STABLE HOMOLOGY OVER ASSOCIATIVE RINGS

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Abstract. We analyze *stable homology* over associative rings and obtain results over Artin algebras and commutative noetherian rings. Our study develops similarly for these classes; for simplicity we only discuss the latter here.

Stable homology is a broad generalization of Tate homology. Vanishing of stable homology detects classes of rings—among them Gorenstein rings, the original domain of Tate homology. Closely related to gorensteinness of rings is Auslander's G-dimension for modules. We show that vanishing of stable homology detects modules of finite G-dimension. This is the first characterization of such modules in terms of vanishing of (co)homology *alone*.

Stable homology, like absolute homology, Tor, is a theory in two variables. It can be computed from a flat resolution of one module together with an injective resolution of the other. This betrays that stable homology is not balanced in the way Tor is balanced. In fact, we prove that a ring is Gorenstein if and only if stable homology is balanced.

Introduction

The homology theory studied in this paper was introduced by P. Vogel in the 1980s. Vogel did not publish his work, but the theory appeared in print in a 1992 paper by Goichot [18], who called it *Tate–Vogel homology.* As the name suggests, the theory is a generalization of Tate homology for modules over finite group rings. Vogel and Goichot also considered a generalization of Tate cohomology, which was studied in detail by Avramov and Veliche [5]. In that paper, the theory was called *stable cohomology*, to emphasize a relation to stabilization of module categories. To align terminology, we henceforth refer to the homology theory as *stable homology.*

For modules M and N over a ring R, stable homology is a $\mathbb{Z}$-indexed family of abelian groups $\widetilde{\operatorname{Tor}}_i^R(M,N)$. These fit into an exact sequence

$$(\dagger)\qquad \cdots \to \widetilde{\operatorname{Tor}}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N) \to \overline{\operatorname{Tor}}_i^R(M,N) \to \widetilde{\operatorname{Tor}}_{i-1}^R(M,N) \to \cdots$$

where the groups $\overline{\operatorname{Tor}}_i^R(M,N)$ and $\operatorname{Tor}_i^R(M,N)$ form, respectively, the *unbounded homology* and the standard *absolute homology* of M and N. We are thus led to study stable and unbounded homology simultaneously. Our investigation takes cues from the studies of stable cohomology [5] and absolute homology, but our results look quite different. This comes down to an inherent asymmetry in the definition

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of stable homology that is not present in either of these precursors. It manifests itself in different ways, but it is apparent in most of our results.

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When we consider stable homology $\widetilde{\mathrm{Tor}}^R(M,N)$, the ring acts on M from the right and on N from the left. In this paper an R-module is a left R-module; to distinguish right R-modules, we speak of modules over the opposite ring R°. We study stable homology over associative rings and obtain conclusive results for Artin algebras and commutative noetherian rings. In the following overview, R denotes an Artin algebra or a commutative noetherian local ring. The results we discuss are special cases of results obtained within the paper; internal references are given in parentheses.

One expects a homology theory to detect finiteness of homological dimensions. Our first two results reflect the asymmetry in the definition of stable homology.

Right vanishing (3.1, 3.2)**.** *For a finitely generated R°-module M, the following conditions are equivalent.*

(*i*) *M has finite projective dimension.*

(*ii*) $\widetilde{\mathrm{Tor}}^R_i(M,-)=0$ *for all $i \in \mathbb{Z}$.*

(*iii*) *There is an $i \geqslant 0$ with* $\widetilde{\mathrm{Tor}}^R_i(M,-)=0$.

Left vanishing (5.1, 5.12)**.** *For a finitely generated R-module N, the following conditions are equivalent.*

(*i*) *N has finite injective dimension.*

(*ii*) $\widetilde{\mathrm{Tor}}^R_i(-,N)=0$ *for all $i \in \mathbb{Z}$.*

(*iii*) *There is an $i \leqslant 0$ with* $\widetilde{\mathrm{Tor}}^R_i(-,N)=0$.

These two vanishing results reveal that stable homology cannot be balanced in the way absolute homology, Tor, is balanced.

Balancedness (4.5, 4.6, 4.7)**.** *The following conditions on R are equivalent.*

(*i*) *R has finite injective dimension over R and over R°.*

(*ii*) *For all finitely generated R°-modules M, all finitely generated R-modules N, and all $i \in \mathbb{Z}$ there are isomorphisms* $\widetilde{\mathrm{Tor}}^R_i(M,N) \cong \widetilde{\mathrm{Tor}}^{R^\circ}_i(N,M)$.

(*iii*) *For all R°-modules M, all R-modules N, and all $i \in \mathbb{Z}$ there are isomorphisms* $\widetilde{\mathrm{Tor}}^R_i(M,N) \cong \widetilde{\mathrm{Tor}}^{R^\circ}_i(N,M)$.

Another way to phrase part (*i*) above is to say that R is *Iwanaga-Gorenstein.* On that topic: a commutative noetherian ring A is *Gorenstein* (*regular* or *Cohen–Macaulay*) if the local ring $A_\mathfrak{p}$ is Iwanaga-Gorenstein (regular or Cohen–Macaulay) for every prime ideal $\mathfrak{p}$ in A. It follows that a commutative noetherian ring of finite self-injective dimension is Gorenstein, but a Gorenstein ring need not have finite self-injective dimension; consider, for example, Nagata's regular ring of infinite Krull dimension [28, appn. exa. 1].

It came as a surprise to us that balancedness of stable homology detects gorensteinness outside of the local situation (4.6): A commutative noetherian ring A is Gorenstein if and only if $\widetilde{\mathrm{Tor}}^A_i(M,N)$ and $\widetilde{\mathrm{Tor}}^A_i(N,M)$ are isomorphic for all finitely generated A-modules M and N and all $i \in \mathbb{Z}$. This turns out to be only one of several ways in which stable homology captures global properties of rings. Indeed, vanishing of stable homology detects if a commutative noetherian (not necessarily local) ring is regular (3.3), Gorenstein (3.12), or Cohen–Macaulay (5.11).

For finitely generated modules over a noetherian ring, Auslander and Bridger [1] introduced a homological invariant called the *G-dimension*. A noetherian ring is Iwanaga-Gorenstein if and only if there is an integer d such that every finitely generated left or right module has G-dimension at most d; see Avramov and Martsinkovsky [4, 3.2]. In [18, sec. 3] Tate homology was defined for modules over Iwanaga-Gorenstein rings and shown to agree with stable homology. Iacob [24] generalized Tate homology further to a setting that includes the case where M is a finitely generated R°-module of finite G-dimension. We prove:

Tate homology (6.4). *Let M be a finitely generated R°-module of finite G-dimension. For every $i \in \mathbb{Z}$ the stable homology functor $\widetilde{\mathrm{Tor}}_i^R(M,-)$ is isomorphic to the Tate homology functor $\widehat{\mathrm{Tor}}_i^R(M,-)$.*

There are two primary generalizations of G-dimension to modules that are not finitely generated: *Gorenstein projective dimension* and *Gorenstein flat dimension*. They are defined in terms of resolutions by Gorenstein projective (flat) modules, notions which were introduced by Enochs, Jenda, and collaborators; see Holm [21][1]. In general, it is not known if Gorenstein projective modules are Gorenstein flat.

Iacob's definition of Tate homology $\widehat{\mathrm{Tor}}^R(M,-)$ is for R°-modules M of finite Gorenstein projective dimension. We show (6.7) that Tate homology agrees with stable homology $\widetilde{\mathrm{Tor}}^R(M,-)$ for all such R°-modules if and only if every Gorenstein projective R°-module is Gorenstein flat.

Here is an example to illustrate how widely the various homology theories differ.

Example. Let $(R,\mathfrak{m})$ be a commutative artinian local ring with $\mathfrak{m}^2 = 0$, and assume that R is not Gorenstein. If k denotes a field, then $k[x,y]/(x^2,xy,y^2)$ is an example of such a ring. Let E be the injective hull of the residue field $R/\mathfrak{m}$. Then:

(a) Absolute homology $\mathrm{Tor}_i^R(E,E)$ is non-zero for every $i \geq 2$.

(b) Stable homology $\widetilde{\mathrm{Tor}}_i^R(E,E)$ is zero for every i.

(c) Unstable homology $\overline{\mathrm{Tor}}_i^R(E,E)$ agrees with $\mathrm{Tor}_i^R(E,E)$ for every i.

(d) Tate homology "$\widehat{\mathrm{Tor}}^R(E,E)$" is not defined.

Since R is not Gorenstein, E has infinite G-dimension; see [22, thm. 3.2]. That explains (d); see 6.1 and 6.2. It also explains (a), as the first syzygy of E is a k-vector space, whence vanishing of $\mathrm{Tor}_i^R(E,E)$ for some $i \geqslant 2$ would imply that E has finite projective dimension and hence finite G-dimension; see [4, thm. 4.9]. Left vanishing, see p. 2, yields (b), and then the exact sequence (†) on p. 1 gives (c).

Finally we return to the topic of vanishing. A much studied question in Gorenstein homological algebra is how to detect finiteness of G-dimension. A finitely generated module G over a commutative noetherian ring A has G-dimension zero if it satisfies $\mathrm{Ext}_A^i(G,A) = 0 = \mathrm{Ext}_A^i(\mathrm{Hom}_A(G,A),A)$ for all $i > 0$ *and* the canonical map $G \to \mathrm{Hom}_A(\mathrm{Hom}_A(G,A),A)$ is an isomorphism. This is the original definition [1], and the last requirement cannot be dispensed with, as shown by Jorgensen

[1] Enochs and Jenda consolidated their work in [16, chaps. 10–12], which deal almost exclusively with Gorenstein rings. In [21] Holm does Gorenstein homological algebra over associative rings, so that is our standard reference for this topic. Basic definitions are recalled in 6.6.

and Şega [26]. There are other characterizations of modules of finite G-dimension in the literature, but none that can be expressed solely in terms of vanishing of (co)homology functors. Hence, we were excited to discover:

G-dimension (3.11)**.** *Let A be a commutative noetherian ring. A finitely generated A-module M has finite G-dimension if and only if $\widetilde{\mathrm{Tor}}_i^A(M,A)=0$ holds for all $i\in\mathbb{Z}$.*

1. Tensor products of complexes

All rings in this paper are assumed to be associative algebras over a commutative ring $\Bbbk$, where $\Bbbk=\mathbb{Z}$ is always possible, and $\Bbbk=R$ works when R is commutative. For an Artin algebra R, one can take $\Bbbk$ to be an artinian ring such that R is finitely generated as a $\Bbbk$-module. We recall that in this situation the functor $\mathrm{D}(-)=\mathrm{Hom}_{\Bbbk}(-,E)$, where E is the injective hull of $\Bbbk/\mathrm{Jac}(\Bbbk)$, yields a duality between the categories of finitely generated R-modules and finitely generated R°-modules.

By $\mathsf{M}(R)$ we denote the category of R-modules. We work with complexes, index these homologically, and follow standard notation (see the appendix in [9] for what is not covered below).

1.1. For a complex X with differential ∂^X and an integer $n\in\mathbb{Z}$ the symbol $\mathrm{C}_n(X)$ denotes the cokernel of ∂^X_{n+1}, and $\mathrm{H}_n(X)$ denotes the homology of X in degree n, i.e., $\mathrm{Ker}\,\partial^X_n/\mathrm{Im}\,\partial^X_{n+1}$. A complex with $\mathrm{H}(X)=0$ is called *acyclic,* and a morphism of complexes $X\to Y$ that induces an isomorphism $\mathrm{H}(X)\to\mathrm{H}(Y)$ is called a *quasi-isomorphism.* The symbol $\simeq$ is used to decorate quasi-isomorphisms; it is also used for isomorphisms in derived categories.

1.2. For an R°-complex X and an R-complex Y, the tensor product $X\otimes_R Y$ is the $\Bbbk$-complex with degree n term $(X\otimes_R Y)_n=\coprod_{i\in\mathbb{Z}}(X_i\otimes_R Y_{n-i})$ and differential given by $\partial(x\otimes y)=\partial^X_i(x)\otimes y+(-1)^i x\otimes\partial^Y_{n-i}(y)$ for $x\in X_i$ and $y\in Y_{n-i}$.

