Syzygies of algebraic curves

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Setup

- We work over the field $k = \mathbb{C}$.
- $S = k[x_0, \ldots, x_r]$ is the polynomial ring.
- An algebraic variety X = zero locus of (finitely many) homogeneous polynomials (defining equations). All defining equations form an ideal homogeneous ideal I_X of S.
- The coordinate ring of X is

$$S_X = S/I_X,$$

- S_X is a finitely generated graded S-module.

I. Syzygies, Betti diagram and N_p-property

Hilbert's syzygies theorem

Theorem (Hilbert's syzygy theorem, 1890)

Let M be a finitely generated graded S-module. Then M has a minimal graded free resolution of the form

$$0 \longrightarrow F_m \xrightarrow{d_m} \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$$

where $m \leq r+1$ and each

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}} \text{ for finitely many nonzero integers } \beta_{i,j}.$$

- "minimal" means that in the matrix d_i each entry is either zero or a non-constant homogeneous polynomial.

Betti Diagram

Given a minimal graded free resolution

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "d_2"] \arrow[d, equal] & F_1 \arrow[r, "d_1"] \arrow[d, equal] & F_0 \arrow[r] \arrow[d, equal] & M \arrow[r] & 0, \\
 & \bigoplus_j S(-j)^{\beta_{2,j}} & \bigoplus_j S(-j)^{\beta_{1,j}} & \bigoplus_j S(-j)^{\beta_{0,j}} & &
\end{tikzcd}$$

the Betti diagram of M has the form

$$\begin{array}{r|rrrcr}
 & 0 & 1 & 2 & \cdots & m \\
\hline
i & \beta_{0,i} & \beta_{1,i+1} & \beta_{2,i+2} & \cdots & \beta_{m,i+m} \\
i+1 & \beta_{0,i+1} & \beta_{1,i+2} & \beta_{2,i+3} & \cdots & \beta_{m,i+1+m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
j & \beta_{0,j} & \beta_{1,j+1} & \beta_{2,j+2} & \cdots & \beta_{m,j+m}
\end{array}$$

Example: complete intersections

Consider $S = k[x_0, x_1, x_2, x_3, x_4]$ and X is defined by the equations

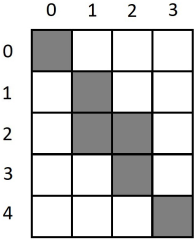
$$f_1 = x_0^2,\ \ f_2 = x_1^2,\ \ f_3 = x_4^3$$

The coordinate ring $S_X = S/(f_1, f_2, f_3)$ has a resolution

$$0 \to S(-7) \to S(-4) \oplus S(-5)^2 \to S(-2)^2 \oplus S(-3) \to S \to S_X \to 0.$$

The Betti diagram is

	0	1	2	3
0	1	–	–	
1	–	2	–	–
2	–	1	1	–
3	–	–	2	–
4	–	–	–	1



Example: rational normal curves

Let $S = k[x_0, x_1, x_2, x_3]$ and let X be the image of the morphism

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ [s,t] & \mapsto & [s^3, s^2t, st^2, t^3]. \end{array}$$

The defining equations of X are

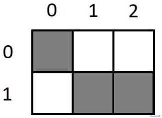
$$q_1 = x_1x_3 - x_2^2,\ q_2 = x_0x_3 - x_1x_2,\ \text{and } q_3 = x_0x_2 - x_1^2.$$

The coordinate ring $S_X = S/(q_1, q_2, q_3)$ has a minimal resolution

$$0 \longrightarrow S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} S^3(-2) \xrightarrow{(q_1, -q_2, q_3)} S \longrightarrow S_X \longrightarrow 0$$

and the Betti diagram of S_X is

	0	1	2
0	1	–	–
1	–	3	2



Geometric Setup.

- An embedding of a projective variety

  $$X \hookrightarrow \mathbb{P}^r$$

  into a projective space can be described by a very ample line bundle L. More often, the coordinate ring of X agrees with the *section ring* of L, defined by

  $$R(L) = \bigoplus_{q \geq 0} H^0(L^q).$$

- Write $V = H^0(L)$ for global sections and $r(L) = \dim V - 1$. Then the symmetric algebra

  $$S = \mathrm{Sym}^\bullet V = k \oplus V \oplus S^2 V \oplus S^3 V \oplus \dots$$

  is a polynomial ring in $r + 1$ variables.

