TRIMMING A GORENSTEIN IDEAL

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Abstract. Let Q be a regular local ring of dimension 3. We show how to trim a Gorenstein ideal in Q to obtain an ideal that defines a quotient ring that is close to Gorenstein in the sense that its Koszul homology algebra is a Poincaré duality algebra P padded with a non-zero graded vector space on which $P_{\geq 1}$ acts trivially. We explicitly construct an infinite family of such rings.

1. Introduction

Let Q be a regular local ring with maximal ideal $\mathfrak{n}$. Quotient rings of Q that have projective dimension at most 3 as Q-modules have been classified based on the multiplicative structure of their Koszul homology algebras. To be precise, let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal such that the minimal free resolution of $R = Q/\mathfrak{a}$ over Q has length at most 3. By a result of Buchsbaum and Eisenbud [4], the resolution carries a structure of an associative differential graded commutative algebra, and based on that structure Avramov, Kustin, and Miller [3] and Weyman [9] established a classification in terms of the induced multiplicative structure on $\mathrm{Tor}_*^Q(R, \Bbbk)$, where $\Bbbk$ is the residue field of Q. Finally, as graded $\Bbbk$-algebras, the Koszul homology algebra of R and $\mathrm{Tor}_*^Q(R, \Bbbk)$ are isomorphic; see Avramov [1] for an in-depth treatment.

An ideal $\mathfrak{a} \subset Q$ is called *Gorenstein* if the quotient $R = Q/\mathfrak{a}$ is a Gorenstein ring. By a classic result of Avramov and Golod [2], a Gorenstein ring is characterized by the fact that its Koszul homology algebra $A = \mathrm{H}(K^R)$ has Poincaré duality. In the classification mentioned above, a Gorenstein ring that is not complete intersection belongs to a parametrized family $\mathbf{G}(r)$, where r is the rank of the canonical map

$$\delta\colon A_2 \longrightarrow \mathrm{Hom}_{\Bbbk}(A_1, A_3)\,;$$

see [1, 1.4.2]. It was conjectured in [1] that all rings of class $\mathbf{G}(r)$ are Gorenstein, but Christensen and Veliche [5] gave sporadic examples of rings of class $\mathbf{G}(r)$ that are not Gorenstein. In this paper we present a systematic construction and achieve:

(1.1) **Theorem.** *Let Q be the power series algebra in three variables over a field. For every $r \geq 3$ there is a quotient ring of Q that is of class $\mathbf{G}(r)$ and not Gorenstein.*

The quotient rings in Theorem (1.1) are obtained as follows: Let $\mathfrak{n}$ be the maximal ideal of Q and start with a graded Gorenstein ideal $\mathfrak{g} \subseteq \mathfrak{n}^2$ generated by $2m+1$

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elements. Trim $\mathfrak{g}$ by replacing one minimal generator g by $\mathfrak{n}g$; this removes a 1-dimensional subspace from $\mathfrak{g}$. The quotient of Q by the resulting ideal is a ring of type 2; in particular, it is not Gorenstein, and for $m \geq 3$ it is of class $\mathbf{G}(r)$. Theorem (1.1) is a consequence of Proposition (3.5), which builds on a more general but slightly less precise statement about local rings, Theorem (2.4).

2. Local rings

Let Q be a d-dimensional regular local ring with maximal ideal $\mathfrak{n}$ and residue field $\Bbbk$. For an ideal $\mathfrak{a}$ in Q, we denote by $\mu(\mathfrak{a})$ the minimal number of generators of $\mathfrak{a}$. Let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal and set $R = Q/\mathfrak{a}$. We denote by K^R the Koszul complex on a minimal set of generators for the maximal ideal $\mathfrak{n}/\mathfrak{a}$ of R; one has $K^R = R \otimes_Q K^Q$. The Koszul complex is an exterior algebra, and the homology algebra $A = \mathrm{H}(K^R)$ is a graded-commutative $\Bbbk$-algebra. Denote by c the projective dimension of R as a Q-module; by the Auslander–Buchsbaum Formula and depth sensitivity of the Koszul complex one has $c = \max\{i \mid A_i \neq 0\}$. The number $\mathrm{rank}_\Bbbk(A_c)$ is called the *type* of R. If the ideal $\mathfrak{a}$ is $\mathfrak{n}$-primary, then one has $c = d$ and the type of R is the socle rank, i.e. $\mathrm{type}(R) = \mathrm{rank}_\Bbbk(0 :_R \mathfrak{n}/\mathfrak{a})$.

(2.1) **Classification.** Let Q be as above, and let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be an ideal such that $R = Q/\mathfrak{a}$ has projective dimension 3 as a Q-module. The possible multiplicative structures on the graded-commutative $\Bbbk$-algebra $A = \mathrm{H}(K^R) \cong \mathrm{Tor}_*^Q(R, \Bbbk)$ were identified in [3]. By assumption one has $A_{\geq 4} = 0$, and the possible structures are described by the invariants

$$p = \mathrm{rank}_\Bbbk(A_1 \cdot A_1)\,, \quad q = \mathrm{rank}_\Bbbk(A_1 \cdot A_2)\,, \quad \text{and} \quad r = \mathrm{rank}_\Bbbk(A_2 \xrightarrow{\delta} \mathrm{Hom}_\Bbbk(A_1, A_3))\,.$$

From [1, thm. 3.1] one extracts the following description of all the possible classes of rings that are not Gorenstein.