Contrast the standard tensor product complex in 1.2 with the construction in 1.3, which first appeared in [18]. Similar constructions for Hom were also treated in [18] and in great detail by Avramov and Veliche [5]. We recall the Hom constructions in the Appendix, where we also study their interactions with the one below.

1.3 Definition. For an R°-complex X and an R-complex Y, consider the graded $\Bbbk$-module $X\mathbin{\overline{\otimes}}_R Y$ with degree n component

$$(X\mathbin{\overline{\otimes}}_R Y)_n=\prod_{i\in\mathbb{Z}}X_i\otimes_R Y_{n-i}\,.$$

Endowed with the degree -1 homomorphism ∂ defined on elementary tensors as in 1.2, it becomes a complex called the *unbounded* tensor product. It contains the tensor product, $X\otimes_R Y$, as a subcomplex. The quotient complex $(X\mathbin{\overline{\otimes}}_R Y)/(X\otimes_R Y)$ is denoted $X\mathbin{\widetilde{\otimes}}_R Y$, and it is called the *stable* tensor product.

1.4. The definition above yields an exact sequence of $\Bbbk$-complexes,

$$0\longrightarrow X\otimes_R Y\longrightarrow X\mathbin{\overline{\otimes}}_R Y\longrightarrow X\mathbin{\widetilde{\otimes}}_R Y\longrightarrow 0\,. \tag{1.4.1}$$

Notice that if X or Y is bounded, or if both complexes are bounded on the same side (above or below), then the unbounded tensor product coincides with the usual tensor product, and the stable tensor product $X\mathbin{\widetilde{\otimes}}_R Y$ is zero.

Note that the unbounded tensor product $X \mathbin{\overline{\otimes}}_R Y$ is the product totalization of the double complex $(X_i \otimes_R Y_j)$.

1.5. We collect some basic results about the unbounded and stable tensor products. For the unbounded tensor product the proofs mimic the proofs for the tensor product, and one obtains the results for the stable tensor product via (1.4.1). As above X is an R°-complex and Y is an R-complex.

(a) There are isomorphisms of $\Bbbk$-complexes

$$X \mathbin{\overline{\otimes}}_R Y \;\cong\; Y \mathbin{\overline{\otimes}}_{R^\circ} X \qquad \text{and} \qquad X \mathbin{\widetilde{\otimes}}_R Y \;\cong\; Y \mathbin{\widetilde{\otimes}}_{R^\circ} X\,.$$

(b) The functors $X \mathbin{\overline{\otimes}}_R -$ and $X \mathbin{\widetilde{\otimes}}_R -$ are additive and right exact.

(c) The functors $X \mathbin{\overline{\otimes}}_R -$ and $X \mathbin{\widetilde{\otimes}}_R -$ preserve degree-wise split exact sequences.

(d) The functors $X \mathbin{\overline{\otimes}}_R -$ and $X \mathbin{\widetilde{\otimes}}_R -$ preserve homotopy.

(e) A morphism $\alpha\colon Y \to Y'$ of R-complexes yields isomorphisms of $\Bbbk$-complexes

$$\operatorname{Cone}(X \mathbin{\overline{\otimes}}_R \alpha) \;\cong\; X \mathbin{\overline{\otimes}}_R \operatorname{Cone}\alpha \quad \text{and} \quad \operatorname{Cone}(X \mathbin{\widetilde{\otimes}}_R \alpha) \;\cong\; X \mathbin{\widetilde{\otimes}}_R \operatorname{Cone}\alpha\,.$$

It follows that $X \mathbin{\overline{\otimes}}_R \alpha$ is a quasi-isomorphism if and only if $X \mathbin{\overline{\otimes}}_R \operatorname{Cone}\alpha$ is acyclic, and similarly for $X \mathbin{\widetilde{\otimes}}_R \alpha$.

1.6 Lemma. *Let $D = (D_{i,j})$ be a double $\Bbbk$-complex. Let z be a cycle in the product totalization of D. Assume z contains a component $z_{m,n}$ that satisfies $z_{m,n} = \partial^h(x') + \partial^v(x'')$ for some $x' \in D_{m+1,n}$ and $x'' \in D_{m,n+1}$. If both*

(1) $\mathrm{H}^h_{m+k}(D_{*,n-k}) = 0$ *for every* $k > 0$ *and*

(2) $\mathrm{H}^v_{n+k}(D_{m-k,*}) = 0$ *for every* $k > 0$,

then z is a boundary in the product totalization of D.

Proof. The goal is to prove the existence of a sequence (x_i) with $x_{m+1} = x'$ and $x_m = x''$ such that

$$z_{m+k,n-k} = \partial^h(x_{m+k+1}) + \partial^v(x_{m+k})$$

holds for all $k \in \mathbb{Z}$; i.e., one has $z = \partial(x)$ in the product totalization of D. Assume k is positive and that $x_m, \dots, x_{m+k}$ have been constructed. Then

$$z_{m+k-1,n-k+1} = \partial^h(x_{m+k}) + \partial^v(x_{m+k-1})$$

holds. As z is a cycle in the product totalization of D, one has

$$\begin{aligned}
0 &= \partial^h(z_{m+k,n-k}) + \partial^v(z_{m+k-1,n-k+1})\\
&= \partial^h(z_{m+k,n-k}) + \partial^v(\partial^h(x_{m+k}) + \partial^v(x_{m+k-1}))\\
&= \partial^h(z_{m+k,n-k}) - \partial^h(\partial^v(x_{m+k}))\,.
\end{aligned}$$

Thus, $z_{m+k,n-k} - \partial^v(x_{m+k})$ is a horizontal cycle, and hence by (1) it is a horizontal boundary. There exists, therefore, an element x_{m+k+1} with $\partial^h(x_{m+k+1}) + \partial^v(x_{m+k}) = z_{m+k,m-k}$. By induction, this argument yields the elements x_{m+k} for $k > 1$; a symmetric argument using (2) yields x_{m+k} for $k < 0$. $\square$

1.7 Proposition. *Let X be an R°-complex and let Y be an R-complex.*

(a) *If X is bounded above and $X_i \otimes_R Y$ is acyclic for all i, then $X \mathbin{\overline{\otimes}}_R Y$ is acyclic.*

(b) *If Y is bounded below and $X_i \otimes_R Y$ is acyclic for all i, then $X \mathbin{\overline{\otimes}}_R Y$ is acyclic.*

Proof. The product totalization of the double complex $(X_i \otimes Y_j)$ is $X \mathbin{\overline{\otimes}_R} Y$.

(a) Let m be an upper bound for X, let n be an integer, and let $z = (z_{m+k,n-k})_k$ be a cycle in $X \mathbin{\overline{\otimes}_R} Y$ of degree $m+n$. The component $z_{m,n}$ is a cycle in $X_m \otimes_R Y$ and hence a boundary as $X_m \otimes_R Y$ is assumed to be acyclic. Moreover, the assumption that $X_i \otimes_R Y$ is acyclic for every i ensures that condition (2) in Lemma 1.6 is met. Condition (1) is satisfied due to the boundedness of X; thus z is a boundary in $X \mathbin{\overline{\otimes}_R} Y$. As n is arbitrary, it follows that $X \mathbin{\overline{\otimes}_R} Y$ is acyclic.

(b) A similar argument with the roles of m and n exchanged handles this case. □

2. Homology

Now we consider the homology of unbounded and stable tensor product complexes. Our notation differs slightly from the one employed by Goichot [18].

2.1 Definition. Let M be an R°-module and N be an R-module. Let $P \xrightarrow{\simeq} M$ be a projective resolution and let $N \xrightarrow{\simeq} I$ be an injective resolution. The $\Bbbk$-modules

$$\overline{\mathrm{Tor}}_i^R(M,N) \;=\; \mathrm{H}_i(P \mathbin{\overline{\otimes}_R} I)$$

are the *unbounded* homology modules of M and N over R, and the *stable* homology modules of M and N over R are

$$\widetilde{\mathrm{Tor}}_i^R(M,N) \;=\; \mathrm{H}_{i+1}(P \mathbin{\widetilde{\otimes}_R} I)\,.$$

2.2. As any two projective resolutions of M, and similarly any two injective resolutions of N, are homotopy equivalent, it follows from 1.5(d) that the definitions in 2.1 are independent of the choices of resolutions.

2.3. Notice from 1.4 that if M has finite projective dimension, or if N has finite injective dimension, then for every $i \in \mathbb{Z}$ one has $\mathrm{Tor}_i^R(M,N) = \overline{\mathrm{Tor}}_i^R(M,N)$ and hence $\widetilde{\mathrm{Tor}}_i^R(M,N) = 0$.

2.4. The standard liftings of module homomorphisms to morphisms of resolutions imply that the definitions of $\overline{\mathrm{Tor}}_i^R(M,N)$ and $\widetilde{\mathrm{Tor}}_i^R(M,N)$ are functorial in either argument; that is, for every $i \in \mathbb{Z}$ there are functors

$$\overline{\mathrm{Tor}}_i^R(-,-),\ \widetilde{\mathrm{Tor}}_i^R(-,-)\colon \mathsf{M}(R^\circ) \times \mathsf{M}(R) \;\longrightarrow\; \mathsf{M}(\Bbbk)\,.$$

These functors are homological in the sense that every short exact sequence of R°-modules $0 \to M' \to M \to M'' \to 0$ and every R-module N give rise to a connected exact sequence of stable homology modules

(2.4.1)
$$\cdots \to \widetilde{\mathrm{Tor}}_{i+1}^R(M'',N) \to \widetilde{\mathrm{Tor}}_i^R(M',N) \to \widetilde{\mathrm{Tor}}_i^R(M,N) \to \widetilde{\mathrm{Tor}}_i^R(M'',N) \to \cdots\,,$$

and to an analogous sequence of $\overline{\mathrm{Tor}}$ modules. Similarly, every short exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules and every R°-module M give rise to a connected exact sequence,

(2.4.2)
$$\cdots \to \widetilde{\mathrm{Tor}}_{i+1}^R(M,N'') \to \widetilde{\mathrm{Tor}}_i^R(M,N') \to \widetilde{\mathrm{Tor}}_i^R(M,N) \to \widetilde{\mathrm{Tor}}_i^R(M,N'') \to \cdots\,,$$

and an analogous sequence of $\overline{\mathrm{Tor}}$ modules. For details see [18, sec. 1].

Stable and unbounded homology entwine with absolute homology.

2.5. For every R°-module M and every R-module N the exact sequence of complexes (1.4.1) yields an exact sequence of homology modules:

$$(2.5.1) \quad \cdots \to \widetilde{\operatorname{Tor}}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N) \to \overline{\operatorname{Tor}}_i^R(M,N) \to \widetilde{\operatorname{Tor}}_{i-1}^R(M,N) \to \cdots$$

Flat resolutions. Just like absolute homology, unbounded and stable homology can be computed using flat resolutions in place of projective resolutions.

2.6 Proposition. *Let M be an R°-module and N be an R-module. If $F \xrightarrow{\simeq} M$ is a flat resolution and $N \xrightarrow{\simeq} I$ is an injective resolution, then there are isomorphisms,*

$$\overline{\operatorname{Tor}}_i^R(M,N) \cong \operatorname{H}_i(F \mathbin{\overline{\otimes}_R} I) \quad \textit{and} \quad \widetilde{\operatorname{Tor}}_i^R(M,N) \cong \operatorname{H}_{i+1}(F \mathbin{\widetilde{\otimes}_R} I)$$

for every $i \in \mathbb{Z}$, and they are natural in either argument.

