The Binomial Distribution

Math 3342

Sect 3.4

The Binomial Setting

- 'n' identical trials
  - Examples:
    - 15 tosses of a coin
    - 10 CDs chosen from a warehouse
- Two mutually exclusive outcomes on each trial ("success" or "failure")
  - e.g.: Head or tail in each toss of a coin; CD defective or not defective

The Binomial Setting

- Trials are independent
  - The outcome of one trial does not affect the outcome of the other
- Constant probability for each trial
  - Probability of a tail is the same each time we toss the coin
  - Probability of getting a defective CD is the same each time we select one

A Binomial Distribution

- The RV X counts the number of successes.
- Parameters:
  - n: reflects the total number of observations or trials.
  - p: reflects the probability of success on any one observation.
- X can be any integer between 0 and n.

Example: Assembly Process

- A company assembles a component used in DVD players.
- It has determined that there are 3 **independent** *opportunities for error* in assembling the component.
- Each opportunity has P(Defect) = 0.1

Assumptions

- n is 3
  - Because the *opportunities for error* are the number of observations.
- Either an error occurs or it does not.
- Must assume the chance of an error is the same for each opportunity.

Example: Assembly Process

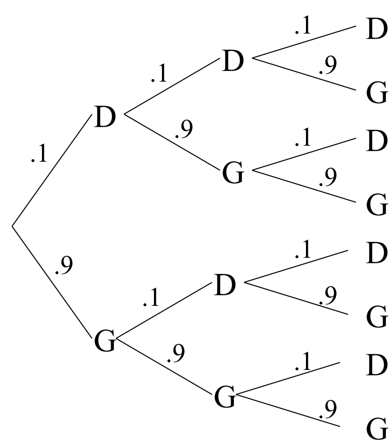
- Of interest to the consumer:
  - The RV X, the number of defects in the component
- How likely is it that the product has exactly one defect?
  - One or fewer defects?

Tree Representation

1st opportunity | 2nd opportunity | 3rd opportunity

D - Defective

G - Good



We know that any one error occurs with probability 0.1

Using the Multiplication Rule

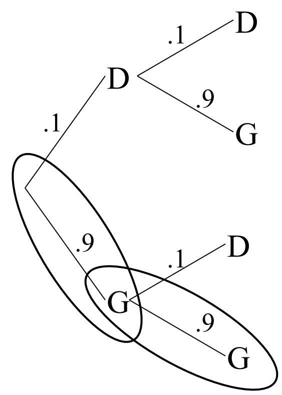
Event A = 1st opportunity no error

Event B = 2nd opportunity no error

Probability of no error in the first two opportunities

= P(A and B) = P(A)P(B)

= (0.9)(0.9) = 0.81

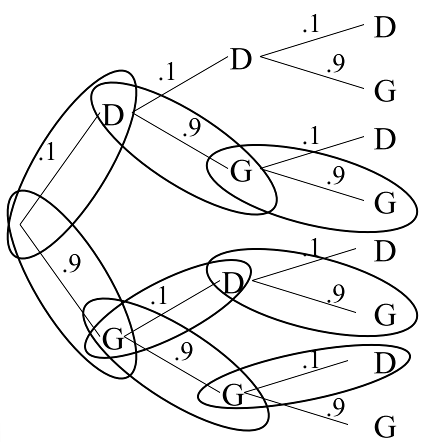


Multiply within paths:

(.9)(.9) = .81

Binomial Probabilities

What is the probability of observing exactly one defect?



Multiply within paths:

.1(.9)(.9) = .081

.9(.1)(.9) = .081

.9(.9)(.1) = .081

Add results together (disjoint events):

.081
+.081
+.081

0.243

Binomial Coefficient

- Counts the number of "paths."
- Number of ways of getting *k* successes in *n* observations.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

For $k = 0, 1, 2, \dots, n$.

Example: One defect in three opportunities

$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{3\times 2\times 1}{1\times 2\times 1} = 3$$

Binomial pmf

- *X* has a binomial distribution
- *p* is probability of success on each observation
- $x = 0, 1, 2, \dots,$ or n.

$$b(x;n,p) = P(X = x) = \binom{n}{x} p^{x}(1-p)^{n-x}$$

- For all other x, $b(x;n,p) = 0$

Assembly Process Example

- How likely is it that the product has exactly one defect?
  - One or fewer defects?
- Recall: n = 3 k = 1 p = 0.1

$$P(X=1)=\binom{3}{1}0.1^1(1-0.1)^{3-1}=3(0.1)(.9)^2=0.243$$

Binomial cdf

- For X ~ Bin(*n,p*), the cdf will be denoted by:

$$B(x;n,p)=P(X\le x)=\sum_{y=0}^{x}\binom{n}{y}p^y(1-p)^{n-y}$$

$$=\sum_{y=0}^{x}b(y;n,p)$$

Example: Assembly Process

How likely is it that the product has one or fewer defects?

$$P(X=1)=0.243$$

$$P(X=0)=\binom{3}{0}0.1^0(1-0.1)^{3-0}=1(1)(.9)^3=0.729$$

$$B(1;3,0.1)=P(X\le 1)=P(X=0)+P(X=1)$$

$$=0.729+0.243=0.972$$

Mean of the Binomial Distribution

$E(X) = \mu = np$

- Assembly Process:
  - n = 3
  - p = 0.1

  $E(X) = \mu = 3*0.1 = 0.3$ errors

Variance of the Binomial Distribution

$$V(X) = \sigma^2 = np(1-p) = npq$$

$$\sigma = \sqrt{npq}$$

- Assembly Process:

$$V(X) = 3 * 0.1(1-0.1) = 0.27 \text{ errors}^2$$

$$\sigma = \sqrt{0.27} \approx 0.52 \text{ errors}$$