(2.1.1)

Class	p	q	r	Restrictions
$\mathbf{B}$	1	1	2	
$\mathbf{G}(r)$	0	1	r	$2 \leq r \leq \mu(\mathfrak{a}) - 2$
$\mathbf{H}(p,q)$	p	q	q	$q \leq \mathrm{type}(R)$
$\mathbf{T}$	3	0	0	

In [3] the multiplication tables for the different structures are given. In particular, if $R = Q/\mathfrak{a}$ is a ring of class $\mathbf{G}(r)$, then with $m = \mu(\mathfrak{a})$ and $t = \mathrm{type}(R)$ there exist bases for A_1, A_2, and A_3:

$$\mathbf{e}_1, \ldots, \mathbf{e}_m\,, \quad \mathbf{f}_1, \ldots, \mathbf{f}_{m+t-1}\,, \quad \text{and} \quad \mathbf{g}_1, \ldots, \mathbf{g}_t$$

such that the only non-zero products are $\mathbf{e}_i\mathbf{f}_i = \mathbf{g}_1 = -\mathbf{f}_i\mathbf{e}_i$ for $1 \leq i \leq r$. That is, the subalgebra P of A spanned by 1, $\mathbf{e}_1, \ldots, \mathbf{e}_r$, $\mathbf{f}_1, \ldots, \mathbf{f}_r$, and $\mathbf{g}_1$ is a pure Poincaré duality algebra, in the sense that the only non-trivial products are those from the perfect pairing. Moreover, $P_{\geq 1}$ acts trivially on the rest of A.

The next result is proved in [6, thm. 4.5 and 5.4]; the argument is based on linkage theory and cannot be reproduced here without significant overhead.

(2.2) **Proposition.** *Let $(Q, \mathfrak{n})$ be a regular local ring and let $\mathfrak{a} \subseteq \mathfrak{n}^2$ be a perfect ideal of grade 3 that is minimally generated by 5 elements and not Gorenstein. If, with the notation above, the ring $Q/\mathfrak{a}$ has $p = 0$, then it has $r \leq 1$.* □

(2.3) **Lemma.** *Let $(Q,\mathfrak{n})$ be a regular local ring and consider an $\mathfrak{n}$-primary ideal $\mathfrak{g} \subseteq \mathfrak{n}^2$, minimally generated by elements $g_0,\dots,g_k$. Let $s_1,\dots,s_t$ be elements of Q whose classes in $Q/\mathfrak{g}$ form a basis for the socle. The ideal $\mathfrak{a} = \mathfrak{n}g_0 + (g_1,\dots,g_k)$ is $\mathfrak{n}$-primary, and if $\mathfrak{n}s_i \subseteq \mathfrak{a}$ holds for all $i = 1,\dots,t$, then the classes of $g_0, s_1,\dots,s_t$ in $Q/\mathfrak{a}$ form a basis for the socle; in particular one has* $\operatorname{type}(Q/\mathfrak{a}) = \operatorname{type}(Q/\mathfrak{g}) + 1$.

Proof. As $\mathfrak{g}$ is $\mathfrak{n}$-primary, it follows from the containment $\mathfrak{n}\mathfrak{g} \subseteq \mathfrak{a}$ that $\mathfrak{a}$ is $\mathfrak{n}$-primary. Consider the rings $R = Q/\mathfrak{a}$ and $S = Q/\mathfrak{g}$; there is an exact sequence

$$0 \longrightarrow \mathfrak{g}/\mathfrak{a} \longrightarrow R \longrightarrow S \longrightarrow 0\,,$$

and an isomorphism of Q-modules $\mathfrak{g}/\mathfrak{a} \cong \Bbbk$, where $\Bbbk$ is the residue field of Q. Tensoring with the Koszul complex K^Q one gets an exact sequence of Q-complexes,

$$(*) \qquad 0 \longrightarrow \Bbbk \otimes_Q K^Q \xrightarrow{\ \alpha\ } K^R \xrightarrow{\ \beta\ } K^S \longrightarrow 0\,.$$

Let d be the dimension of Q. From the sequence in homology associated to $(*)$ one gets the following exact sequence

$$0 \longrightarrow \Bbbk \xrightarrow{\mathrm{H}_d(\alpha)} \mathrm{H}_d(K^R) \xrightarrow{\mathrm{H}_d(\beta)} \mathrm{H}_d(K^S)\,.$$

The rings R and S are artinian, and a rank count yields

$$\operatorname{type}(R) \;=\; \operatorname{rank}_\Bbbk(\mathrm{H}_d(K^R)) \;\le\; \operatorname{rank}_\Bbbk(\mathrm{H}_d(K^S)) + 1 \;=\; \operatorname{type}(S) + 1\,.$$

It is clear that the residue classes $[g_0]$ and $[s_1],\dots,[s_t]$ in R are non-zero socle elements. Moreover, they are $\Bbbk$-linearly independent: Indeed, the elements $[s_1],\dots,[s_t]$ are $\Bbbk$-linearly independent, because of the inclusion $\mathfrak{a} \subset \mathfrak{g}$. Further, suppose one has $[g_0] = \sum_{i=1}^t [u_i][s_i]$ where the elements u_i are units in Q. It follows that $g_0 - \sum_{i=1}^t u_i s_i$ is in $\mathfrak{a} \subseteq \mathfrak{g}$, and as $g_0 \in \mathfrak{g}$ one gets $\sum_{i=1}^t u_i s_i \in \mathfrak{g}$, a contradiction. Thus, there are $t+1$ $\Bbbk$-linearly independent elements in the socle of R. □

For the next result, recall from work of J. Watanabe [8] that a grade 3 Gorenstein ideal in a regular ring is minimally generated by an odd number of elements.