Proof. Let $P \xrightarrow{\simeq} M$ be a projective resolution, then there is a quasi-isomorphism $\alpha\colon P \to F$. For every R-complex E the complex $\operatorname{Cone}(\alpha) \otimes_R E$ is acyclic by [11, cor. 7.5]. In particular, $\operatorname{Cone}(\alpha) \otimes_R I$ is acyclic, and it follows from Proposition 1.7 that $\operatorname{Cone}(\alpha) \mathbin{\overline{\otimes}_R} I$ is acyclic; now $\operatorname{Cone}(\alpha) \mathbin{\widetilde{\otimes}_R} I$ is acyclic in view of (1.4.1). Thus, the morphisms $\alpha \mathbin{\overline{\otimes}_R} I$ and $\alpha \mathbin{\widetilde{\otimes}_R} I$ are quasi-isomorphisms by 1.5(e), and the desired isomorphisms follow from 2.1; it is standard to verify that they are natural. □

In view of 1.4 the next result is now immediate.

2.7 Corollary. *For every R°-module M of finite flat dimension, $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$ holds for every $i \in \mathbb{Z}$.* □

2.8 Remark. If every flat R°-module has finite projective dimension, then Corollary 2.7 is covered by 2.3. There are wide classes of rings over which flat modules have finite projective dimension: Iwanaga-Gorenstein rings, see [16, thm. 9.1.10], and more generally rings of finite finitistic projective dimension, see Jensen [25, prop. 6]; in a different direction, rings of cardinality at most $\aleph_n$ for some $n \in \mathbb{N}$, see Gruson and Jensen [20, thm. 7.10].

On the other hand, recall that any product of fields is a von Neumann regular ring, i.e., every module over such a ring is flat. Osofsky [29, 3.1] shows that a sufficiently large product of fields has infinite global dimension and hence must have flat modules of infinite projective dimension.

The next result is an analogue for stable homology of [5, thm. 2.2].

2.9 Proposition. *Let M be an R°-module and let $n \in \mathbb{Z}$. The following conditions are equivalent.*

- (i) *The connecting morphism $\widetilde{\operatorname{Tor}}_i^R(M,-) \to \operatorname{Tor}_i^R(M,-)$ is an isomorphism for every $i \geqslant n$.*
- (ii) $\operatorname{Tor}_i^R(M,E) = 0$ *for every injective R-module E and every $i \geqslant n$.*
- (iii) $\overline{\operatorname{Tor}}_i^R(M,-) = 0$ *for every $i \geqslant n$.*

Proof. For every injective R-module E and for every $i \in \mathbb{Z}$ one has $\widetilde{\operatorname{Tor}}_i^R(M,E) = 0$ by 2.3. Thus (i) implies (ii).

In the following we use the notation $(-)_{\supset s}$ for the soft truncation below at s.

$(ii) \Longrightarrow (iii)$: Let N be an R-module and $N \xrightarrow{\simeq} I$ be an injective resolution. Let $P \xrightarrow{\simeq} M$ be a projective resolution and let P^+ denote its mapping cone. Let E be an injective R-module; by assumption one has $\operatorname{Tor}_i^R(M,E) = 0$ for all $i \geqslant n$. By right exactness of the tensor product, $P^{+}_{\supset n-2} \otimes_R E$ is acyclic and hence $P^{+}_{\supset n-2} \mathbin{\overline{\otimes}}_R I$ is acyclic by 1.5(a) and Proposition 1.7. Thus, for every $i \geqslant n$ one has

$$\overline{\operatorname{Tor}}_i^R(M,N) = \mathrm{H}_i(P \mathbin{\overline{\otimes}}_R I) = \mathrm{H}_i(P^{+}_{\supset n-2} \mathbin{\overline{\otimes}}_R I) = 0.$$

(Notice that for all $n \leqslant 0$ one has $P^{+}_{\supset n-2} = P^{+}$.)

(iii) $\Longrightarrow$ *(i)*: This is immediate from the exact sequence (2.5.1). $\square$

As vanishing of $\operatorname{Tor}^R_{\gg 0}(M,-)$ detects finite flat dimension of M, the next result is an immediate consequence of Proposition 2.9. In the parlance of [15] it says that the flat and copure flat dimensions agree for R°-modules M with $\widetilde{\operatorname{Tor}}^R(M,-) = 0$.

2.10 Corollary. *Let M be an R°-module. If $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$ for all $i \in \mathbb{Z}$, then*

$$\operatorname{fd}_{R^\circ} M = \sup\{i \in \mathbb{Z} \mid \operatorname{Tor}_i^R(M,E) \neq 0 \text{ for some injective } R\text{-module } E\}\,. \quad \square$$

Dimension shifting is a useful tool in computations with stable homology.

2.11. For a module X, we denote by $\Omega_m X$ the m^{th} syzygy in a projective resolution of X and by $\Omega^m X$ the m^{th} cosyzygy in an injective resolution of X. By Schanuel's lemma a syzygy/cosyzygy is uniquely determined up to a projective/injective summand. In view of 2.3 and the exact sequences in 2.4 one gets

$$\widetilde{\operatorname{Tor}}_i^R(M,N) \cong \widetilde{\operatorname{Tor}}_{i-m}^R(\Omega_m M, N) \quad \text{for } m \geqslant 0 \quad \text{and} \tag{2.11.1}$$

$$\widetilde{\operatorname{Tor}}_i^R(M,N) \cong \widetilde{\operatorname{Tor}}_{i+m}^R(M, \Omega^m N) \quad \text{for } m \geqslant 0\,. \tag{2.11.2}$$

Moreover, if $F \xrightarrow{\simeq} M$ is a flat resolution of M, then there are isomorphisms,

$$\widetilde{\operatorname{Tor}}_i^R(M,N) \cong \widetilde{\operatorname{Tor}}_{i-m}^R(\mathrm{C}_m(F), N) \quad \text{for } m \geqslant 0\,; \tag{2.11.3}$$

this follows from (2.4.1) and Corollary 2.7.

2.12. Suppose R is commutative and S is a flat R-algebra. For every S°-module M, a projective resolution $P \xrightarrow{\simeq} M$ over S° is a flat resolution of M as an R-module. Thus, it follows from Proposition 2.6 that stable homology $\widetilde{\operatorname{Tor}}_i^R(M,-)$ is a functor from $\mathsf{M}(R)$ to $\mathsf{M}(S^\circ)$. Similarly, an injective resolution $N \xrightarrow{\simeq} I$ of an S-module is also an injective resolution of N as an R-module, whence $\widetilde{\operatorname{Tor}}_i^R(-,N)$ is a functor from $\mathsf{M}(R)$ to $\mathsf{M}(S)$.

3. Vanishing of stable homology $\widetilde{\operatorname{Tor}}(M,-)$

We open with a partial converse to Corollary 2.7; recall, for example from [16, prop. 3.2.12], that a finitely presented module is flat if and only if it is projective.

3.1 Theorem. *Let M be an R°-module that has a degree-wise finitely generated projective resolution. The following conditions are equivalent.*

(i) $\operatorname{pd}_{R^\circ} M$ *is finite.*

(ii) $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$ *for all* $i \in \mathbb{Z}$.

(iii) $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$ *for some* $i \geqslant 0$.

(iv) $\widetilde{\operatorname{Tor}}_0^R(M, \operatorname{Hom}_{\Bbbk}(M,E)) = 0$ *for some faithfully injective* $\Bbbk$*-module* E.

Moreover, if R is left noetherian with $\operatorname{id}_R R$ finite, then (i)–(iv) are equivalent to

(iii') $\widetilde{\operatorname{Tor}}_i^R(M,-) = 0$ *for some* $i \in \mathbb{Z}$.

Proof. The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iii')$ are clear; see 2.3. Part (iv) follows from (iii) by dimension shifting (2.11.2).

For $(iv) \Longrightarrow (i)$ use Theorem A.8 to get

$$0 = \widetilde{\operatorname{Tor}}_0^R(M, \operatorname{Hom}_{\Bbbk}(M, E)) \cong \operatorname{Hom}_{\Bbbk}(\widetilde{\operatorname{Ext}}_{R^\circ}^0(M, M), E)\,.$$

Thus $\widetilde{\operatorname{Ext}}_{R^\circ}^0(M, M)$ is zero, hence M has finite projective dimension by [5, prop. 2.2].

Finally, assume that R is left noetherian with $\operatorname{id}_R R$ finite. Let E be a faithfully injective $\Bbbk$-module; in view of what has already been proved, it is sufficient to show that $\widetilde{\operatorname{Tor}}_{-i}^R(M, -) = 0$ for some $i > 0$ implies $\widetilde{\operatorname{Tor}}_0^R(M, \operatorname{Hom}_{\Bbbk}(M, E)) = 0$. It follows from the assumptions on R that every free R-module has finite injective dimension. Thus every projective R-module P has finite injective dimension, and 2.3 yields $\widetilde{\operatorname{Tor}}_i^R(-, P) = 0$ for all $i \in \mathbb{Z}$. From the exact sequence (2.4.2) it follows that there are isomorphisms $\widetilde{\operatorname{Tor}}_0^R(M, \operatorname{Hom}_{\Bbbk}(M, E)) \cong \widetilde{\operatorname{Tor}}_{-i}^R(M, \Omega_i \operatorname{Hom}_{\Bbbk}(M, E))$ for $i > 0$, so vanishing of $\widetilde{\operatorname{Tor}}_{-i}^R(M, -)$ implies $\widetilde{\operatorname{Tor}}_0^R(M, \operatorname{Hom}_{\Bbbk}(M, E)) = 0$. □

3.2 Corollary. *Let R be an Artin algebra with duality functor* $\mathrm{D}(-)$*. For a finitely generated R°-module M the following conditions are equivalent.*

- (i) $\operatorname{pd}_{R^\circ} M$ *is finite.*
- (ii) *There is an $i \geqslant 0$ with $\widetilde{\operatorname{Tor}}_i^R(M, N) = 0$ for all finitely generated R-modules N.*
- (iii) $\widetilde{\operatorname{Tor}}_0^R(M, \mathrm{D}(M)) = 0$.

Proof. Part (ii) follows from (i) in view of 2.3. Part (iii) follows from (ii) by dimension shifting (2.11.2) in a degree-wise finitely generated injective resolution of $\mathrm{D}(M)$. Finally, one has $\mathrm{D}(M) = \operatorname{Hom}_{\Bbbk}(M, E)$, see the first paragraph in Section 1, so (i) follows from (iii) by Theorem 3.1, as the injective hull E of $\Bbbk/\operatorname{Jac}(\Bbbk)$ is faithfully injective. □

By a result of Bass and Murthy [6, lem. 4.5], a commutative noetherian ring is regular if and only if every finitely generated module has finite projective dimension.

3.3 Corollary. *For a commutative noetherian ring R, the following are equivalent:*

- (i) *R is regular.*
- (ii) $\widetilde{\operatorname{Tor}}_i^R(M, -) = 0$ *for all finitely generated R-modules M and all $i \in \mathbb{Z}$.*
- (iii) *There is an $i \in \mathbb{Z}$ with $\widetilde{\operatorname{Tor}}_i^R(M, -) = 0$ for all finitely generated R-modules M.*

Proof. The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ are clear; see 2.3.

$(iii) \Longrightarrow (i)$: If $i \geqslant 0$, then every finitely generated R-module has finite projective dimension by Theorem 3.1; i.e., R is regular. If $i < 0$, then for any finitely generated R-module M one has $\widetilde{\operatorname{Tor}}_0^R(M, -) \cong \widetilde{\operatorname{Tor}}_i^R(\Omega_{-i} M, -) = 0$ by dimension shifting (2.11.1), and as above it follows that R is regular. □

Tate flat resolutions. We show that under extra assumptions on M one can compute $\widetilde{\operatorname{Tor}}^R(M, N)$ without resolving the second argument.

3.4. Let N be an R-module; choose an injective resolution $N \xrightarrow{\simeq} I$ and a projective resolution $P \xrightarrow{\simeq} N$. The composite $P \to N \to I$ is a quasi-isomorphism, so the complex $\mathrm{C}_{PI}^N = \operatorname{Cone}(P \to N \to I)$ is acyclic, and up to homotopy equivalence it is independent of the choice of resolutions. This construction is functorial in N.