- The section ring $R(L)$ is a finitely generated graded S-module.

Example

Let $X = \mathbb{P}^1$. The coordinate ring of X is $S = k[x_0, x_1]$. Let $L = \mathscr{O}_{\mathbb{P}^1}(d)$. Then the global sections

$$V = H^0(L) = S_d = \langle x_0^d, x_0^{d-1}x_1, \ldots, x_0 x_1^{d-1}, x_1^d \rangle, \; r(L) = d.$$

Thus L gives an embedding

$$\mathbb{P}^1 \xrightarrow{|L|} \mathbb{P}(V) = \mathbb{P}^d$$

$$[x_0 : x_1] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_0 x_1^{d-1} : x_1^d]$$

The section ring $R(L)$ of L is the same as the coordinate ring of the image in $\mathbb{P}^d$.

Definition of Koszul group $K_{p,q}$

Consider $\mathbb{C} = S/(x_0, \cdots, x_r)$ and it has a Koszul resolution

$$\cdots \longrightarrow \bigoplus^{r+1} S(-1) \longrightarrow S \longrightarrow \mathbb{C} \longrightarrow 0$$

So for a finitely generated graded S-module M, we define

$$K_{p,q}(M) = \operatorname{Tor}_p^S(M, \mathbb{C})_{p+q}.$$

It turns out

$$k_{p,q}(M) := \dim K_{p,q}(M) = \beta_{p,p+q}.$$

For a line bundle L, we write $K_{p,q}(L)$ for $K_{p,q}(R(L))$.

Koszul cohomology

- M. Green (1980s) developed Koszul cohomolgoy to compute $K_{p,q}(L)$ for the section ring of a line bundle.

- In particular, *Koszul cohomology* group $K_{p,q}(L)$ is defined as the homology at the middle of the Koszul-type complex

$$\wedge^{p+1} H^0(L) \otimes H^0(L^{q-1}) \longrightarrow \wedge^{p} H^0(L) \otimes H^0(L^{q}) \longrightarrow \wedge^{p-1} H^0(L) \otimes H^0(L^{q+1})$$

- In this approach, we can have a geometric way to compute Betti numbers for a section ring.

N_p-property

Definition

We say L satisfies Property N_p for $p \geq 0$, if in a minimal resolution of $R(L)$, one has

$$E_0 = S \text{ and } E_i = \bigoplus S(-i-1) \text{ for } 1 \leq i \leq p,$$

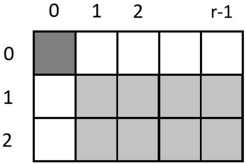
i.e.,

$$\cdots \to \oplus S(-p-1) \to \cdots \to \oplus S(-3) \to \oplus S(-2) \to S \to R(L) \to 0$$

Equivalently, the Koszul group $K_{p,q}(L)$ of the section ring $R(L)$ satisfies

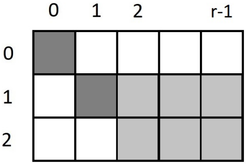
$$K_{i,j}(L) = 0, \text{ for } 0 \leq i \leq p, \ j \geq 2.$$

- Classically, N_0 property is also called projectively normal. N_1 means the variety is defined by quadrics.
- We illustrate N_p property using Betti diagram



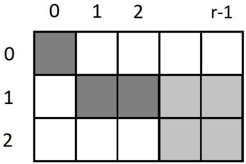
	0	1	2		r-1
0	■				
1		▨	▨	▨	▨
2		▨	▨	▨	▨

N_0



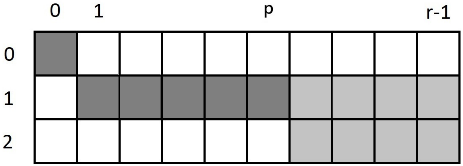
	0	1	2		r-1
0	■				
1		■	▨	▨	▨
2			▨	▨	▨

N_1



	0	1	2		r-1
0	■				
1		■	■	▨	▨
2				▨	▨

N_2



	0	1				p				r-1
0	■									
1		■	■	■	■	■	▨	▨	▨	▨
2							▨	▨	▨	▨

Example: Rational normal scrolls

Consider a vector bundle on $\mathbb{P}^1$:

$$E = \bigoplus_{0}^{d} \mathscr{O}_{\mathbb{P}^1}(a_i), \text{ with } 0 \le a_0 \le a_1 \le \cdots \le a_d, \text{ and } a_d \ge 1.$$