(2.4) **Theorem.** *Let $(Q,\mathfrak{n})$ be a regular local ring of dimension 3 and let $\mathfrak{g} \subseteq \mathfrak{n}^2$ be an $\mathfrak{n}$-primary Gorenstein ideal minimally generated by elements $g_0,\dots,g_{2m}$. The ideal $\mathfrak{a} = \mathfrak{n}g_0 + (g_1,\dots,g_{2m})$ is $\mathfrak{n}$-primary, one has $\operatorname{type}(Q/\mathfrak{a}) = 2$ and:*

(a) *If $m = 1$, then $\mu(\mathfrak{a}) = 5$ and $Q/\mathfrak{a}$ is of class* **B**.

(b) *If $m = 2$, then one of the following holds:*
- $\mu(\mathfrak{a}) = 4$ *and $Q/\mathfrak{a}$ is of class* $\mathbf{H}(3,2)$.
- $\mu(\mathfrak{a}) = 5$ *and $Q/\mathfrak{a}$ is of class* **B**.
- $\mu(\mathfrak{a}) \in \{6,7\}$ *and $Q/\mathfrak{a}$ is of class $\mathbf{G}(r)$ with* $\mu(\mathfrak{a}) - 2 \ge r \ge \mu(\mathfrak{a}) - 3$.

(c) *If $m \ge 3$, then $Q/\mathfrak{a}$ is of class $\mathbf{G}(r)$ with* $\mu(\mathfrak{a}) - 2 \ge r \ge \mu(\mathfrak{a}) - 3$.

Proof. As $\mathfrak{g}$ defines a Gorenstein ring, one has $\mathfrak{g} : (\mathfrak{g} : \mathfrak{b}) = \mathfrak{b}$ for every ideal $\mathfrak{b}$ in Q that contains $\mathfrak{g}$. Let $s \in Q$ be a representative of the socle of $Q/\mathfrak{g}$; in Q one has

$$\mathfrak{g} \subseteq (\mathfrak{a} : \mathfrak{n}) \subseteq (\mathfrak{g} : \mathfrak{n}) = \mathfrak{g} + (s)\,.$$

Forming colon ideals one gets $\mathfrak{g} : (\mathfrak{a} : \mathfrak{n}) \supseteq \mathfrak{g} : (\mathfrak{g} : \mathfrak{n}) = \mathfrak{n}$ and hence $\mathfrak{g} : (\mathfrak{a} : \mathfrak{n}) = \mathfrak{n}$. Forming colon ideals a second time now yields $(\mathfrak{a} : \mathfrak{n}) = (\mathfrak{g} : \mathfrak{n}) = \mathfrak{g} + (s)$; in particular, one has $\mathfrak{n}s \subseteq \mathfrak{a}$, so it follows from Lemma (2.3) that $\mathfrak{a}$ is $\mathfrak{n}$-primary and $R = Q/\mathfrak{a}$ has type 2; in particular, R is not Gorenstein.

Note that one has

$$2m \le \mu(\mathfrak{a}) \le 2m+3\,.$$

Set $S = Q/\mathfrak{g}$; there is an exact sequence of Q-modules

$$0 \longrightarrow \mathfrak{g}/\mathfrak{a} \longrightarrow R \longrightarrow S \longrightarrow 0$$

and an isomorphism $\mathfrak{g}/\mathfrak{a} \cong \Bbbk$. Tensor with the Koszul complex K^Q to get an exact sequence of Q-complexes,

$$(*) \qquad 0 \longrightarrow \Bbbk \otimes_Q K^Q \xrightarrow{\ \alpha\ } K^R \xrightarrow{\ \beta\ } K^S \to 0\,,$$

where β is a morphism of DG Q-algebras. Set $A = \mathrm{H}(K^R)$ and $G = \mathrm{H}(K^S)$, one has

$$\operatorname{rank}_{\Bbbk}(G_0) = 1 = \operatorname{rank}_{\Bbbk}(G_3) \qquad \text{and} \qquad \operatorname{rank}_{\Bbbk}(G_1) = 2m+1 = \operatorname{rank}_{\Bbbk}(G_2)$$

$$\operatorname{rank}_{\Bbbk}(A_0) = 1,\ \operatorname{rank}_{\Bbbk}(A_1) = \mu(\mathfrak{a}),\ \operatorname{rank}_{\Bbbk}(A_2) = \mu(\mathfrak{a})+1, \ \text{ and } \ \operatorname{rank}_{\Bbbk}(A_3) = 2\,.$$

Consider the exact sequence in homology associated to $(*)$

$$\begin{array}{ccccccc}
0 \longrightarrow & \Bbbk & \xrightarrow[1]{\mathrm{H}_3(\alpha)} & A_3 & \xrightarrow[1]{\mathrm{H}_3(\beta)} & G_3 & \\
 & & & \downarrow{\scriptstyle 0} & & & \\
\longrightarrow & \Bbbk^3 & \xrightarrow[3]{\mathrm{H}_2(\alpha)} & A_2 & \xrightarrow[\mu(\mathfrak{a})-2]{\mathrm{H}_2(\beta)} & G_2 & \\
 & & & \downarrow{\scriptstyle 2m+3-\mu(\mathfrak{a})} & & & \\
\longrightarrow & \Bbbk^3 & \xrightarrow[\mu(\mathfrak{a})-2m]{\mathrm{H}_1(\alpha)} & A_1 & \xrightarrow[2m]{\mathrm{H}_1(\beta)} & G_1 & \\
 & & & \downarrow{\scriptstyle 1} & & & \\
\longrightarrow & \Bbbk & \xrightarrow[0]{\mathrm{H}_0(\alpha)} & A_0 & \xrightarrow[1]{\mathrm{H}_0(\beta)} & G_0 & \longrightarrow 0
\end{array}$$

where the numbers below the arrows indicate the ranks of the maps. As $\mathrm{H}(\beta)$ is a homomorphism of graded $\Bbbk$-algebras, there is a commutative diagram