3.5 Lemma. *Let F be a bounded below complex of flat R°-modules; let N be an R-module with a projective resolution $P \xrightarrow{\simeq} N$ and an injective resolution $N \xrightarrow{\simeq} I$. There is an isomorphism $F \mathbin{\widetilde{\otimes}}_R I \simeq F \mathbin{\overline{\otimes}}_R \mathrm{C}^N_{PI}$ in the derived category $\mathsf{D}(\Bbbk)$, and it is functorial in the second argument; cf. 3.4.*

Proof. There is by 1.5(c) a short exact sequences of $\Bbbk$-complexes

$$0 \longrightarrow F \mathbin{\overline{\otimes}}_R I \longrightarrow F \mathbin{\overline{\otimes}}_R \mathrm{C}^N_{PI} \longrightarrow F \mathbin{\overline{\otimes}}_R \Sigma P \longrightarrow 0\,.$$

With (1.4.1) it forms a commutative diagram whose rows are triangles in $\mathsf{D}(\Bbbk)$,

$$\begin{tikzcd}
F \mathbin{\overline{\otimes}}_R P \arrow[r] \arrow[d, "\simeq"'] & F \mathbin{\overline{\otimes}}_R I \arrow[r] \arrow[d, equal] & F \mathbin{\overline{\otimes}}_R \mathrm{C}^N_{PI} \arrow[r] & \Sigma(F \mathbin{\overline{\otimes}}_R P) \arrow[d, "\simeq"'] \\
F \otimes_R I \arrow[r] & F \mathbin{\overline{\otimes}}_R I \arrow[r] & F \mathbin{\widetilde{\otimes}}_R I \arrow[r] & \Sigma(F \otimes_R I)\,,
\end{tikzcd}$$

where the top triangle is rotated back once. The isomorphism is induced by the composite quasi-isomorphism $P \to N \to I$, as one has $F \mathbin{\overline{\otimes}}_R P = F \otimes_R P$; cf. 1.4. Completion of the morphism of triangles yields the desired isomorphism via the Triangulated Five Lemma; see [23, prop. 4.3]. Functoriality in the second argument is straightforward to verify. $\square$

3.6 Lemma. *Let N be an R-module with a projective resolution $P \xrightarrow{\simeq} N$ and an injective resolution $N \xrightarrow{\simeq} I$. Let T be a complex of flat R°-modules such that $T \otimes_R E$ is acyclic for every injective R-module E. There is then an isomorphism $T \mathbin{\overline{\otimes}}_R \mathrm{C}^N_{PI} \simeq \Sigma(T \otimes_R N)$ in $\mathsf{D}(\Bbbk)$ which is functorial in the second argument; cf. 3.4.*

Proof. By the assumptions and Proposition 1.7(a) the complex $T \mathbin{\overline{\otimes}}_R I$ is acyclic, so application of $T \mathbin{\overline{\otimes}}_R -$ to the exact sequence $0 \to I \to \mathrm{C}^N_{PI} \to \Sigma P \to 0$ yields a quasi-isomorphism $T \mathbin{\overline{\otimes}}_R \mathrm{C}^N_{PI} \to T \mathbin{\overline{\otimes}}_R \Sigma P$; cf. 1.5(c). Now, let P^+ denote the mapping cone of the quasi-isomorphism $P \to N$. The complex $T \mathbin{\overline{\otimes}}_R P^+$ is acyclic by Proposition 1.7(b), so application of $T \mathbin{\overline{\otimes}}_R -$ to the mapping cone sequence $0 \to N \to P^+ \to \Sigma P \to 0$ yields an isomorphism $T \mathbin{\overline{\otimes}}_R \Sigma P \simeq \Sigma(T \mathbin{\overline{\otimes}}_R N)$ in $\mathsf{D}(\Bbbk)$, and the desired isomorphism follows as N is a module; cf. 1.4. Functoriality in the second argument is straightforward to verify; cf. 3.4. $\square$

3.7 Remark. The quasi-isomorphisms in Lemmas 3.5 and 3.6 hold with C^N_{PI} replaced by C^N_{FI} where $F \xrightarrow{\simeq} N$ is a flat resolution.

The objects discussed in the next paragraph appear in the literature under a variety of names; here we stick with the terminology from [27].

3.8. An acyclic complex T of flat R°-modules is called *totally acyclic* if $T \otimes_R E$ is acyclic for every injective R-module E. A *Tate flat resolution* of an R°-module M is a pair (T, F), where $F \xrightarrow{\simeq} M$ is a flat resolution and T is a totally acyclic complex of flat R°-modules with $T_{\geqslant n} \cong F_{\geqslant n}$ for some $n \geqslant 0$. Tate's name is invoked here because these resolutions can be used to compute Tate homology; see [27, thm. A].

3.9. Over a left coherent ring R, an R°-module M has a Tate flat resolution if and only if it has finite Gorenstein flat dimension; see [27, prop. 3.4].

If R is noetherian, then every finitely generated R°-module M of finite G-dimension has a Tate flat resolution. Here is a direct argument: By [4, thm. 3.1] there exists a pair (T, P) where $P \xrightarrow{\simeq} M$ is a projective resolution and T is an acyclic complex of finitely generated projective R°-modules with the following properties: the complex $\operatorname{Hom}_{R^\circ}(T,R)$ is acyclic, and there is an $n \geqslant 0$ with $T_{\geqslant n} \cong P_{\geqslant n}$. For every injective R-module E one now has

$$T \otimes_R E \cong T \otimes_R \operatorname{Hom}_R(R,E) \cong \operatorname{Hom}_R(\operatorname{Hom}_{R^\circ}(T,R),E) \tag{3.9.1}$$

where the last isomorphism holds because T is degree-wise finitely generated; see [7, prop. VI.5.2]. Thus T is a totally acyclic complex of flat R°-modules and (T,P) is a Tate flat resolution of M.

The next result shows that stable homology can be computed via Tate flat resolutions. We apply it in Section 6 to compare stable homology to Tate homology, but before that it is used in the proofs of Theorems 3.11 and 4.2.

3.10 Theorem. *Let M be an R°-module that has a Tate flat resolution (T,F). For every R-module N and every $i \in \mathbb{Z}$ there is an isomorphism,*

$$\widetilde{\operatorname{Tor}}{}_i^R(M,N) \;\cong\; \operatorname{H}_i(T \otimes_R N)\,,$$

and it is functorial in the second argument.

Proof. Choose $n \geqslant 0$ such that $T_{\geqslant n}$ and $F_{\geqslant n}$ are isomorphic, and consider the degree-wise split exact sequence

$$0 \longrightarrow T_{\leqslant n-1} \longrightarrow T \longrightarrow F_{\geqslant n} \longrightarrow 0\,.$$

Let $P \xrightarrow{\simeq} N$ be a projective resolution and $N \xrightarrow{\simeq} I$ be an injective resolution; set $C = \mathrm{C}^N_{PI}$; cf. 3.4. There is, by 1.5(c), an exact sequence of $\Bbbk$-complexes,

$$0 \longrightarrow T_{\leqslant n-1} \mathbin{\overline{\otimes}}_R C \longrightarrow T \mathbin{\overline{\otimes}}_R C \longrightarrow F_{\geqslant n} \mathbin{\overline{\otimes}}_R C \longrightarrow 0\,.$$

As C is acyclic, it follows from Proposition 1.7(a) that $T_{\leqslant n-1} \mathbin{\overline{\otimes}}_R C$ is acyclic, whence the surjective morphism above is a quasi-isomorphism. Since T is a totally acyclic complex of flat modules, there is by Lemma 3.6 an isomorphism $T \mathbin{\overline{\otimes}}_R C \simeq \Sigma(T \otimes_R N)$ in $\mathsf{D}(\Bbbk)$. Moreover, Lemma 3.5 yields an isomorphism $F_{\geqslant n} \mathbin{\overline{\otimes}}_R C \simeq F_{\geqslant n} \mathbin{\widetilde{\otimes}}_R I$ in $\mathsf{D}(\Bbbk)$. Thus one has $\Sigma(T \otimes_R N) \simeq F_{\geqslant n} \mathbin{\widetilde{\otimes}}_R I$ in $\mathsf{D}(\Bbbk)$. This explains the second isomorphism in the next computation.

$$\begin{aligned}
\widetilde{\operatorname{Tor}}{}_i^R(M,N) &\cong \widetilde{\operatorname{Tor}}{}_{i-n}^R(\mathrm{C}_n(F),N)\\
&= \operatorname{H}_{i-n+1}((\Sigma^{-n} F_{\geqslant n}) \mathbin{\widetilde{\otimes}}_R I)\\
&= \operatorname{H}_{i+1}(F_{\geqslant n} \mathbin{\widetilde{\otimes}}_R I)\\
&\cong \operatorname{H}_{i+1}(\Sigma(T \otimes_R N))\\
&= \operatorname{H}_i(T \otimes_R N)
\end{aligned}$$

The first isomorphism holds by (2.11.3); the first equality follows from Proposition 2.6 as the canonical surjection $\Sigma^{-n} F_{\geqslant n} \to \mathrm{C}_n(F)$ is a flat resolution. $\square$

As discussed in the introduction, the G-dimension is a homological invariant for finitely generated modules over noetherian rings. Characterizations of modules of finite G-dimension have traditionally involved both vanishing of (co)homology and invertibility of a certain morphism; see for example [9, (2.1.6), (2.2.3), (3.1.5), and (3.1.11)]. More recently, Avramov, Iyengar, and Lipman [3] showed that a finitely generated module M over a commutative noetherian ring R has finite G-dimension if and only M is isomorphic to the complex $\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(M,R),R)$ in the derive category $\mathsf{D}(R)$. The crucial step in our proof of the next theorem is to show that vanishing of stable homology $\widetilde{\operatorname{Tor}}^R(M,R)$ implies such an isomorphism.

3.11 Theorem. *Let R be a commutative noetherian ring. For a finitely generated R-module M, the following conditions are equivalent.*

(i) $\operatorname{G-dim}_R M < \infty$.

(ii) $\widetilde{\operatorname{Tor}}_i^R(M,N) = 0$ *for every R-module N of finite flat dimension and all $i \in \mathbb{Z}$.*

(iii) $\widetilde{\operatorname{Tor}}_i^R(M,R) = 0$ *for all $i \in \mathbb{Z}$.*

Proof. The implication (ii)$\Longrightarrow$(iii) is trivial.

(i)$\Longrightarrow$(ii): By 3.9 the R° module M has a Tate flat resolution (T,F). Theorem 3.10 yields isomorphisms $\widetilde{\operatorname{Tor}}_i^R(M,-) \cong \operatorname{H}_i(T \otimes_R -)$ for all $i \in \mathbb{Z}$. It follows by induction on the flat dimension of N that $T \otimes_R N$ is acyclic; cf. [10, lem. 2.3].

(iii)$\Longrightarrow$(i): Let $P \xrightarrow{\simeq} M$ be a degree-wise finitely generated projective resolution, and let $R \xrightarrow{\simeq} I$ be an injective resolution. By assumption, the complex $P \mathbin{\widetilde{\otimes}}_R I$ is acyclic, which explains the third $\simeq$ below:

$$M \simeq P \otimes_R R \simeq P \otimes_R I \simeq P \mathbin{\overline{\otimes}}_R I \cong P \mathbin{\overline{\otimes}}_R \operatorname{Hom}_R(R,I) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(P,R),I).$$

The last isomorphism holds by Proposition A.6 as $\overline{\operatorname{Hom}}_R(P,R)$ equals $\operatorname{Hom}_R(P,R)$. Now it follows from [3, thm. 2] and [9, thm. (2.2.3)] that $\operatorname{G-dim}_R M$ is finite. □

3.12 Corollary. *For a commutative noetherian ring R the following are equivalent.*

(i) *R is Gorenstein.*

(ii) $\widetilde{\operatorname{Tor}}_i^R(M,R) = 0$ *for all finitely generated R-modules M and all $i \in \mathbb{Z}$.*

(iii) $\widetilde{\operatorname{Tor}}_i^R(M,R) = 0$ *for all finitely generated R-modules M and all $i < 0$.*

Proof. By a result of Goto, R is Gorenstein if and only if every finitely generated R-module has finite G-dimension [19, cor. 2]. Combining this with Theorem 3.11 one gets the the equivalence of (i) and (ii). The implication (ii) $\Longrightarrow$ (iii) is clear. Finally, (ii) follows from (iii) by dimension shifting (2.11.1). □

4. Balancedness of stable homology

Absolute homology is balanced over any ring: there are always isomorphisms $\operatorname{Tor}^R(M,N) \cong \operatorname{Tor}^{R^\circ}(N,M)$. It follows already from Corollary 3.2 that stable homology can be balanced only over special rings. Indeed, if R is an Artin algebra and stable homology is balanced over R, then the dual module of R has finite projective dimension over both R and R°, whence R is Iwanaga-Gorenstein. The converse is part of Corollary 4.5.

