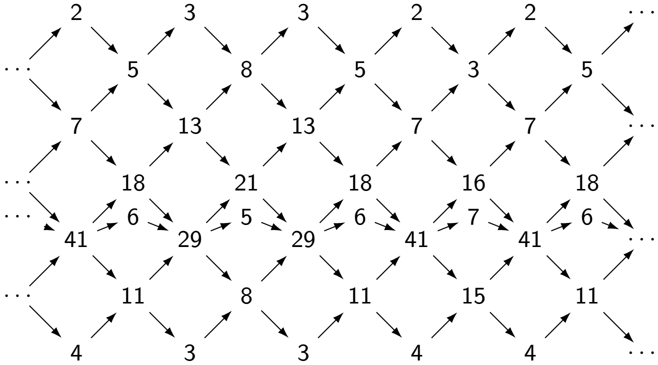
Greg Muller (OU) February 28th, 2026

Friezes of Dynkin Type



Classical friezes

An example before the definition

Consider the following infinite strip of positive integers.

$$
\begin{array}{cccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 3 & & 1 & & 3 & & 2 & & 1 & & 4 & & 1 & & 3 & & 1 & \\
 & 2 & & 2 & & 2 & & 5 & & 1 & & 3 & & 3 & & 2 & & 2 & & \cdots \\
\cdots & & 1 & & 3 & & 3 & & 2 & & 2 & & 2 & & 5 & & 1 & & 3 & \\
 & 2 & & 1 & & 4 & & 1 & & 3 & & 1 & & 3 & & 2 & & 1 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

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Consider the following infinite strip of positive integers.

$$
\begin{array}{cccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 3 & & 1 & & 3 & & 2 & & 1 & & 4 & & 1 & & 3 & & 1 & \\
 & 2 & & 2 & & 2 & & 5 & & 1 & & 3 & & 3 & & 2 & & 2 & & \cdots \\
\cdots & & 1 & & 3 & & 3 & & 2 & & 2 & & 2 & & 5 & & 1 & & 3 & \\
 & 2 & & 1 & & 4 & & 1 & & 3 & & 1 & & 3 & & 2 & & 1 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

What makes this configuration of numbers special?

Every 2×2 diamond has determinant 1 (as a matrix rotated 45°).

Definition: Friezes of type A

A **frieze of type A_n** of consists of $n+2$ rows of positive integers, stacked vertically and offset in a diamond grid, such that

1. the top and bottom rows consist entirely of 1s,and
2. each 2×2-diamond has determinant 1.

Ex: A frieze of type A_2

	1		1		1		1		1		1		1		1		1		1		1		1		1		1		⋯
⋯		3		1		2		2		1		3		1		2		2		1		3		1		2		2	
	2		2		1		3		1		2		2		1		3		1		2		2		1		3		⋯
⋯		1		1		1		1		1		1		1		1		1		1		1		1		1		1	

Also called Coxeter friezes, Coxeter-Conway friezes, or $\mathrm{SL}(2)$-friezes.

Introduced by Coxeter in '71 to generalize Gauss' *pentagramma mirificum*, and classified by Coxeter and Conway in '73.

Let's try to construct friezes of type A_2!

Start with two rows of 1s and two arbitrary values a and b.

$$
\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & a & & & & & & & & & & & & \\
 & & & & & & & & b & & & & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

Let's try to construct friezes of type A_2!