$$\begin{array}{ccc}
G_2 & \xrightarrow[2m+1]{\delta_G} & \operatorname{Hom}_{\Bbbk}(G_1, G_3) \\
{\scriptstyle \mathrm{H}_2(\beta)}\uparrow{\scriptstyle \mu(\mathfrak{a})-2} & & {\scriptstyle 2m}\downarrow{\scriptstyle \operatorname{Hom}_{\Bbbk}(\mathrm{H}_1(\beta), G_3)} \\
A_2 & \xrightarrow{\ \varepsilon\ } & \operatorname{Hom}_{\Bbbk}(A_1, G_3) \\
\| & & \uparrow{\scriptstyle \operatorname{Hom}_{\Bbbk}(A_1, \mathrm{H}_3(\beta))} \\
A_2 & \xrightarrow[r]{\delta_A} & \operatorname{Hom}_{\Bbbk}(A_1, A_3)
\end{array}$$

The rank of δ_A is at least the rank of $\varepsilon = \operatorname{Hom}_{\Bbbk}(A_1, \mathrm{H}_3(\beta)) \circ \delta_A$. The rank of ε is at least $(\mu(\mathfrak{a})-2)-1$ as δ_G is an isomorphism, see [1, 1.4.2], and the kernel of $\operatorname{Hom}_{\Bbbk}(\mathrm{H}_1(\beta), G_3)$ has rank 1. Thus, one has $r = \operatorname{rank}(\delta_A) \ge \mu(\mathfrak{a}) - 3$.

In case $\mu(\mathfrak{a}) \ge 6$, one has $r \ge 3$, and since the type of R is 2, this implies that R is of class $\mathbf{G}(r)$; see (2.1.1). This proves part (c) and the last case of part (b). For $m = 2$ one has $4 \le \mu(\mathfrak{a}) \le 7$. If $\mu(\mathfrak{a}) = 5$ it follows from Proposition (2.2) and (2.1.1) that R is of class $\mathbf{B}$. If $\mu(\mathfrak{a}) = 4$, then R is of class $\mathbf{H}(3,2)$ by [1, 3.4.2.(a)]. Finally, part (a) is a result of Faucett [7]. □

3. A family of graded local rings of class $\mathbf{G}(r)$

A grade 3 Gorenstein ideal of a local ring is by a result of Buchsbaum and Eisenbud [4, thm. 2.1] minimally generated by the sub-maximal Pfaffians of a $(2m+1) \times (2m+1)$ skew-symmetric matrix. Thus, skew-symmetric matrices are a source of Gorenstein rings and, via Theorem (2.4), also a source of rings of class $\mathbf{G}(r)$ that are not Gorenstein. In this section, we construct an infinite family of such rings.

(3.1) Let $\Bbbk$ be a field and set $Q = \Bbbk[\![x,y,z]\!]$; let m be a positive integer.
Denote by U_m the $m \times m$ matrix over Q whose i^{th} row has entries

$$u_{i,m-i} = x\,, \quad u_{i,m-i+1} = z\,, \quad \text{and} \qquad u_{i,m-i+2} = y$$

and 0 elsewhere; set

$$d_{-1} = 0\,, \quad d_0 = 1\,, \quad \text{and} \quad d_m = \det(U_m)\,.$$

That is,

$$U_1 = [z]\,, \quad U_2 = \begin{bmatrix} x & z \\ z & y \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & x & z \\ x & z & y \\ z & y & 0 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 0 & 0 & x & z \\ 0 & x & z & y \\ x & z & y & 0 \\ z & y & 0 & 0 \end{bmatrix}, \quad \ldots$$

$$d_1 = z\,, \qquad d_2 = xy - z^2\,, \quad d_3 = 2xyz - z^3\,, \qquad d_4 = -3xyz^2 + x^2y^2 + z^4\,, \quad \ldots$$

Notice that for every i in the range $2,\ldots,m$ one has,

$$U_m \;=\; \left[\begin{array}{c|c} O_x & U_{i-1} \\ \hline U_{m-i+1} & {}^yO \end{array}\right], \tag{3.1.1}$$

where O_x is the appropriately sized matrix with x in the lower right corner and 0 elsewhere, and ${}^yO$ is the matrix with y in the top left corner and 0 elsewhere.

Let V_m be the $(2m+1) \times (2m+1)$ skew-symmetric matrix given by

$$V_m \;=\; \left[\begin{array}{c|c|c} O & O_x & U_m \\ \hline -(O_x)^T & 0 & {}^yO \\ \hline -U_m & -({}^yO)^T & O \end{array}\right], \tag{3.1.2}$$

where O is the $m \times m$ zero-matrix and, as above, O_x and ${}^yO$ are appropriately sized matrices with 0 everywhere but in the lower left and upper right corner, respectively. That is,

$$V_1 = \begin{bmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 & 0 & 0 & x & z \\ 0 & 0 & x & z & y \\ 0 & -x & 0 & y & 0 \\ -x & -z & -y & 0 & 0 \\ -z & -y & 0 & 0 & 0 \end{bmatrix}, \quad \ldots \tag{3.1.3}$$

The sub-maximal Pfaffians of V_m are determined (up to a sign) by minors, $\operatorname{pf}_i(V_m)^2 = \det((V_m)_{ii})$. Consider the ideal of Q generated by these Pfaffians,

$$\mathfrak{g}_m \;=\; (\operatorname{pf}_1(V_m),\ldots,\operatorname{pf}_{2m+1}(V_m))\,. \tag{3.1.4}$$

(3.2) **Lemma.** *In the notation from* (3.1) *the next equalities hold for every* $m \geq 1$.

$$\begin{aligned} d_m &= (-1)^{m-1} z d_{m-1} + xy d_{m-2} \quad \text{and} \\ d_m &= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-j}{j} (-1)^{\lfloor \frac{m-2j}{2} \rfloor} x^j y^j z^{m-2j} \,. \end{aligned}$$