Let $X = \mathbb{P}(E)$ and $L = \mathscr{O}(1)$ be the tautological bundle. The section ring $R(L)$ only has the following nonzero Koszul groups

$$\begin{aligned} K_{0,q}(L) \neq 0 \quad &\Longleftrightarrow \quad q = 0 \\ K_{1,q}(L) \neq 0 \quad &\Longleftrightarrow \quad 1 \le q \le d+1 \end{aligned}$$

II. Syzygies of curves

- canonical curves
- curves of large degree

Syzygies of canonical curves

Let C be a nonsingular projective curve of genus $g \geq 2$, let

$$K_C = \Omega^1_C$$

be its canonical bundle that is ample, and let

$$\varphi_K : C \longrightarrow \mathbb{P}^{g-1}$$

be the canonical map. Define the canonical ring of C to be

$$R(K_C) = \bigoplus_{d \geq 0} H^0(dK_C).$$

Noether's theorem and Petri's theorem

Theorem (M. Noether, Castelnuovo, Enriques, 1880s-1890s)

If K_C is very ample, then φ_C embeds C projectively normal, i.e,

$$C \text{ is non-hyperelliptic} \iff K_C \text{ satisfies } N_0.$$

Theorem (Petri, 1922, also Noether-Enriques-Petri)

If C is a non-hyperelliptic canonical curve, then C is defined by quadrics unless

(1) *C is trigonal (i.e, $\operatorname{gon}(C) = 3$), or*

(2) *C is a plane quintic (i.e., degree 5 plane curve).*

Clifford index

Definition

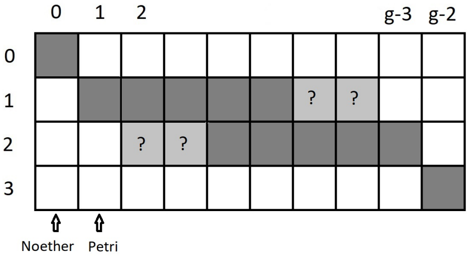
Let C be a nonsingular projective curve. The Clifford index of the curve C is defined by

$$\mathrm{Cliff}(C) = \min_{\text{line bundle } A} \{\deg A - 2(h^0(A) - 1) \mid h^0(A) \geq 2, h^1(A) \geq 2\}.$$

- Many people made research on this invariant, including Martens, Eisenbud, Lange, Schreyer, Lopez, etc.

$\mathrm{Cliff}(C) \geq 0$		Clifford Theorem
$\mathrm{Cliff}(C) = 0$	$\Leftrightarrow$	hyerelliptic
$\mathrm{Cliff}(C) = 1$	$\Leftrightarrow$	trigonal, plane quintic (degree 5)
$\mathrm{Cliff}(C) = 2$	$\Leftrightarrow$	tetragonal (4-gon), plane sextic (degree 6)
$\mathrm{Cliff}(C) = 3$	$\Leftrightarrow$	pentagonal, plane septic (degree 7), c.i. of cubices in $\mathbb{P}^3$

- Noether theorem: if $\mathrm{Cliff}(C) > 0$, then K_C satisfies N_0.
- Petri's theorem: if $\mathrm{Cliff}(C) > 1$, then K_C satisfies N_1.



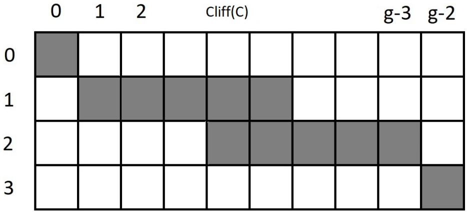
	0	1	2					g-3	g-2
0	▨								
1		▨	▨	▨	▨	▨	?	?		
2			?	?	▨	▨	▨	▨	▨	
3										▨

Green's conjecture

Conjecture (M. Green, 1984)