We open this section with a technical lemma; it is similar to a result of Enochs, Estrada, and Iacob [14, thm. 3.6]. Recall the notation $\operatorname{C}_m(-)$ from 1.1.

4.1 Lemma. *Let T and T' be acyclic complexes of flat R°-modules and flat R-modules, respectively. For all integers m and n there are isomorphisms in $\mathsf{D}(\Bbbk)$,*

$$T_{\leqslant n-1} \otimes_R \operatorname{C}_m(T') \simeq \Sigma^{n-m}(\operatorname{C}_n(T) \otimes_R T'_{\leqslant m-1})$$
$$T_{\geqslant n} \otimes_R \operatorname{C}_m(T') \simeq \Sigma^{n-m}(\operatorname{C}_n(T) \otimes_R T'_{\geqslant m})$$
$$T \otimes_R \operatorname{C}_m(T') \simeq \Sigma^{n-m}(\operatorname{C}_n(T) \otimes_R T')\,.$$

Proof. Because the complexes T and T' are acyclic, there are quasi-isomorphisms $\mathrm{C}_n(T) \to \Sigma^{1-n} T_{\leqslant n-1}$ and $\mathrm{C}_m(T') \to \Sigma^{1-m} T'_{\leqslant m-1}$. Hence [10, prop. 2.14(b)] yields

$$\begin{aligned} T_{\leqslant n-1} \otimes_R \mathrm{C}_m(T') &\simeq T_{\leqslant n-1} \otimes_R \Sigma^{1-m} T'_{\leqslant m-1} \\ &\simeq \Sigma^{n-m}(\Sigma^{1-n} T_{\leqslant n-1} \otimes_R T'_{\leqslant m-1}) \\ &\simeq \Sigma^{n-m}(\mathrm{C}_n(T) \otimes_R T'_{\leqslant m-1}) \,. \end{aligned}$$

This demonstrates the first isomorphism in the statement, and the second is proved similarly. Finally, these two isomorphisms connect the exact sequences

$$0 \to T_{\leqslant n-1} \otimes_R \mathrm{C}_m(T') \to T \otimes_R \mathrm{C}_m(T') \to T_{\geqslant n} \otimes_R \mathrm{C}_m(T') \to 0$$

and

$$0 \to \mathrm{C}_n(T) \otimes_R T'_{\leqslant m-1} \to \mathrm{C}_n(T) \otimes_R T' \to \mathrm{C}_n(T) \otimes_R T'_{\geqslant m} \to 0 \,,$$

and one obtains the last isomorphism via the Triangulated Five Lemma in $\mathsf{D}(\Bbbk)$; see [23, prop. 4.3]. □

4.2 Theorem. *Let M be an R°-module and N be an R-module. If they both have Tate flat resolutions, then for each $i \in \mathbb{Z}$ there is an isomorphism*

$$\widetilde{\mathrm{Tor}}_i^R(M,N) \;\cong\; \widetilde{\mathrm{Tor}}_i^{R^\circ}(N,M) \,.$$

Proof. Let (T,F) and (T',F') be Tate flat resolutions of M and N, respectively. Choose $n \in \mathbb{Z}$ such that there are isomorphisms $T_{\geqslant n} \cong F_{\geqslant n}$ and $T'_{\geqslant n} \cong F'_{\geqslant n}$. For every $i \in \mathbb{Z}$ one has

$$\begin{aligned} \widetilde{\mathrm{Tor}}_i^R(M,N) &\cong \mathrm{H}_i(T \otimes_R N) \\ &\cong \mathrm{H}_{i-n}(T \otimes_R \mathrm{C}_n(T')) \\ &\cong \mathrm{H}_{i-n}(\mathrm{C}_n(T) \otimes_R T') \\ &\cong \mathrm{H}_i(M \otimes_R T') \\ &\cong \widetilde{\mathrm{Tor}}_i^{R^\circ}(N,M) \,, \end{aligned}$$

where the first and the last isomorphisms follow from Theorem 3.10, the second and fourth isomorphisms follow by dimension shifting, and the third isomorphism holds by the last isomorphism in Lemma 4.1. □

4.3 Definition. Let M be an R°-module and N be an R-module. Stable homology is *balanced* for M and N if one has $\widetilde{\mathrm{Tor}}_i^R(M,N) \cong \widetilde{\mathrm{Tor}}_i^{R^\circ}(N,M)$ for all $i \in \mathbb{Z}$.

4.4. Theorem 4.2 says that stable homology is balanced for all (pairs of) R°- and R-modules that have Tate flat resolutions. If R is Iwanaga–Gorenstein, then every R°-module and every R-module has a Tate flat resolution, see 3.9 and [16, thm. 12.3.1], so stable homology is balanced for all (pairs of) R°- and R-modules.

4.5 Corollary. *For an Artin algebra R the following conditions are equivalent.*

(*i*) *R is Iwanaga-Gorenstein.*
(*ii*) *Stable homology is balanced for all R°- and R-modules.*
(*iii*) *Stable homology is balanced for all finitely generated R°- and R-modules.*

Proof. Per 4.4, part (i) implies (ii), which clearly implies (iii). Let $E = \mathrm{D}(R)$ be the dual module of R; it is injective and finitely generated over R° and over R. Thus, if stable homology is balanced for finitely generated R°- and R modules, then it follows from 2.3 and Corollary 3.2 that $\mathrm{pd}_R\, E$ as well as $\mathrm{pd}_{R^\circ}\, E$ is finite. By duality, both $\mathrm{id}_R\, R$ and $\mathrm{id}_{R^\circ}\, R$ are then finite. $\square$

4.6 Corollary. *A commutative noetherian ring R is Gorenstein if and only if stable homology is balanced for all finitely generated R-modules.*

Proof. Over a Gorenstein ring, the G-dimension of every finitely generated module is finite by [19, cor. 2], so every finitely generated module has a Tate flat resolution; see 3.9. Thus, balancedness of stable homology follows from Theorem 4.2. Conversely, balancedness of stable homology implies $\widetilde{\mathrm{Tor}}_i^R(M,R) = 0$ for every finitely generated R-module M and all $i \in \mathbb{Z}$, so R is Gorenstein by Corollary 3.12. $\square$

4.7 Corollary. *A commutative noetherian ring R of finite Krull dimension is Gorenstein if and only if stable homology is balanced over R.*

Proof. If R is Gorenstein of finite Krull dimension, then it is Iwanaga-Gorenstein. Therefore, stable homology is balanced over R per 4.4. The converse holds by Corollary 4.6. $\square$

5. Vanishing of stable homology $\widetilde{\mathrm{Tor}}(-,N)$

Vanishing of stable homology $\widetilde{\mathrm{Tor}}_i^R(-,N)$ over an Artin algebra can by duality be understood via vanishing of $\widetilde{\mathrm{Tor}}_i^R(M,-)$.

5.1 Proposition. *Let R be an Artin algebra with duality functor* $\mathrm{D}(-)$. *For a finitely generated R-module N the following conditions are equivalent.*

(i) $\mathrm{id}_R\, N$ *is finite.*

(ii) $\widetilde{\mathrm{Tor}}_i^R(-,N) = 0$ *for all* $i \in \mathbb{Z}$.

(iii) *There is an integer $i \leqslant 0$ with $\widetilde{\mathrm{Tor}}_i^R(M,N) = 0$ for all finitely generated R°-modules M.*

(iv) $\widetilde{\mathrm{Tor}}_0^R(\mathrm{D}(N),N) = 0$.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are clear; see 2.3. Part (iv) follows from (iii) by dimension shifting (2.11.1), as $\mathrm{D}(N)$ is finitely generated. Finally, vanishing of $\widetilde{\mathrm{Tor}}_0^R(\mathrm{D}(N),N) \cong \widetilde{\mathrm{Tor}}_0^R(\mathrm{D}(N),\mathrm{D}(\mathrm{D}(N)))$ implies by Corollary 3.2 that $\mathrm{pd}_{R^\circ}\,\mathrm{D}(N)$ is finite, whence $\mathrm{id}_R\, N$ is finite, and so (i) follows from (iv). $\square$

Local rings. To analyze vanishing of stable homology $\widetilde{\mathrm{Tor}}_i^R(-,N)$ over commutative noetherian rings, we start locally.

5.2. Let R be a commutative noetherian local ring with residue field k. For an R-module M, the *depth* invariant can be defined as

$$\mathrm{depth}_R\, M = \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(k,M) \neq 0\}\,,$$

and if M is finitely generated, then its injective dimension can be computed as

$$\mathrm{id}_R\, M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(k,M) \neq 0\}\,.$$

The depth is finite for $M \neq 0$. The ring R is Cohen–Macaulay if there exists a finitely generated module $M \neq 0$ of finite injective dimension; this is a consequence of the New Intersection Theorem due to Peskine and Szpiro [30] and Roberts [32].

The following is an analogue of [5, thm. 6.1].

5.3 Lemma. *Let R be a commutative noetherian local ring with residue field k. For every finitely generated R-module N and for every $i \in \mathbb{Z}$ there is an isomorphism*

$$\overline{\mathrm{Tor}}_i^R(k,N) \;\cong\; \prod_{j\in\mathbb{Z}} \mathrm{Hom}_k(\mathrm{Ext}_R^j(k,R), \mathrm{Ext}_R^{j-i}(k,N))$$

of R-modules, and in particular, $\overline{\mathrm{Tor}}_i^R(k,N)$ is a k-vector space.

Proof. Let $P \xrightarrow{\simeq} k$ be a degree-wise finitely generated projective resolution and let $N \xrightarrow{\simeq} I$ and $R \xrightarrow{\simeq} J$ be injective resolutions. By definition $\overline{\mathrm{Tor}}_i^R(k,N)$ is the i^{th} homology of the complex $P \mathbin{\overline{\otimes}}_R I$ which we compute using Proposition A.6 as follows $P \mathbin{\overline{\otimes}}_R I \;\cong\; P \mathbin{\overline{\otimes}}_R \mathrm{Hom}_R(R,I) \;\cong\; \mathrm{Hom}_R(\mathrm{Hom}_R(P,R),I)$. Next we simplify using quasi-isomorphisms and Hom-tensor adjointness:

$$\begin{aligned}
\mathrm{Hom}_R(\mathrm{Hom}_R(P,R),I) &\simeq \mathrm{Hom}_R(\mathrm{Hom}_R(P,J),I)\\
&\simeq \mathrm{Hom}_R(\mathrm{Hom}_R(k,J),I)\\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_R(k,J)\otimes_k k,I)\\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_R(k,J),\mathrm{Hom}_R(k,I))\,.
\end{aligned}$$

Finally, pass to homology. □

5.4 Remark. Lemma 5.3 suggests that stable homology $\widetilde{\mathrm{Tor}}_i^R(k,N)$ may be a k-vector space, and indeed it is. If R is a ring and if x annihilates the R°-module M, or the R-module N, then there are two homotopic lifts—zero and multiplication by x—to the projective resolution in the case of M or to the injective resolution in the case of N; see Definition 2.1 and [34, 22.6 and 2.3.7]. Hence multiplication by x is zero on unbounded and on stable homology of M against N. In particular, stable homology $\widetilde{\mathrm{Tor}}_i^R(k,N)$ is a k-vector space.