Proof. Per (3.1.1) with $i = 2$, expansion of the determinant of U_m along the first row yields

$$d_m = (-1)^m x \det((U_m)_{1,m-1}) + (-1)^{m+1} z \det(U_{m-1}) \,.$$

From (3.1.1) with $i = 3$ it follows that expansion along the last column yields

$$\det((U_m)_{1,m-1}) = (-1)^m y \det(U_{m-2}) \,.$$

Combining these two expressions, one gets the first equality. The second equality now follows by induction. ☐

Evidently, the ideal $\mathfrak{g}_m$ from (3.1.4) is contained in $\mathfrak{n}^m$; in fact, one has $\mathfrak{g}_1 = \mathfrak{n}$. One can check that, though the generating matrices are different, the family of ideals $\{\mathfrak{g}_m\}_{m \geq 2}$ is the same as that provided by [4, prop. 6.2]. To understand what happens when one trims these ideals, we provide a more detailed description.

(3.3) **Proposition.** *Adopt the notation from* (3.1) *and let* $\mathfrak{n}$ *denote the maximal ideal of* Q. *For every* $m \geq 2$ *the ideal* $\mathfrak{g}_m \subseteq \mathfrak{n}^2$ *is an* $\mathfrak{n}$*-primary Gorenstein ideal minimally generated by the elements*

$$x^{m-i} d_i \quad \text{and} \quad y^{m-i} d_i \quad \text{for} \quad 0 \leq i \leq m-1 \quad \text{and} \quad d_m \,.$$

The ring $Q/\mathfrak{g}_m$ *has socle generated by the class of* $x^{m-1}y^{m-1}$ *and Hilbert series*

$$\operatorname{Hilb}_{Q/\mathfrak{g}_m}(t) = \sum_{i=0}^{m-2} \binom{i+2}{2} \left(t^i + t^{2m-2-i}\right) + \binom{m+1}{2} t^{m-1} \,.$$

Proof. Per (3.1.3) the Pfaffians of V_1 are, up to signs,

$$\operatorname{pf}_1(V_1) = y = y d_0 \,, \quad \operatorname{pf}_2(V_1) = z = d_1 \,, \quad \text{and} \quad \operatorname{pf}_3(V_1) = x = x d_0 \,.$$

For $m \geq 2$ we argue that, up to signs, one has

$$\begin{aligned} \operatorname{pf}_i(V_m) &= y^{m-i+1} d_{i-1} \quad \text{for } 1 \leq i \leq m \,, \\ \operatorname{pf}_{m+1}(V_m) &= d_m \,, \text{ and} \\ \operatorname{pf}_{2m+2-i}(V_m) &= x^{m-i+1} d_{i-1} \quad \text{for } 1 \leq i \leq m \,. \end{aligned}$$

First notice that the equality $\operatorname{pf}_{m+1}(V_m) = d_m$ is immediate from (3.1.2). Further, note that by symmetry in x and y it is sufficient to prove that $\operatorname{pf}_i(V_m) = y^{m-i+1} d_{i-1}$ holds for $1 \leq i \leq m$. To compute $\operatorname{pf}_1(V_m)$ notice that the matrix $(V_m)_{11}$ is a $2m \times 2m$-matrix with $\pm y$ on the anti-diagonal and zeros below it. Thus, one has $\operatorname{pf}_1(V_m) = y^m = y^m d_0$. Now, for i in the range $2, \dots, m$ consider the matrix $(V_m)_{ii}$ as a 2×2 block matrix with blocks of size $m \times m$,

$$(V_m)_{ii} = \left[\begin{array}{c|c} X & W_i \\ \hline -W_i^T & O \end{array} \right],$$

where O is as in (3.1.2), i.e. it is zero. Thus, one has

$$\det((V_m)_{ii}) = \left|\begin{array}{c|c} X & W_i \\ \hline -W_i^T & O \end{array}\right| = (-1)^m \left|\begin{array}{c|c} W_i & X \\ \hline O & -W_i^T \end{array}\right| = (\det(W_i))^2 .$$

Next, notice that W_i is obtained from U_m by removing row i and adding a row ${}^yO$ at the bottom. Thus, per (3.1.1) it has the form

$$W_i = \left[\begin{array}{c|c} O_x & U_{i-1} \\ \hline Y & O \end{array}\right],$$

where Y is the matrix obtained from U_{m-i+1} by removing the first row and adding a row ${}^yO$ at the bottom. In particular, it is a $(m-i+1)\times(m-i+1)$-matrix with $\pm y$ on the anti-diagonal and zeros below it. Thus, computing the determinant of W_i by successive expansion on the last $m-i+1$ rows one gets, up to a sign, $\mathrm{pf}_i(V_m) = y^{m-i+1}d_{i-1}$. It follows that $\mathfrak{g}_m$ is generated by the listed elements.

The elements x^m, y^m, d_m form a Q-regular sequence in $\mathfrak{g}_m$, so it follows from [4, thm. 2.1] that $\mathfrak{g}_m$ is a Gorenstein ideal minimally generated by the listed elements. In particular, $\mathfrak{g}_m$ is $\mathfrak{n}$-primary. In fact, in this case it is elementary to see that the generating set is minimal: Notice from Lemma (3.2) that d_i is a linear combination of monomials of the form $x^jy^jz^{i-2j}$. Hence, each generator $x^{m-i}d_i$ is a linear combination of monomials of the form $x^{m-i+j}y^jz^{i-2j}$ while the generators $y^{m-i}d_i$ are linear combinations of monomials $x^jy^{m-i+j}z^{i-2j}$. Thus the generators are linear combinations of disjoint sets of degree m monomials and hence linearly independent.