Then we know 3 of the 4 numbers in the following diamond:

$$
\begin{array}{ccccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & & & & & & a & & ? & & & & & & & & & & & \\
 & & & & & & & & b & & & & & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots &
\end{array}
$$

$$? = \frac{b+1}{a}$$

Let's try to construct friezes of type A_2!

$$\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & a & & \frac{b+1}{a} & & & & & & & & & & \\
 & & & & & & & & b & & ? & & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}$$

$$? = \frac{\frac{b+1}{a} + 1}{b} = \frac{a+b+1}{ab}$$

Let's try to construct friezes of type A_2!

$$\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & & & a & & \frac{b+1}{a} & & ? & & & & & & \\
 & & & & & & & & & & b & & \frac{a+b+1}{ab} & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}$$

$$? = \frac{\frac{a+b+1}{ab}+1}{\frac{b+1}{a}} = \frac{(a+1)(b+1)}{b(b+1)} = \frac{a+1}{b}$$

Let's try to construct friezes of type A_2!

$$
\begin{array}{ccccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & & & & & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & & & & & & & & \\
 & & & & & & & & b & & \frac{a+b+1}{ab} & & ? & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots &
\end{array}
$$

$$? = \frac{\frac{a+1}{b}+1}{\frac{a+b+1}{ab}} = \frac{\frac{a+b+1}{b}}{\frac{a+b+1}{ab}} = a$$

Let's try to construct friezes of type A_2!

$$
\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & ? & & & & \\
 & & & & & & & & & & b & & \frac{a+b+1}{ab} & & a & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

$$? = \frac{a+1}{\frac{a+1}{b}} = b$$

Let's try to construct friezes of type A_2!

$$
\begin{array}{ccccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & & & & & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b & & & & & & & \\
 & & & & & & & & b & & \frac{a+b+1}{ab} & & a & & & & & & & & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots &
\end{array}
$$

If we keep going to the right, the computations just repeat!

Let's try to construct friezes of type A_2!

$$
\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & & & & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b & & \frac{a+b+1}{ab} & & a & & \cdots \\
 & & & & & & & & b & & \frac{a+b+1}{ab} & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

If we repeat this approach to the left, the same thing happens!

Let's try to construct friezes of type A_2!

$$
\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & \frac{a+1}{b} & & b & & \frac{a+b+1}{ab} & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b & & \frac{a+b+1}{ab} & & a & & \cdots \\
\cdots & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b & & \frac{a+b+1}{ab} & & a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

- This yields an A_2 frieze iff a, b are positive integers such that

$$\frac{b+1}{a} \qquad \frac{a+b+1}{ab} \qquad \frac{a+1}{b}$$

  are positive integers. (These fractions are **cluster variables**)
- Every such frieze can be constructed in this way.

Observation: Periodicity

Every frieze of type A_2 is **5-periodic**; that is, each number is the same as the number 5 places to the right.

This turns out to be a general phenomenon!

Theorem [Coxeter, 1971]

Every frieze of type A_n is $(n+3)$-periodic.

So when does this construction give a frieze?

Problem

For which pairs of positive integers a, b are

$$\frac{b+1}{a} \qquad \frac{a+b+1}{ab} \qquad \frac{a+1}{b}$$

positive integers?

Dumb observation

Positive integers are greater than or equal to 1.

Weaker Problem

For which real numbers $a \geq 1$ and $b \geq 1$ are

$$\frac{b+1}{a} \geq 1 \qquad \frac{a+b+1}{ab} \geq 1 \qquad \frac{a+1}{b} \geq 1$$

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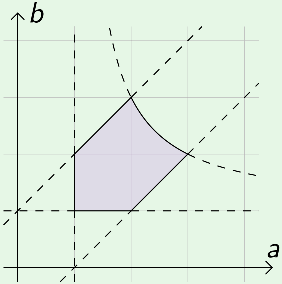
$$b+1 \geq a \qquad a+b+1 \geq ab \qquad a+1 \geq b$$

Five inequalities

We can plot the region carved out by these five inequalities:

$$a \geq 1$$
$$b \geq 1$$
$$b + 1 \geq a$$
$$a + b + 1 \geq ab$$
$$a + 1 \geq b$$



Any choice of (a, b) that gives a frieze must lie in this region.