5.5 Proposition. *Let R be a commutative noetherian local ring with residue field k and let N be a finitely generated R-module. If for some $i \in \mathbb{Z}$ the k-vector space $\widetilde{\mathrm{Tor}}_i^R(k,N)$ has finite rank, then N has finite injective dimension, or R is Gorenstein.*

Proof. Each k-vector space $\mathrm{Tor}_j^R(k,N)$ has finite rank, so $\overline{\mathrm{Tor}}_{i+1}^R(k,N)$ has finite rank by the assumption and the exact sequence (2.5.1). For a finitely generated R-module $M \neq 0$ the vector spaces $\mathrm{Ext}_R^j(k,M)$ are non-zero for all j between $\mathrm{depth}_R M < \infty$ and $\mathrm{id}_R M$; see Roberts [31, thm. 2]. When $\overline{\mathrm{Tor}}_{i+1}^R(k,N)$ has finite rank, it follows from Lemma 5.3 that R or N has finite injective dimension. □

Compared to the characterization of globally Gorenstein rings in Corollary 3.12, condition (*iii*) below is sharper.

5.6 Theorem. *Let R be a commutative noetherian local ring with residue field k. The following conditions are equivalent.*

(*i*) *R is Gorenstein.*

(*ii*) *$\widetilde{\mathrm{Tor}}_i^R(-,R) = 0$ for all $i \in \mathbb{Z}$.*

(*iii*) $\widetilde{\mathrm{Tor}}_i^R(k,R)$ *has finite rank for some* $i \in \mathbb{Z}$.

(*iv*) *There exists a finitely generated* R*-module* M *such that* $\widetilde{\mathrm{Tor}}_i^R(k,M)$ *has finite rank for some* $i \in \mathbb{Z}$ *and* $\widetilde{\mathrm{Tor}}_i^R(M,R) = 0$ *holds for all* $i \in \mathbb{Z}$.

Proof. The implications $(i) \implies (ii) \implies (iii) \implies (iv)$ are clear; see 2.3. Let M be a module as specified in (*iv*); Theorem 3.11 yields $\operatorname{G-dim}_R M < \infty$. By a result of Holm [22, thm. 3.2], R is Gorenstein if $\operatorname{id}_R M$ is also finite. Now it follows from Proposition 5.5 that R is Gorenstein. □

5.7 Remark. Let R be a commutative noetherian local ring with residue field k, and let E denote the injective hull of k; it is a faithfully injective R-module. A computation based on Theorem A.8 shows that the ranks of $\widetilde{\mathrm{Tor}}_i^R(k,k)$ and $\widetilde{\mathrm{Ext}}_R^i(k,k)$ are simultaneously finite:

$$\begin{aligned}
\operatorname{rank}_k \widetilde{\mathrm{Tor}}_i^R(k,k) &= \operatorname{rank}_k \widetilde{\mathrm{Tor}}_i^R(k, \operatorname{Hom}_R(k,E)) \\
&= \operatorname{rank}_k \operatorname{Hom}_R(\widetilde{\mathrm{Ext}}_R^i(k,k), E) \\
&= \operatorname{rank}_k \widetilde{\mathrm{Ext}}_R^i(k,k)\,.
\end{aligned}$$

Combined with this equality of ranks, [5, thm. 6.4, 6.5, and 6.7] yield characterizations of regular, complete intersection, and Gorenstein local rings in terms of the size of the stable homology spaces $\widetilde{\mathrm{Tor}}_i^R(k,k)$. For example, R is regular if and only if $\widetilde{\mathrm{Tor}}_i^R(k,k) = 0$ holds for some (equivalently, all) $i \in \mathbb{Z}$, and R is Gorenstein if and only if $\widetilde{\mathrm{Tor}}_i^R(k,k)$ has finite rank for some (equivalently, all) $i \in \mathbb{Z}$.

Commutative rings. If R is commutative and $\mathfrak{p}$ is a prime ideal in R, then the local ring $R_\mathfrak{p}$ is a flat R-algebra, and for an $R_\mathfrak{p}$-module M it follows from 2.12 that the stable homology modules $\widetilde{\mathrm{Tor}}_i^R(M,N)$ are $R_\mathfrak{p}$-modules.

5.8 Lemma. *Let* R *be a commutative noetherian ring and let* N *be an* R*-module; let* $\mathfrak{p}$ *be a prime ideal in* R *and let* M *be an* $R_\mathfrak{p}$*-module. For every* $i \in \mathbb{Z}$ *there is a natural isomorphism of* $R_\mathfrak{p}$*-modules,*

$$\widetilde{\mathrm{Tor}}_i^R(M,N) \;\cong\; \widetilde{\mathrm{Tor}}_i^{R_\mathfrak{p}}(M,N_\mathfrak{p})\,.$$

Hence, $\widetilde{\mathrm{Tor}}_i^R(-,N) = 0$ *implies* $\widetilde{\mathrm{Tor}}_i^{R_\mathfrak{p}}(-,N_\mathfrak{p}) = 0$ *for all prime ideals* $\mathfrak{p}$ *in* R.

Proof. Let $N \xrightarrow{\simeq} I$ be an injective resolution. Let $P \xrightarrow{\simeq} M$ be a projective resolution over $R_\mathfrak{p}$; it is a flat resolution of M as an R-module. The second isomorphism below follows from Proposition A.4.

$$\begin{aligned}
P \mathbin{\overline{\otimes}}_R I &\cong (P \mathbin{\overline{\otimes}}_{R_\mathfrak{p}} R_\mathfrak{p}) \mathbin{\overline{\otimes}}_R I \\
&\cong P \mathbin{\overline{\otimes}}_{R_\mathfrak{p}} (R_\mathfrak{p} \mathbin{\overline{\otimes}}_R I) \\
&\cong P \mathbin{\overline{\otimes}}_{R_\mathfrak{p}} (R_\mathfrak{p} \otimes_R I)
\end{aligned}$$

The computation gives $P \mathbin{\overline{\otimes}}_R I \cong P \mathbin{\overline{\otimes}}_{R_\mathfrak{p}} I_\mathfrak{p}$; similarly one gets $P \otimes_R I \cong P \otimes_{R_\mathfrak{p}} I_\mathfrak{p}$. Now (1.4.1) and the Five Lemma yield $P \mathbin{\widetilde{\otimes}}_R I \cong P \mathbin{\widetilde{\otimes}}_{R_\mathfrak{p}} I_\mathfrak{p}$, and the desired isomorphisms follow as $N_\mathfrak{p} \to I_\mathfrak{p}$ is an injective resolution by Matlis Theory. □

The proof of the next result is similar. Compared to Lemma 5.8 the noetherian hypothesis on R has been dropped, as it was only used to invoke Matlis Theory.

5.9 Lemma. *Let R be a commutative ring and let M be an R-module; let $\mathfrak{p}$ be a prime ideal in R and let N be an $R_\mathfrak{p}$-module. For every $i \in \mathbb{Z}$ there is a natural isomorphism of $R_\mathfrak{p}$-modules,*

$$\widetilde{\operatorname{Tor}}_i^R(M,N) \;\cong\; \widetilde{\operatorname{Tor}}_i^{R_\mathfrak{p}}(M_\mathfrak{p},N)\,.$$

Hence, $\widetilde{\operatorname{Tor}}_i^R(M,-)=0$ implies $\widetilde{\operatorname{Tor}}_i^{R_\mathfrak{p}}(M_\mathfrak{p},-)=0$ for all prime ideals $\mathfrak{p}$ in R. $\square$

5.10 Theorem. *Let R be a commutative noetherian ring and let N be a finitely generated R-module. If $\widetilde{\operatorname{Tor}}_i^R(-,N)=0$ holds for some $i \in \mathbb{Z}$, then $\operatorname{id}_{R_\mathfrak{p}} N_\mathfrak{p}$ is finite for every prime ideal $\mathfrak{p}$ in R.*

Proof. From the hypotheses and Lemma 5.8 one has $\widetilde{\operatorname{Tor}}_i^{R_\mathfrak{p}}(-,N_\mathfrak{p})=0$. It follows from Proposition 5.5 that the local ring $R_\mathfrak{p}$ is Gorenstein, or $\operatorname{id}_{R_\mathfrak{p}} N_\mathfrak{p}$ is finite. However, if $R_\mathfrak{p}$ is Gorenstein, then vanishing of $\widetilde{\operatorname{Tor}}_i^{R_\mathfrak{p}}(-,N_\mathfrak{p})=0$ implies that $\operatorname{pd}_{R_\mathfrak{p}} N_\mathfrak{p}$ is finite by Corollary 4.7 and Theorem 3.1, and then $\operatorname{id}_{R_\mathfrak{p}} N_\mathfrak{p}$ is finite. $\square$

The next corollary is now immediate per the remarks in 5.2.

5.11 Corollary. *A commutative noetherian ring R is Cohen–Macaulay if there is a finitely generated R-module $N \neq 0$ with $\widetilde{\operatorname{Tor}}_i^R(-,N)=0$ for some $i \in \mathbb{Z}$.* $\square$

5.12 Corollary. *Let R be a commutative noetherian ring of finite Krull dimension. For a finitely generated R-module N, the following conditions are equivalent.*

(i) $\operatorname{id}_R N$ *is finite.*

(ii) $\widetilde{\operatorname{Tor}}_i^R(-,N)=0$ *for all $i \in \mathbb{Z}$.*

(iii) $\widetilde{\operatorname{Tor}}_i^R(-,N)=0$ *for some $i \in \mathbb{Z}$.*

Proof. The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ are clear; see 2.3. Part (i) follows from (iii) as $\operatorname{id}_R N$ equals $\sup\{\operatorname{id}_{R_\mathfrak{p}} N_\mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal in } R\} \leqslant \dim R$. $\square$

5.13 Remark. We do not know if the assumption of finite Krull dimension in Corollary 5.12 is necessary. By Theorem 5.10, vanishing of $\widetilde{\operatorname{Tor}}^R(-,N)$ implies that N is locally of finite injective dimension, but that does not imply finite injective dimension over R: Just consider a Gorenstein ring R of infinite Krull dimension. On the other hand, we do not know if $\widetilde{\operatorname{Tor}}^R(-,R)$ vanishes for such a ring.

6. Comparison to Tate homology

In this section we compare stable homology to Tate homology. We parallel some of the findings of Avramov and Veliche [5]. First we recall a few definitions.

6.1. An acyclic complex T of projective R°-modules is called *totally acyclic* if $\operatorname{Hom}_{R^\circ}(T,P)$ is acyclic for every projective R°-module P; cf. 3.8. A *complete projective resolution* of an R°-module M is a diagram $T \xrightarrow{\varpi} P \xrightarrow{\simeq} M$, where T is a totally acyclic complex of projective R°-modules, $P \xrightarrow{\simeq} M$ is a projective resolution, and ϖ_i is an isomorphism for $i \gg 0$; see [33, sec. 2].

Let M be an R°-module with a complete projective resolution $T \to P \to M$. For an R-module N, the *Tate homology* of M and N over R are the $\Bbbk$-modules $\widehat{\operatorname{Tor}}_i^R(M,N)=\operatorname{H}_i(T \otimes_R N)$ for $i \in \mathbb{Z}$; see Iacob [24, sec. 2].

6.2. An R°-module has a complete projective resolution if and only if it has finite Gorenstein projective dimension; see [33, thm. 3.4].

If R is noetherian and M is a finitely generated R°-module with a complete projective resolution, then M has finite G-dimension and it has a complete projective resolution $T \to P \to M$ with T and P degree-wise finitely generated and $T \to P$ surjective; see [33, 2.4.1] and [4, thm. 3.1].