The Hilbert series of the power series ring Q is $\mathrm{Hilb}_Q(t) = \sum_{i=0}^{\infty}\binom{i+2}{2}t^i$. Since $\mathfrak{g}_m$ is Gorenstein and minimally generated by $2m+1$ elements of degree m, the Hilbert series of the ring $S_m = Q/\mathfrak{g}_m$ is symmetric and given by

$$\mathrm{Hilb}_{S_m}(t) = \sum_{i=0}^{m-2}\binom{i+2}{2}\left(t^i + t^{2m-2-i}\right) + \binom{m+1}{2}t^{m-1} .$$

In particular, the socle degree of S_m is $2m-2$. Evidently, one has $(x^{m-1}y^{m-1})\mathfrak{n} \subseteq \mathfrak{g}_m$, so it is sufficient to show that the element $x^{m-1}y^{m-1}$ is not in $\mathfrak{g}_m$, i.e. that it yields a non-zero socle element in S_m. If it were in $\mathfrak{g}_m$, then one would have $x(x^{m-2}y^{m-1})$ in $\mathfrak{g}_m$ along with $x^{m-2}(y^md_0) = y(x^{m-2}y^{m-1})$ and $x^{m-2}(y^{m-1}d_1) = z(x^{m-2}y^{m-1})$. Thus, $x^{m-2}y^{m-1}$ would yield a socle element in S_m of degree $2m-3$, whence it must be 0; i.e. one would have $x^{m-2}y^{m-1} \in \mathfrak{g}_m$. Reiterating this argument, one arrives at the conclusion that y^{m-1} is in $\mathfrak{g}_m$, which is absurd as the generators of $\mathfrak{g}_m$ have degree m. □

Finally, we apply the trimming procedure from Theorem (2.4) to the ideals $\mathfrak{g}_m$.

(3.4) Adopt the notation from (3.1). By Proposition (3.3) one has

$$\mathfrak{g}_2 = (x^2, xz, xy - z^2, yz, y^2) .$$

Trimming the generators xz and yz one gets the following ideals of Q,

$$(x,y,z)xz + (x^2, xy - z^2, yz, y^2) = (x^2, xy - z^2, yz, y^2) \quad \text{and}$$
$$(x,y,z)yz + (x^2, xz, xy - z^2, y^2) = (x^2, xz, xy - z^2, y^2) .$$

They are both minimally generated by 4 elements, so they define quotient rings of class $\mathbf{H}(3,2)$; see Theorem (2.4)(b). Moreover, one has

$$\begin{aligned}(x,y,z)x^2+(xz,xy-z^2,yz,y^2) &= (x^3,xz,xy-z^2,yz,y^2)\,,\\ (x,y,z)y^2+(x^2,xz,xy-z^2,yz) &= (x^2,xz,xy-z^2,yz,y^3)\,, \quad \text{and}\\ (x,y,z)(xy-z^2)+(x^2,xz,yz,y^2) &= (x^2,xz,z^3,yz,y^2)\,,\end{aligned}$$

so by Theorem (2.4)(b) these ideals define rings of class $\mathbf{B}$.

From the next result one immediately gets the statement of Theorem (1.1) about existence of infinite families of rings of class $\mathbf{G}(r)$ that are not Gorenstein.

(3.5) **Proposition.** *Adopt the notation from* (3.1) *and let* $\mathfrak{n}$ *denote the maximal ideal of* Q. *Let* g *be one of the generators of* $\mathfrak{g}_m$ *listed in* (3.3), *let* $\mathfrak{b}$ *be the ideal generated by the remaining* $2m$ *generators of* $\mathfrak{g}_m$, *and set* $\mathfrak{a}=\mathfrak{n}g+\mathfrak{b}$. *For* $m\geq 3$ *the ring* $R=Q/\mathfrak{a}$ *has the following properties.*

- (a) *R is an artinian local ring of type 2 with socle generated by the classes of the elements g and $x^{m-1}y^{m-1}$.*
- (b) *If g is $x^{m-i}d_i$ or $y^{m-i}d_i$ for some $i\in\{1,\dots,m-1\}$, then $\mathfrak{a}$ is minimally generated by $2m$ elements and R is of class $\mathbf{G}(2m-3)$.*
- (c) *If g is x^m, y^m, or d_m, then $\mathfrak{a}$ is minimally generated by $2m+1$ elements and R is of class $\mathbf{G}(2m-2)$.*

Proof. Fix $m\geq 3$; for brevity the class in R or $S=Q/\mathfrak{g}_m$ of an element u in Q is also written u.

Part (a) is immediate from Lemma (2.3). We prove parts (b) and (c) together. First we describe the generators of $\mathfrak{a}$ using the recurrence formula from Lemma (3.2). For $1\leq i\leq m$ one has

$$\begin{aligned}x(x^{m-i}d_i) &= x^{m-(i-1)}((-1)^{i-1}zd_{i-1}+xyd_{i-2})\\ &= (-1)^{i-1}z(x^{m-(i-1)}d_{i-1})+y(x^{m-(i-2)}d_{i-2})\,.\end{aligned} \tag{1}$$

For $0\leq i\leq m-2$ one has

$$\begin{aligned}y(x^{m-i}d_i) &= x^{m-(i+1)}(xyd_i)\\ &= x^{m-(i+1)}(d_{i+2}-(-1)^{i+1}zd_{i+1})\\ &= x(x^{m-(i+2)}d_{i+2})+(-1)^iz(x^{m-(i+1)}d_{i+1}) \quad \text{and moreover}\\ y(xd_{m-1}) &= x(yd_{m-1})\,.\end{aligned} \tag{2}$$

For $0\leq i\leq m-1$ one has

$$\begin{aligned}z(x^{m-i}d_i) &= x^{m-i}(-1)^i(d_{i+1}-xyd_{i-1})\\ &= (-1)^ix(x^{m-(i+1)}d_{i+1})-(-1)^iy(x^{m-(i-1)}d_{i-1})\,.\end{aligned} \tag{3}$$

For $g=x^{m-i}d_i$ with $1\leq i\leq m-1$ it follows immediately from (1)–(3) that $\mathfrak{n}g$ is contained in $\mathfrak{b}$, so $\mathfrak{a}=\mathfrak{b}$ is minimally generated by $2m$ elements. By symmetry the same is true for $g=y^{m-i}d_i$ with $1\leq i\leq m-1$.