Let C be a nonsingular projective curve, then

$$\mathrm{Cliff}(C) > l \iff K_C \textit{ satisfies } N_l$$

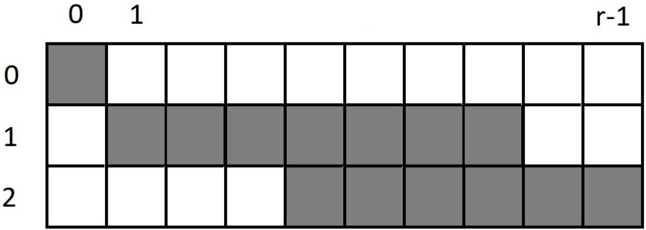


	0	1	2		Cliff(C)				g-3	g-2
0	▨									
1		▨	▨	▨	▨	▨				
2					▨	▨	▨	▨	▨	
3										▨

There a lot of work on this conjecture including Schreyer, Voisin, Kemeny, Aprodu, Farkas, Ortega, Park, etc.

Syzygies of curves of large degree

- Consider a curve C and a line bundle L.
- If $\deg L$ is large enough, then $H^1(C, L) = 0$. This basically means the Betti diagram of $R(L)$ has only three rows and looks like in the following shape



	0	1								r-1
0	▨									
1		▨	▨	▨	▨	▨	▨	▨		
2					▨	▨	▨	▨	▨	▨

Theorem (Mumford, 1970)

Let L_1 and L_2 be two line bundles on a curve C of genus g such that $\deg L_1 \geq 2g+1$ and $\deg L_2 \geq 2g$. Then the multiplication map

$$H^0(L_1) \otimes H^0(L_2) \longrightarrow H^0(L_1 \otimes L_2)$$

is surjective. In particular, if L is a line bundle on a curve C of degree $\geq 2g+1$, then L satisfies Property N_0, i.e., the morphism

$$S^m H^0(L) \longrightarrow H^0(L^m)$$

is surjective for all $m \geq 0$.

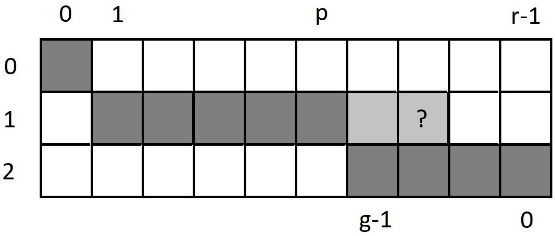
- Castelnuovo (1893) N_0 if $\deg L \geq 2g+1$
- Mattuck-Mayer (1963) N_1 if $\deg L \geq 2g+2$.

Green's $2g + 1 + p$-theorem

Theorem (Green, 1984)

Let L be a line bundle on a curve of degree $\geq 2g + 1 + p$, $p \geq 0$, then L satisfies Property N_p.

- This theorem greatly extended and revealed a full picture of many classic results.
- The theorem also initiated the study of N_p property for higher dimensional varieties.
- The Betti diagram of the line bundle in the theorem



Understand

$$K_{i,1}(L), \text{ for } r-g \leq i \leq r-1.$$

Observation: Suppose C carries a free pencil g^1_e, i.e.,

$$C \xrightarrow{e:1} \mathbb{P}^1$$

Fibers span $(e-1)$-linear spaces in $\mathbb{P}^r$ and swipe out a rational normal scroll S of dimension e containing C, i.e.,

$$C \hookrightarrow S \hookrightarrow \mathbb{P}^r$$

This means

$$K_{p,1}(S) \hookrightarrow K_{p,1}(C)$$

Thus $K_{r-e}(C) \neq 0$.

Gonality

Definition

Let C be a nonsingular projective curve. The gonality $\text{gon}(C)$ of C is the least degree of non constant morphisms $C \to \mathbb{P}^1$.

$$\text{gon}(C) = \min\{d \mid C \text{ carries a } g^1_d\}.$$

Example

- Let $C \subset \mathbb{P}^2$ be a degree d plan curve, then $\text{gon}(C) = d - 1$.
- If $\text{gon}(C) = 2$, then C is a hyperelliptic curve.
- Using Brill-Noether theory, one has

$$\text{gon}(X) \leq [\frac{g+3}{2}]$$

- $\text{Cliff}(C) + 2 \leq \text{gon}(C) \leq \text{Cliff}(C) + 3.$

Green-Lazarsfeld, 1986

Gonality conjecture: For $\deg L \gg 0$,

$$K_{p,1}(L) \neq 0 \iff 1 \leq p \leq r - \operatorname{gon}(C),$$

which is reduced to

$$K_{p,1}(L) = 0 \text{ for } r - \operatorname{gon}(C) + 1 \leq p \leq r - 1.$$

- Very small value of gonality: Green (1984), Teixidor Bigas (2007).
- A general curve of each gonality: Aprodu-Voisin (2003), Aprodu (2004).