6.3 Lemma. *Let M be an R°-module that has a complete projective resolution $T \to P \to M$. The following conditions are equivalent.*

(*i*) *$T \otimes_R E$ is acyclic for every injective R-module E.*

(*ii*) *There are isomorphisms of functors $\widetilde{\operatorname{Tor}}{}_i^R(M,-) \cong \widehat{\operatorname{Tor}}{}_i^R(M,-)$ for all $i \in \mathbb{Z}$.*

Proof. Assume that $T \otimes_R E$ is acyclic for every injective R-module E. The pair (T,P) is then a Tate flat resolution of M, see 3.8, and it follows from Theorem 3.10 that the functors $\widetilde{\operatorname{Tor}}{}_i^R(M,-)$ and $\widehat{\operatorname{Tor}}{}_i^R(M,-)$ are isomorphic for all $i \in \mathbb{Z}$. For the converse, let E be an injective R-module. By 2.3 one then has

$$0 = \widetilde{\operatorname{Tor}}{}_i^R(M,E) \cong \widehat{\operatorname{Tor}}{}_i^R(M,E)$$

for all $i \in \mathbb{Z}$, and hence $\operatorname{H}(T \otimes_R E) = 0$. □

6.4 Theorem. *Let R be noetherian, and let M be a finitely generated R°-module that has a complete projective resolution. There are isomorphisms of functors*

$$\widetilde{\operatorname{Tor}}{}_i^R(M,-) \;\cong\; \widehat{\operatorname{Tor}}{}_i^R(M,-) \quad \textit{for all } i \in \mathbb{Z}\,.$$

Proof. The module M has a complete projective resolution $T \to P \to M$ with T and P degree-wise finitely generated; see 6.2. The isomorphisms (3.9.1) show that the complex $T \otimes_R E$ is acyclic for every injective R-module E, and Lemma 6.3 finishes the argument. □

6.5 Remark. The isomorphisms of homology modules in Theorem 6.4 actually follow from *one* isomorphism in $\mathsf{D}(\Bbbk)$, but this is unapparent in the proof, which rests on Theorem 3.10. The finitely generated module M has a complete projective resolution $T \to L \to M$ with T and L degree-wise finitely generated and $T \to L$ surjective; see 6.2. Thus the kernel $K = \operatorname{Ker}(T \to L)$ is a bounded above complex of projective modules. Given a module N, let C be the cone as one would construct in 3.4. By Proposition 1.7(a) the complex $K \mathbin{\overline{\otimes}}_R C$ is acyclic, so that Lemma 3.5 and 1.5(c) yield $\Sigma^{-1}(L \mathbin{\widetilde{\otimes}}_R I) \simeq \Sigma^{-1}(L \mathbin{\overline{\otimes}}_R C) \simeq \Sigma^{-1}(T \mathbin{\overline{\otimes}}_R C)$. As in the proof above, $T \otimes_R E$ is acyclic, so Lemma 3.6 gives $\Sigma^{-1}(T \mathbin{\overline{\otimes}}_R C) \simeq T \otimes_R N$, and combining these isomorphisms in $\mathsf{D}(\Bbbk)$ gives the desired one.

Without extra assumptions on the ring, we do not know if stable homology agrees with Tate homology whenever the latter is defined. In general, the relation between stable homology and Tate homology is tied to an unresolved problem in Gorenstein homological algebra. Theorem 6.7 explains how.

6.6. An R°-module G is called *Gorenstein projective* if there exists a totally acyclic complex T of projective R°-modules with $\operatorname{C}_0(T) \cong G$; see 6.1. Similarly, an R°-module G is called *Gorenstein flat* if there exists a totally acyclic complex T of flat R°-modules with $\operatorname{C}_0(T) \cong G$; see 3.8.

6.7 Theorem. *The following conditions on R are equivalent.*

(*i*) *Every Gorenstein projective R°-module is Gorenstein flat.*

(*ii*) *For every R°-module M that has a complete projective resolution there are isomorphisms of functors $\widetilde{\mathrm{Tor}}_i^R(M,-) \cong \widehat{\mathrm{Tor}}_i^R(M,-)$ for all $i \in \mathbb{Z}$.*

Proof. Assume that every Gorenstein projective R°-module is Gorenstein flat. Let $T \to P \to M$ be a complete projective resolution. It follows that T is a totally acyclic complex of flat modules; see Emmanouil [13, thm. 2.2]. Thus stable homology and Tate homology coincide by Lemma 6.3.

For the converse, let M be a Gorenstein projective R°-module and let T be a totally acyclic complex of projective R°-modules with $M \cong \mathrm{C}_0(T)$. Since there are isomorphisms of functors $\widetilde{\mathrm{Tor}}_i^R(M,-) \cong \widehat{\mathrm{Tor}}_i^R(M,-)$ for all $i \in \mathbb{Z}$, it follows from 2.3 that $T \otimes_R E$ is acyclic for every injective R-module E. Thus T is a totally acyclic complex of flat R°-modules, and so M is Gorenstein flat. □

6.8 Remark. As Holm notes [21, prop. 3.4], the obvious way to achieve that every Gorenstein projective R°-module is Gorenstein flat is to ensure that (1) the Pontryagin dual of every injective R-module is flat, and (2) that every flat R°-module has finite projective dimension. The first condition is satisfied if R is left coherent, and the second is discussed in Remark 2.8. A description of the rings over which Gorenstein projective modules are Gorenstein flat seems elusive; see [13, sec. 2].

Complete homology. In his thesis, Triulzi considers the *J-completion* of the homological functor $\mathrm{Tor}^R(M,-) = \{\mathrm{Tor}_i^R(M,-) \mid i \in \mathbb{Z}\}$. His construction is similar to Mislin's P-completion of covariant Ext and Nucinkis' I-completion of contravariant Ext. The resulting homology theory is called *complete homology.* Like stable homology, it is a generalization of Tate homology. We compare these two generalizations in [8]. From the point of view of stable homology, it is interesting to know when it agrees with complete homology, because the latter has a universal property. In this direction the main results in [8] are that these two homology theories agree over Iwanaga-Gorenstein rings, and for finitely generated modules over Artin algebras and complete commutative local rings. Moreover, the two theories agree with Tate homology, whenever it is defined, under the exact same condition; that is, if and only if every Gorenstein projective module is Gorenstein flat; see Theorem 6.7.

Appendix

We start by recalling the definition of stable cohomology.

A.1. Let X and Y be R-complexes, following [5, 18] we denote by $\overline{\mathrm{Hom}}_R(X,Y)$ the subcomplex of $\mathrm{Hom}_R(X,Y)$ with degree n term

$$\overline{\mathrm{Hom}}_R(X,Y)_n = \coprod_{i\in\mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+n})\,.$$

It is called the *bounded* Hom complex, and the quotient complex

$$\widetilde{\mathrm{Hom}}_R(X,Y) = \mathrm{Hom}_R(X,Y)/\overline{\mathrm{Hom}}_R(X,Y)$$

is called the *stable* Hom complex.

For R-modules M and N with projective resolutions $P_M \xrightarrow{\simeq} M$ and $P_N \xrightarrow{\simeq} N$, the $\Bbbk$-modules

$$\overline{\mathrm{Ext}}_R^i(M,N) = \mathrm{H}_{-i}(\overline{\mathrm{Hom}}_R(P_M,P_N))\,,$$

are called the *bounded cohomology* of M and N over R, and the *stable cohomology* modules of M and N over R are

$$\widetilde{\mathrm{Ext}}_R^i(M,N) = \mathrm{H}_{-i}(\widetilde{\mathrm{Hom}}_R(P_M,P_N))\,.$$

Avramov and Veliche [5] use the notation $\widehat{\mathrm{Ext}}_R^i(M,N)$ for the stable cohomology; this notation is standard for Tate cohomology, which coincides with stable cohomology whenever the former is defined; see [5, cor. 2.4].

A.2 Proposition. *Let X and Y be R-complexes.*

(a) *If X or Y is bounded above, and* $\mathrm{Hom}_R(X_i,Y)$ *is acyclic for all i, then the complex* $\overline{\mathrm{Hom}}_R(X,Y)$ *is acyclic.*

(b) *If X or Y is bounded below, and* $\mathrm{Hom}_R(X,Y_i)$ *is acyclic for all i, then the complex* $\overline{\mathrm{Hom}}_R(X,Y)$ *is acyclic.*

Proof. Similar to the proof of Proposition 1.7. □

Standard isomorphisms. We study composites of the functors $\overline{\mathrm{Hom}}$ and $\overline{\otimes}$. To the extent possible, we establish analogs of the standard isomorphisms for composites of Hom and $\otimes$; see [7, sec. II.5 and VI.5]. There seems to be no analog of Hom-tensor adjunction [7, prop. II.5.2].

The setup is the same for Propositions A.4–A.6; namely:

A.3. Let X be a complex of R°-modules, let Y be a complex of (R,S°)-bimodules, and let Z be a complex of S-modules.

Under finiteness conditions, the unbounded tensor product is associative.

A.4 Proposition. *For complexes as in A.3, under either of the following conditions*

- *X and Z are complexes of finitely presented modules*
- *Y is a bounded complex*

there is an isomorphism of $\Bbbk$-complexes,

$$(X \mathbin{\overline{\otimes}}_R Y) \mathbin{\overline{\otimes}}_S Z \longrightarrow X \mathbin{\overline{\otimes}}_R (Y \mathbin{\overline{\otimes}}_S Z)\,,$$

and it is functorial in X, Y, and Z.

Proof. For every $n \in \mathbb{Z}$ one has,

$$((X \mathbin{\overline{\otimes}}_R Y) \mathbin{\overline{\otimes}}_S Z)_n = \prod_{i\in\mathbb{Z}} \Big(\textstyle\prod_{j\in\mathbb{Z}} X_j \otimes_R Y_{i-j}\Big) \otimes_S Z_{n-i}$$

and

$$(X \mathbin{\overline{\otimes}}_R (Y \mathbin{\overline{\otimes}}_S Z))_n = \prod_{j\in\mathbb{Z}} X_j \otimes_R \big(\textstyle\prod_{i\in\mathbb{Z}} Y_{i-j} \otimes_S Z_{n-i}\big)\,,$$

and from each of these modules there is a canonical homomorphism to

$$\prod_{i\in\mathbb{Z}}\prod_{j\in\mathbb{Z}} (X_j \otimes_R Y_{i-j}) \otimes_S Z_{n-i} \;\cong\; \prod_{j\in\mathbb{Z}}\prod_{i\in\mathbb{Z}} X_j \otimes_R (Y_{i-j} \otimes_S Z_{n-i})\,.$$

If Y is bounded, then these homomorphisms are isomorphisms as the inner most products are finite. Recall, e.g. from [16, thm. 3.2.22], that the functor $M \otimes_R -$ commutes with products if M is finitely presented. Thus, the canonical homomorphisms are isomorphisms when X and Z are complexes of finitely presented modules. It is straightforward to verify that these isomorphisms commute with the differentials and form an isomorphism of complexes. □

The model for the following *swap* isomorphism is [7, ex. II.4]. The proof is similar to the proof of Proposition A.4 and uses that the functor $\operatorname{Hom}_R(M,-)$ commutes with coproducts if M is a finitely generated R-module.

A.5 Proposition. *For complexes as in A.3, under either of the following conditions*

- *X and Z are complexes of finitely generated modules*
- *Y is a bounded complex*

there is an isomorphism of $\Bbbk$-complexes,

$$\overline{\operatorname{Hom}}_{R^\circ}(X, \overline{\operatorname{Hom}}_S(Z,Y)) \longrightarrow \overline{\operatorname{Hom}}_S(Z, \overline{\operatorname{Hom}}_{R^\circ}(X,Y))\,,$$

and it is functorial in X, Y, and Z. $\square$

The next result is an analog of [7, prop. VI.5.2 and VI.5.3]; it is used below to establish a duality between stable homology and stable cohomology.

A.6 Proposition. *Let X, Y, and Z be as in A.3, and assume further that X is a complex of finitely presented R°-modules. There is a morphism of $\Bbbk$-complexes,*

$$X \mathbin{\overline{\otimes}}_R \operatorname{Hom}_S(Y,Z) \longrightarrow \operatorname{Hom}_S(\overline{\operatorname{Hom}}_{R^\circ}(X,Y), Z)\,,$$

and it is functorial in X, Y, and Z. Furthermore, it is an isomorphism if X is a complex of projective modules or Z is a complex of injective modules.