For $g=x^m$ one has $yg\in\mathfrak{b}$ and $zg\in\mathfrak{b}$ by (2) and (3), so $\mathfrak{a}$ is generated by the $2m$ generators of $\mathfrak{b}$ and x^{m+1}. To see that this is a minimal set of generators, note that the generators of $\mathfrak{b}$ have degree m and none of them includes the term x^m. The statement for $g=y^m$ follows by symmetry.

For $g = d_m$ one has $xg \in \mathfrak{b}$ by (1) and $yg \in \mathfrak{b}$ by symmetry. Thus $\mathfrak{a}$ is generated by the $2m$ generators of $\mathfrak{b}$ and zd_m. To see that this is a minimal set of generators, note from Lemma (3.2) that zd_m has a z^{m+1} term, while the generators of $\mathfrak{b}$ have degree m and none of them has a z^m term.

To determine the multiplicative structure on $A = \mathrm{H}(K^R)$ we first describe a basis for A_1. The Koszul complex K^R is the exterior algebra of the free R-module with basis $\{\varepsilon_x, \varepsilon_y, \varepsilon_z\}$ endowed with the differential given by $\partial(\varepsilon_x) = x$, $\partial(\varepsilon_y) = y$, and $\partial(\varepsilon_z) = z$. We suppress the wedge in products on K^R and adopt the following shorthands

$$\varepsilon_{xy} = \varepsilon_x\varepsilon_y\,, \quad \varepsilon_{xz} = \varepsilon_x\varepsilon_z\,, \quad \varepsilon_{yz} = \varepsilon_y\varepsilon_z\,, \quad \text{and} \quad \varepsilon_{xyz} = \varepsilon_x\varepsilon_y\varepsilon_z\,.$$

Because of the symmetry in x and y we only consider $g = x^{m-i}d_i$. Given the minimal generating set of $\mathfrak{a}$ described above, one gets:

If $g = x^m$ then the following cycles in K^R_1 yield a basis for A_1

$$\begin{aligned} x^m\varepsilon_x \quad \text{and} \quad & x^{m-j-1}d_j\varepsilon_x \text{ for } 1 \le j \le m-1, \\ & y^{m-j-1}d_j\varepsilon_y \text{ for } 0 \le j \le m-1, \quad \text{and} \\ (-1)^{m-1}z^{m-1}&d_{m-1}\varepsilon_z + xd_{m-2}\varepsilon_y\,. \end{aligned}$$

If $g = x^{m-i}d_i$ for some i in the range $1, \dots, m-1$, then the following cycles in K^R_1 yield a basis for A_1

$$\begin{aligned} & x^{m-j-1}d_j\varepsilon_x \text{ for } 0 \le j \le m-1,\ j \ne i \\ & y^{m-j-1}d_j\varepsilon_y \text{ for } 0 \le j \le m-1, \quad \text{and} \\ (-1)^{m-1}z^{m-1}&d_{m-1}\varepsilon_z + xd_{m-2}\varepsilon_y\,. \end{aligned}$$

If $g = d_m$ then the following cycles in K^R_1 yield a basis for A_1

$$\begin{aligned} & x^{m-j-1}d_j\varepsilon_x \text{ for } 0 \le j \le m-1, \\ & y^{m-j-1}d_j\varepsilon_y \text{ for } 0 \le j \le m-1, \quad \text{and} \\ & d_m\varepsilon_z\,. \end{aligned}$$

From Theorem (2.4) it is known that R is of class $\mathbf{G}(r)$ with $\mu(\mathfrak{a}) - 3 \le r$. To prove that equality holds, which is the claim in (b) and (c), it suffices to show that the kernel of δ has rank at least $(\mu(\mathfrak{a}) + 1) - (\mu(\mathfrak{a}) - 3) = 4$; see (2.1). To this end we first notice that the cycles $g\varepsilon_{xy}$, $g\varepsilon_{xz}$, and $g\varepsilon_{yz}$ yield linearly independent elements of A_2. Assume towards a contradiction that they are not, then there exists an element $h\varepsilon_{xyz}$ in K^Q_3 and elements q_1, q_2, and q_3 in Q and not all in $\mathfrak{n}$ with

$$\partial(h\varepsilon_{xyz}) - (q_1g\varepsilon_{xy} + q_2g\varepsilon_{xz} + q_3g\varepsilon_{yz}) \in \mathfrak{a}K^Q_2\,.$$

That is, one has $zh - q_1g \in \mathfrak{a}$, $yh + q_2g \in \mathfrak{a}$, and $xh - q_3g \in \mathfrak{a}$, and hence $h \notin \mathfrak{n}^m$ as $g \notin \mathfrak{a} + \mathfrak{n}^{m+1}$. Furthermore, the class of h is a socle element in S as one has $\mathfrak{n}h \subseteq \mathfrak{a} + Qg = \mathfrak{g}_m$. Thus, $h \in \mathfrak{g}_m$ or $h = qx^{m-1}y^{m-1}$ for some $q \in Q \setminus \mathfrak{n}$. In either case one has $h \in \mathfrak{n}^m$, which is a contradiction. Thus $g\varepsilon_{xy}$, $g\varepsilon_{xz}$, and $g\varepsilon_{yz}$ yield linearly independent elements in A_2 that clearly belong to the kernel of δ.