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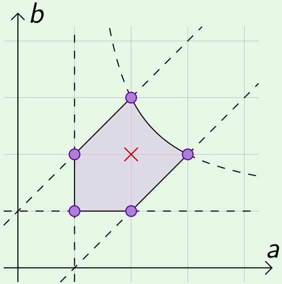
$$a + 1 \geq b$$



Any choice of (a, b) that gives a frieze must lie in this region.

There are 6 choices of integers (a, b) in this 'dented pentagon'. Plugging into the cluster variables, **5** of them give friezes:

$$(1,1) \quad (1,2) \quad (2,1) \quad (2,3) \quad (3,2)$$

The **5** friezes of type A_2 are all translations of:

$$\begin{array}{cccccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
& 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & \cdots \\
\cdots & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}$$

Theorem [Conway-Coxeter, 1973]

Friezes of type A_n biject to triangulations of an $(n+3)$-gon, so the number of such friezes is the $(n+1)$st Catalan number:

$$C_{n+1} := \frac{1}{n+2}\binom{2n+2}{n+1}$$

Ex: Counting friezes of type A_2

# of friezes of type $A_2 = \mathbf{5} =$ # triangulations of pentagon

Friezes of Dynkin type (and other trees)

Where to next?

This is a lovely story, and one that has been generalized in many directions (all dubbed 'friezes') with varying levels of success.

Example: A path not taken

"What if we made the diamonds bigger?" $\Rightarrow$ $\mathrm{SL}(k)$-friezes

The direction we have in mind is not the most obvious one (unless you have already noticed a connection with cluster algebras).

Our path: Friezes from trees

Generalize the notion of 'adjacency between rows'.

First, we need notation more amenable to non-planar friezes.

Modifying our notation for friezes of type A_n

- Delete the rows of 1s
- Add an arrow from each number to the numbers immediately above right and below right.

$$
\begin{array}{ccccccccccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\cdots & & 3 & & 1 & & 3 & & 2 & & 1 & & 4 & & 1 & & 3 & & 1 & \\
 & 2 & & 2 & & 2 & & 5 & & 1 & & 3 & & 3 & & 2 & & 2 & & \cdots \\
\cdots & & 1 & & 3 & & 3 & & 2 & & 2 & & 2 & & 5 & & 1 & & 3 & \\
 & 2 & & 1 & & 4 & & 1 & & 3 & & 1 & & 3 & & 2 & & 1 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

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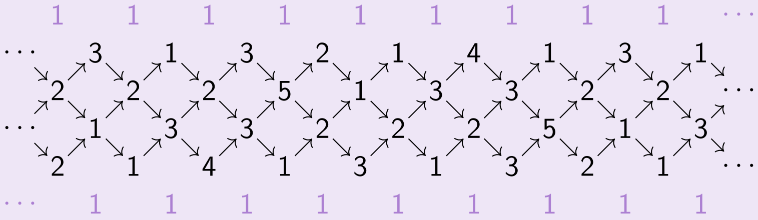
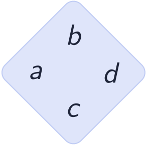
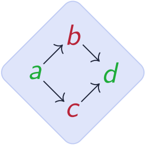


Reformulating the determinant condition using arrows

The product of any two adjacent numbers in the same row is equal to 1 plus the product of the intermediate numbers.



$$ad - bc = 1$$

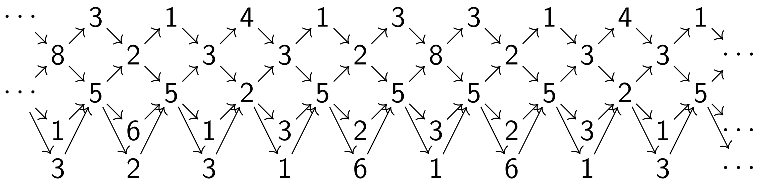


$$ad = 1 + bc$$

This new formula can be generalized in more directions!

An example before the definition

Consider the following configuration of numbers and arrows.