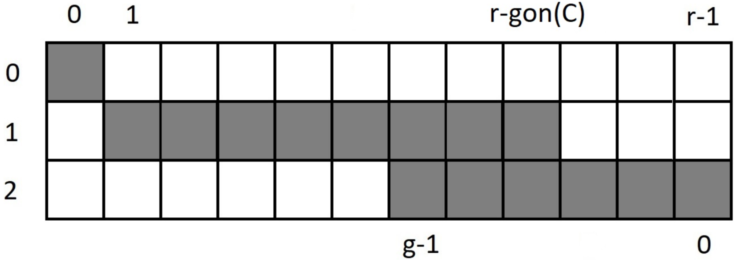
Effective gonality problem: Find optimal bounds for $\deg L$

Theorem (Ein-Lazarsfeld, 2015)

Let L be a line bundle on a curve. If $\deg L \gg 0$, then

$$K_{p,1}(L) \neq 0 \iff 1 \leq p \leq r - \operatorname{gon}(C)$$

where $r = h^0(L) - 1$.



What about the effective gonality problem?

- Green–Lazarsfeld (1986): expected $\deg L \geq 2g + \text{gon}(C) - 1$.
- Ein-Lazarsfeld (2015): proved $\deg L \geq g^3$.
- Rathmann (2016): proved $\deg L \geq 4g - 3$.
- Castryck (2017): showed if $C \subseteq \mathbb{P}H^0(C, H) = \mathbb{P}^2$ is a plane curve and $L := \omega_C \otimes H$, then $\deg L = 2g + \text{gon}(C) - 1$ but $K_{r(L)-\text{gon}(C)+1,1}(C; L) \neq 0$.
- Farkas–Kemeny (2019): showed $\deg L \geq 2g + \text{gon}(C) - 1$ and $g \geq 4$ for general curves. Also notice that $L = \omega_C(\xi)$, where ξ is an effective divisor of degree $\text{gon}(C)$, then $\deg L = 2g + \text{gon}(C) - 2$ but $K_{r(L)-\text{gon}(C)+1,1}(C; L) \neq 0$.

Theorem (**Niu-Park**, 2024)

Let C be a smooth projective curve of genus $g \geq 2$, and L be a line bundle on C. If $\deg L \geq 2g + \mathrm{gon}(C) - 2$, then

$$K_{p,1}(C; L) \neq 0 \iff 1 \leq p \leq r(L) - \mathrm{gon}(C)$$

except for the following two cases:

(1) $C \subseteq \mathbb{P}H^0(C, H) = \mathbb{P}^2$ *is a plane curve of degree* ≥ 4, *and* $L = \omega_C \otimes H$. *In this case,* $\deg L = 2g + \mathrm{gon}(C) - 1$.

(2) C *is arbitrary, and* $L = \omega_C(\xi)$ *for an effective divisor* ξ *of degree* $\mathrm{gon}(C)$ *with* $\dim |\xi| = 1$. *In this case,* $\deg L = 2g + \mathrm{gon}(C) - 2$.

In the exceptional cases, we have

$$K_{p,1}(C; L) \neq 0 \iff 1 \leq p \leq r(L) - \mathrm{gon}(C) + 1.$$

III. Secant varieties of curves and others

Secant varieties of curves

Consider a nonsingular curve

$$C \hookrightarrow \mathbb{P}^r$$

of genus g embedded by a very ample line bundle L. For an integer $k \geq 0$, the k-th secant variety

$$\Sigma_k \subseteq \mathbb{P}^r$$

to the curve is the Zariski closure of the union of $(k+1)$-secant k-planes to C. One has the natural inclusions

$$C = \Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_k \subseteq \mathbb{P}^r.$$

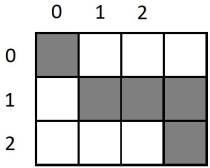
If $\deg L \geq 2g + 2k + 1$, then

$$\dim \Sigma_k = 2k + 1.$$

Syzygies of secant varieties

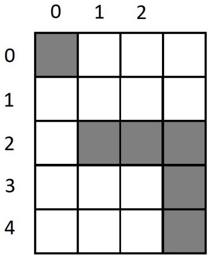
Theorem (Ein-Niu-Park, 2020)

Let C be a nonsingular projective curve of genus g and let L be a line bundle. If $\deg L \geq 2g+2k+1+p$ for integers $k, p \geq 0$, then the k-th secant variety Σ_k of C in $\mathbb{P}(H^0(L))$ satisfies the property $N_{k+2,p}$.



