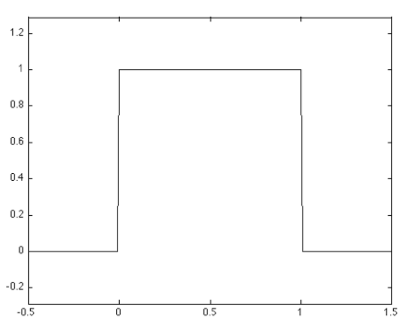
$$=\frac{x-A}{B-A}$$

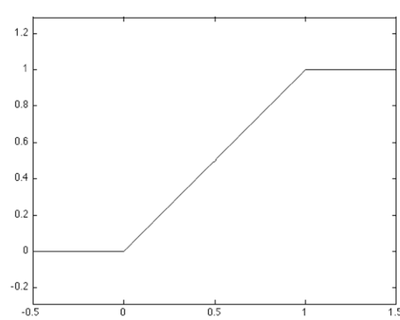
The Uniform cdf

- More completely:

$$F(x) = \begin{cases} 0 & x < A \\ \dfrac{x-A}{B-A} & A \le x < B \\ 1 & B \le x \end{cases}$$

The Uniform Distribution over [0, 1]





Computing Probabilities with F(x)

- Let X be a continuous RV with pdf f(x) and cdf F(x).
- For any number a:
- $$P(X > a) = 1 - F(a)$$
- For any two numbers a and b with $a < b$:

$$P(a \leq X \leq b) = F(b) - F(a)$$

Example

- Let X be a RV denoting the magnitude of a dynamic load on a bridge with pdf given by

$$f(x) = \frac{1}{8} + \frac{3}{8}x, \quad 0 \leq x \leq 2; \quad 0, \textit{otherwise}$$

- Calculate P(0.99 ≤ X ≤ 1.01)
- Calculate P(X > 1.5)

Obtaining the pdf from the cdf

- If X is a continuous RV with pdf f(x) and cdf F(x).
- Then at every x at which the derivative F'(x) exists,
- F'(x) = f(x)

The (100p)th Percentile

- Let p be a number between 0 and 1.
- The (100p)th percentile of the distribution of a continuous RV X, denoted by η(p), is defined as

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)\,dy$$

Example

- Let X be a RV denoting the magnitude of a dynamic load on a bridge with pdf given by

$$f(x) = \frac{1}{8} + \frac{3}{8}x, \quad 0 \le x \le 2; \quad 0, \textit{otherwise}$$

- What is the 95th percentile of this distribution?
- What is the 50th percentile of this distribution?

The Median

- The median of a continuous distribution is the 50th percentile, so

$$0.5 = F(\tilde{\mu})$$

- If a continuous distribution is symmetric, then the median will be equal to the point of symmetry.

Expected Value

- The **expected value** or **mean** of a continuous RV with pdf f(x) is:

$$\mu = \mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

Example

- Let X be a RV denoting the magnitude of a dynamic load on a bridge with pdf given by

$$f(x) = \frac{1}{8} + \frac{3}{8}x, \quad 0 \le x \le 2; \quad 0, \textit{otherwise}$$

- What is the expected value of this distribution?

Expected Value of a Function

- If X is a continuous RV with pdf f(x) and h(X) is any function of X, then

$$E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$$

Variance

- The **variance** of a continuous RV X with pdf f(x) and mean μ is:

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

$$= E[(X-\mu)^2]$$

- The standard deviation (SD) of X is

$$\sigma_X = \sqrt{V(X)}$$

Example

- Let X be a RV denoting the magnitude of a dynamic load on a bridge with pdf given by

$$f(x) = \frac{1}{8} + \frac{3}{8}x, \quad 0 \le x \le 2; \quad 0, \textit{otherwise}$$

- What is the variance of this distribution?

Shortcut Method for Variance

- An often quicker formula to compute variance is given by:

$$\sigma_X^2 = E(X^2) - [E(X)]^2$$

- Try it for the previous example!