Finally we produce a fourth element in the kernel. For $g = x^n$ the element

$$f = y^{m-1}\varepsilon_{yz}$$

is clearly a cycle in K^R_2, and it is not a boundary. Indeed, if one had $f = \partial(h\varepsilon_{xyz}) = hx\varepsilon_{yz} - hy\varepsilon_{xz} + hz\varepsilon_{xy}$ for some homogeneous element $h \in R$, then it would have degree $m-2$ and one would have $hy = 0 = hz$ in R, which is impossible as $\mathfrak{a}$ has generators of degree at least m. The products $(y^{m-j-1}d_j\varepsilon_y)\cdot f$ and $((-1)^{m-1}z^{m-1}\varepsilon_z + xd_{m-2}\varepsilon_y)\cdot f$ in K^R vanish by graded commutativity. Moreover, one has

$$(x^m\varepsilon_x)\cdot f = x(x^{m-1}y^{m-1})\varepsilon_{xyz} = 0 \quad \text{and}$$
$$(x^{m-j-1}d_j\varepsilon_x)\cdot f = x^{m-j-1}y^{j-1}(y^{m-j}d_j)\varepsilon_{xyz} = 0\,.$$

Thus the homology class of f annihilates A_1.

For $g = x^{m-i}d_i$ and $1 \le i \le m-1$ the element

$$f = y^{m-i}d_{i-1}\varepsilon_{xy} + (-1)^{i-1}y^{m-i-1}d_i\varepsilon_{yz}$$

is a cycle in K_2^R; indeed one has

$$\begin{aligned}
\partial(f) &= xy^{m-i}d_{i-1}\varepsilon_y - y^{m-(i-1)}d_{i-1}\varepsilon_x + (-1)^{i-1}y^{m-i}d_i\varepsilon_z + (-1)^i y^{m-i-1}zd_i\varepsilon_y \\
&= y^{m-i-1}((-1)^i zd_i + xyd_{i-1})\varepsilon_y \\
&= y^{m-(i+1)}d_{i+1} \\
&= 0\,,
\end{aligned}$$

where the third equality follows from Lemma (3.2). An argument similar to the one above shows that f is not a boundary. The products $(y^{m-j-1}d_j\varepsilon_y)\cdot f$ in K^R vanish by graded commutativity. Moreover, one has

$$(x^{m-j-1}d_j\varepsilon_x)\cdot f = (-1)^{i-1}x^{m-j-1}d_jy^{m-i-1}d_i\varepsilon_{xyz}\,.$$

If $i > j$ holds, then the element $x^{m-j-1}d_jy^{m-i-1}d_i$ is 0 in R because it is divisible by g, which is a socle element in R. If one has $i < j$, then the element $x^{m-j-1}d_jy^{m-i-1}d_i$ is zero in R because it is divisible in Q by the generator $y^{m-j}d_j$ of $\mathfrak{a}$. Finally, one has

$$\begin{aligned}
((-1)^{m-1}z^{m-1}d_{m-1}\varepsilon_z + xd_{m-2}\varepsilon_y)\cdot f &= (-1)^{m-1}y^{m-i}d_{i-1}z^{m-1}d_{m-1}\varepsilon_{xyz} \\
&= (-1)^{m-1}y^{m-i-1}d_{i-1}z^{m-1}(yd_{m-1})\varepsilon_{xyz} \\
&= 0
\end{aligned}$$

in K^R, so the homology class of f annihilates A_1.

For $g = d_m$ the element

$$f = d_{m-1}\varepsilon_{xy}$$

is evidently a cycle in K_2^R, and as above it is not a boundary. The products $(x^{m-j-1}d_j\varepsilon_x)\cdot f$ and $(y^{m-j-1}d_j\varepsilon_y)\cdot f$ in K^R vanish by graded commutativity. Finally one has,

$$(d_m\varepsilon_z)\cdot f = d_{m-1}d_m\varepsilon_{xyz} = 0\,,$$

as $g = d_m$ is a socle element of R. □

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References

1. Luchezar L. Avramov, *A cohomological study of local rings of embedding codepth 3*, J. Pure Appl. Algebra **216** (2012), no. 11, 2489–2506. MR2927181
2. Luchezar L. Avramov and Evgeniy S. Golod, *On the homology algebra of the Koszul complex of a local Gorenstein ring*, Mat. Zametki **9** (1971), 53–58. MR0279157
3. Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, *Poincaré series of modules over local rings of small embedding codepth or small linking number*, J. Algebra **118** (1988), no. 1, 162–204. MR0961334
4. David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485. MR0453723
5. Lars Winther Christensen and Oana Veliche, *Local rings of embedding codepth 3. Examples*, Algebr. Represent. Theory **17** (2014), no. 1, 121–135. MR3160716
6. Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, *Linkage classes of grade* 3 *perfect ideals*, J. Pure Appl. Algebra, to appear. Preprint arXiv:1812.11552 [math.AC].
7. Jessica Ann Faucett, *Expanding the socle of a codimension 3 complete intersection*, Rocky Mountain J. Math. **46** (2016), no. 5, 1489–1498. MR3580796
8. Junzo Watanabe, *A note on Gorenstein rings of embedding codimension three*, Nagoya Math. J. **50** (1973), 227–232. MR0319985
9. Jerzy Weyman, *On the structure of free resolutions of length* 3, J. Algebra **126** (1989), no. 1, 1–33. MR1023284

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