Proof. For every $n \in \mathbb{Z}$ one can compute as follows,

$$\begin{aligned}
(X \mathbin{\overline{\otimes}}_R \operatorname{Hom}_S(Y,Z))_n &= \prod_{i\in\mathbb{Z}} (X_i \otimes_R \operatorname{Hom}_S(Y,Z)_{n-i}) \\
&= \prod_{i\in\mathbb{Z}} (X_i \otimes_R \prod_{j\in\mathbb{Z}} \operatorname{Hom}_S(Y_j, Z_{n-i+j})) \\
&\cong \prod_{i\in\mathbb{Z}} \prod_{j\in\mathbb{Z}} (X_i \otimes_R \operatorname{Hom}_S(Y_j, Z_{n-i+j}))\,,
\end{aligned}$$

where the isomorphism holds as the module X_i is finitely presented for every $i \in \mathbb{Z}$; see [16, thm. 3.2.22]. On the other hand, for every $n \in \mathbb{Z}$ one has

$$\begin{aligned}
\operatorname{Hom}_S(\overline{\operatorname{Hom}}_{R^\circ}(X,Y), Z)_n &= \prod_{h\in\mathbb{Z}} \operatorname{Hom}_S(\overline{\operatorname{Hom}}_{R^\circ}(X,Y)_h, Z_{n+h}) \\
&= \prod_{h\in\mathbb{Z}} \operatorname{Hom}_S(\coprod_{i\in\mathbb{Z}} \operatorname{Hom}_{R^\circ}(X_i, Y_{i+h}), Z_{n+h}) \\
&\cong \prod_{h\in\mathbb{Z}} \prod_{i\in\mathbb{Z}} \operatorname{Hom}_S(\operatorname{Hom}_{R^\circ}(X_i, Y_{i+h}), Z_{n+h}) \\
&= \prod_{i\in\mathbb{Z}} \prod_{j\in\mathbb{Z}} \operatorname{Hom}_S(\operatorname{Hom}_{R^\circ}(X_i, Y_j), Z_{n-i+j})\,.
\end{aligned}$$

Now set $(\theta_{XYZ})_n = \prod_{i\in\mathbb{Z}} \prod_{j\in\mathbb{Z}} (-1)^{i(n-j)} \theta_{X_iY_jZ_{n-i+j}}$, where

$$\theta_{X_iY_jZ_{n-i+j}} \colon X_i \otimes_R \operatorname{Hom}_S(Y_j, Z_{n-i+j}) \longrightarrow \operatorname{Hom}_S(\operatorname{Hom}_{R^\circ}(X_i, Y_j), Z_{n-i+j})$$

is the homomorphism of $\Bbbk$-modules given by $\theta_{X_iY_jZ_{n-i+j}}(x \otimes \psi)(\phi) = \psi\phi(x)$. It is straightforward to verify that θ_{XYZ} is a morphism of $\Bbbk$-complexes and functorial in X, Y, and Z.

Finally, if each module X_i is projective or each module Z_{n-i+j} is injective, then it follows from [7, prop. VI.5.2 and VI.5.3] that $\theta_{X_iY_jZ_{n-i+j}}$ is invertible for all $i,j,n \in \mathbb{Z}$, and so the morphism θ_{XYZ} is invertible. $\square$

A.7 Lemma. *Let P be a bounded below complex of finitely generated projective R°-modules and let X be a complex of R°-modules with $\mathrm{H}(X)$ bounded. For every injective $\Bbbk$-module E, there is an isomorphism in the derived category $\mathsf{D}(\Bbbk)$:*

$$P \mathbin{\widetilde{\otimes}}_R \operatorname{Hom}_{\Bbbk}(X,E) \longrightarrow \Sigma \operatorname{Hom}_{\Bbbk}(\widetilde{\operatorname{Hom}}_{R^\circ}(P,X),E)\ ,$$

and it is functorial in P, X, and E.

Proof. Set $(-)^\vee = \operatorname{Hom}_{\Bbbk}(-,E)$. In the commutative square of $\Bbbk$-complexes

$$\begin{tikzcd}
P \otimes_R X^\vee \arrow[r, "\tau"] \arrow[d, "\theta"] & P \mathbin{\overline{\otimes}}_R X^\vee \arrow[d, "\alpha", "\cong"'] \\
\operatorname{Hom}_{R^\circ}(P,X)^\vee \arrow[r, "\vartheta^\vee"] & \overline{\operatorname{Hom}}_{R^\circ}(P,X)^\vee
\end{tikzcd}$$

each horizontal morphism is (the dual of) a canonical embedding. The vertical map α is the isomorphism from Proposition A.6. The morphism θ is the standard evaluation map; it is a quasi-isomorphism by [2, 4.4(I)]. The square induces a morphism of triangles in the homotopy category:

$$\begin{tikzcd}
P \otimes_R X^\vee \arrow[r, "\tau"] \arrow[d, "\theta", "\simeq"'] & P \mathbin{\overline{\otimes}}_R X^\vee \arrow[r] \arrow[d, "\alpha", "\cong"'] & \operatorname{Cone}\tau \arrow[r] \arrow[d, "\gamma"] & \Sigma(P \otimes_R X^\vee) \arrow[d, "\Sigma\,\theta", "\simeq"'] \\
\operatorname{Hom}_{R^\circ}(P,X)^\vee \arrow[r, "\vartheta^\vee"] & \overline{\operatorname{Hom}}_{R^\circ}(P,X)^\vee \arrow[r] & \operatorname{Cone}\vartheta^\vee \arrow[r] & \Sigma \operatorname{Hom}_{R^\circ}(P,X)^\vee
\end{tikzcd}$$

The construction of γ is functorial in all three arguments, and it is a quasi-isomorphism because α and θ are quasi-isomorphisms. Recall, say from [17, III.3.4–5], that there are natural quasi-isomorphisms

$$\operatorname{Cone}\tau \ \simeq\ \operatorname{Coker}\tau \ =\ P \mathbin{\widetilde{\otimes}}_R X^\vee \quad \text{and}$$

$$\operatorname{Cone}\vartheta^\vee \ \cong\ \Sigma(\operatorname{Cone}\vartheta)^\vee \ \simeq\ \Sigma(\operatorname{Coker}\vartheta)^\vee \ =\ \Sigma\, \widetilde{\operatorname{Hom}}_{R^\circ}(P,X)^\vee\ .$$

They yield the desired isomorphism in the derived category

$$P \mathbin{\widetilde{\otimes}}_R \operatorname{Hom}_{\Bbbk}(X,E) \longrightarrow \Sigma \operatorname{Hom}_{\Bbbk}(\widetilde{\operatorname{Hom}}_{R^\circ}(P,X),E)\ .$$

With regard to functoriality of γ, notice that a morphism between arguments, $P \to P_1$ say, induces the solid commutative square in the following diagram.

$$\begin{tikzcd}
\operatorname{Coker}\tau \arrow[rrr, dashed] \arrow[ddd, dashed] & & & \operatorname{Coker}\tau_1 \arrow[ddd, dashed] \\
& \operatorname{Cone}\tau \arrow[r] \arrow[d, "\gamma"] \arrow[ul, "\simeq"] & \operatorname{Cone}\tau_1 \arrow[d, "\gamma_1"] \arrow[ur, "\simeq"'] & \\
& \operatorname{Cone}\vartheta^\vee \arrow[r] \arrow[dl, "\simeq"'] & \operatorname{Cone}(\vartheta_1^\vee) \arrow[dr, "\simeq"] & \\
\Sigma(\operatorname{Coker}\vartheta)^\vee \arrow[rrr, dashed] & & & \Sigma(\operatorname{Coker}\vartheta_1)^\vee
\end{tikzcd}$$

Commutativity in the derived category of the dashed square is now a consequence. Functoriality in the other arguments is handles similarly. $\square$

The Lemma immediately yields a useful duality.

A.8 Theorem. *Let M and N be R°-modules and assume that M has a degreewise finitely generated projective resolution. For every injective $\Bbbk$-module E and for every $i \in \mathbb{Z}$ there is an isomorphism of $\Bbbk$-modules,*

$$\mathrm{Hom}_{\Bbbk}(\widetilde{\mathrm{Ext}}^i_{R^\circ}(M,N),E) \;\cong\; \widetilde{\mathrm{Tor}}^R_i(M,\mathrm{Hom}_{\Bbbk}(N,E))\;,$$

and it is functorial in M, N, and E. $\square$

To prove the next two results one proceeds as in the proof of Proposition A.6.

A.9 Proposition. *Let X be a complex of finitely generated R-modules and let Y and Z be as in A.3. There is a morphism of $\Bbbk$-complexes*

$$\overline{\mathrm{Hom}}_R(X,Y)\otimes_S Z \;\longrightarrow\; \overline{\mathrm{Hom}}_R(X,Y\otimes_S Z)\;,$$

and it is functorial in X, Y, and Z. Furthermore, it is an isomorphism under each of the following conditions

- *Z is a complex of finitely generated projective modules*
- *X is a complex of finitely presented modules and Z is a complex of flat modules*
- *X is a complex of projective modules* $\square$

A.10 Proposition. *Let X be a complex of R-modules, let Y be a complex of (R,S°)-bimodules, and let Z be a complex of finitely presented S-modules. There is a morphism of $\Bbbk$-complexes,*

$$\mathrm{Hom}_R(X,Y)\,\overline{\otimes}_S\, Z \;\longrightarrow\; \mathrm{Hom}_R(X,Y\,\overline{\otimes}_S\, Z)\;,$$

and it is functorial in X, Y, and Z. Furthermore, it is an isomorphism if X or Z is a complex of projective modules. $\square$

Pinched tensor products. Christensen and Jorgensen devised in [12] a *pinched* tensor product, $-\otimes^{\bowtie}_R -$, to compute Tate homology. In view of Theorem 3.10 their proof of [12, thm. 3.5] applies verbatim to yield the next result; we refer the reader to [12] for the definition of the pinched tensor product.

A.11 Theorem. *Let M be an R°-module that has a Tate flat resolution (T,F), let A be an acyclic complex of R-modules and set $N = \mathrm{C}_0(A)$. For every $i \in \mathbb{Z}$, there is an isomorphism of $\Bbbk$-modules*

$$\widetilde{\mathrm{Tor}}^R_i(M,N) \;\cong\; \mathrm{H}_i(T\otimes^{\bowtie}_R A)\,. \qquad \square$$

The next corollary is an analogue of [12, Corollary 4.10].

A.12 Corollary. *Let R be commutative and let M and N be Gorenstein flat R-modules with corresponding totally acyclic complexes of flat modules T and T', respectively. If $\widetilde{\mathrm{Tor}}^R_i(M,N) = 0$ holds for all $i \in \mathbb{Z}$, then $T\otimes^{\bowtie}_R T'$ is an acyclic complex of flat R-modules, and the following statements are equivalent:*

(i) *$T\otimes^{\bowtie}_R T'$ is a totally acyclic complex of flat R-modules.*

(ii) *$\widetilde{\mathrm{Tor}}^R_i(M,N\otimes_R E) = 0$ holds for every injective R-module E and all $i \in \mathbb{Z}$.*

When these conditions hold, $M \otimes_R N$ is a Gorenstein flat R-module and $T \otimes_R^{\bowtie} T'$ is a corresponding totally acyclic complex of flat R-modules.

Proof. It follows from the definition of pinched tensor products that $T \otimes_R^{\bowtie} T'$ is a complex of flat R-modules, and if $\widetilde{\operatorname{Tor}}_i^R(M,N) = 0$ holds for all $i \in \mathbb{Z}$, then the complex is acyclic by Theorem A.11. It is totally acyclic if and only if $(T \otimes_R^{\bowtie} T') \otimes_R E \cong T \otimes_R^{\bowtie} (T \otimes_R E)$ is acyclic for every injective R-module E; that is, if and only if $\widetilde{\operatorname{Tor}}_i^R(M, N \otimes_R E) = 0$ holds for every injective R-module E and all $i \in \mathbb{Z}$. Finally, it follows from the definition of pinched tensor products that there is an isomorphism $M \otimes_R N \cong \mathrm{C}_0(T \otimes_R^{\bowtie} T')$. $\square$

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