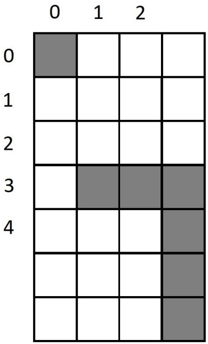
	0	1	2	
0	■			
1		■	■	■
2				■

k=0, p=2



	0	1	2	
0	■			
1				
2		■	■	■
3				■
4				■

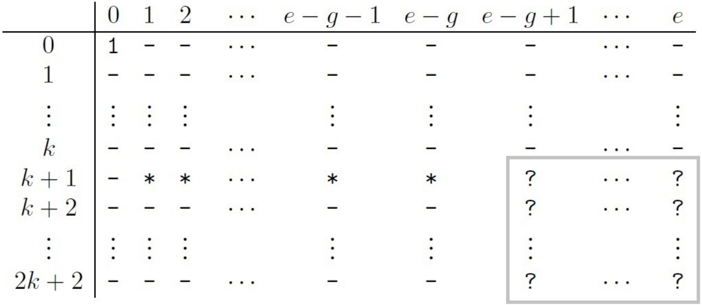
k=1, p=2



	0	1	2	
0	■			
1				
2				
3		■	■	■
4				■
				■
				■

k=3, p=2

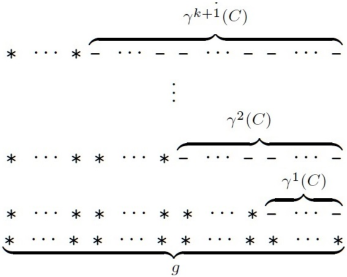
So the overall shape of the Betti diagram for *k*-th secant variety looks like, where $e = \operatorname{codim} \Sigma_k$,



	0	1	2	$\cdots$	$e-g-1$	$e-g$	$e-g+1$	$\cdots$	e
0	1	–	–	$\cdots$	–	–	–	$\cdots$	–
1	–	–	–	$\cdots$	–	–	–	$\cdots$	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
k	–	–	–	$\cdots$	–	–	–	$\cdots$	–
$k+1$	–	$*$	$*$	$\cdots$	$*$	$*$	?	$\cdots$	?
$k+2$	–	–	–	$\cdots$	–	–	?	$\cdots$	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
$2k+2$	–	–	–	$\cdots$	–	–	?	$\cdots$	?

Choe-Kwak-Park, 2023

For $\deg L \gg 0$,



where $\gamma^q(C) := \min\{d - q \mid C$ carries a linear series $g^q_d\}$ (gonality sequence). In particular, $\mathrm{gon}(C) = \gamma^1(C) + 1$.

Ideas and approach

- For an integer $k \geq 0$, denote by C_k the k-th symmetric product of C, which parameterizes effective divisors of degree k on C. It is a smooth projective variety of dimension k.
- There is a quotient map
$$q : C^k \longrightarrow C_k.$$
For a line bundle L on C, the box product
$$L^{\boxtimes k} := \underbrace{L \boxtimes \cdots \boxtimes L}_{k \text{ times}} \quad \text{on } C^k$$
has a S_k-action and descends to a line bundle $T_{k,L}$ on C_k.
- It was observed by Voisin that the Koszul groups $K_{p,q}(L)$ can be computed on the symmetric product C_k for appropriate k using some vector bundles related to $T_{k,L}$. This is the essential ideas used in above results on gonality theorem.

N_p of $T_{k+1,L}$

Collaborated with Peter Yi Wei, we recently show that

Theorem (In progress)

Let C be a nonsingular projective curve of genus g. Let $k \geq 0$ be an integer. For any line bundle L of $\deg L \geq kg + 2g + p + 1$, the line bundle $T_{k+1,L}$ on C_{k+1} satisfy N_p property.

Thank you!