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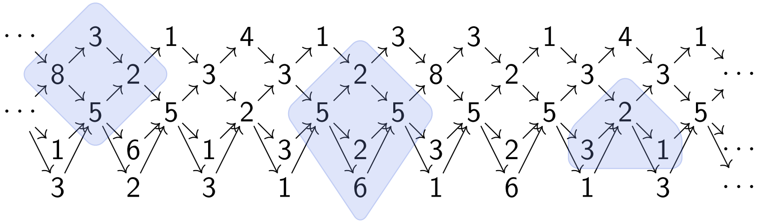


What makes this configuration of numbers special?

The product of any two adjacent numbers in the same row is equal to 1 plus the product of the intermediate numbers.

$$8 \cdot 2 = 1 + 3 \cdot 5 \qquad 5 \cdot 5 = 1 + 2 \cdot 2 \cdot 6 \qquad 3 \cdot 1 = 1 + 2$$

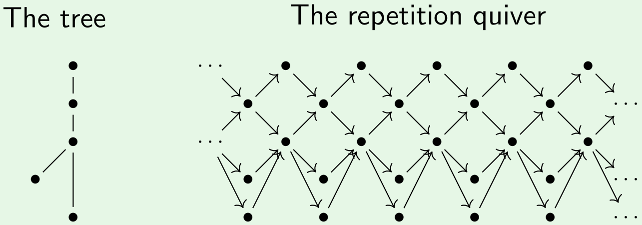
The shape of these non-planar friezes will be determined by a tree.

Rough Definition: The repetition quiver of a tree

Let Γ be a (finite) tree. The **repetition quiver** of Γ has

- an infinite horizontal row of vertices for each vertex in Γ, and
- an infinite zigzag path of arrows for each edge in Γ.

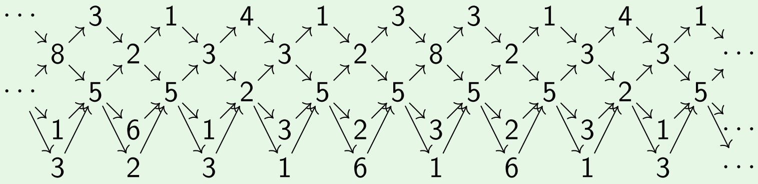
Example: The repetition quiver of type D_5



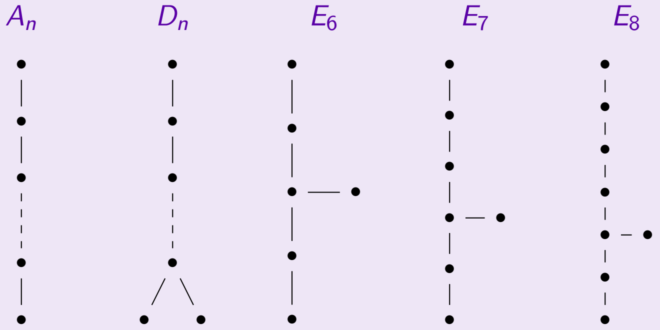
Definition: Friezes of tree type

A **frieze of type Γ** puts a positive integer on each vertex of the repetition quiver of Γ, so the product of two horizontally adjacent values equals 1 plus the product of the intermediate values.

Example: A frieze of type D_5



Definition: The simply-laced Dynkin diagrams



Friezes of type A_n are vertically adjacent rows (our first examples).

What about non-simply laced Dynkin diagrams?

It's too fiddly for us, but one can define friezes for non-simply-laced trees, and get friezes of Dynkin types B_n, C_n, F_4, and G_2.

Theorem: Friezes of Dynkin type are periodic [∼2009]

The friezes of type Γ have a common period p iff Γ is Dynkin.

Both directions were 'folklore' for years before appearing in print.

Dynkin	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
period	$n+3$	$n+1$	$n+1$	$\mathrm{lcm}(2,n)$	14	10	16	7	4

Example

Every frieze of type D_5 is 10-periodic.

Since the translations of an aperiodic frieze are all distinct...

Corollary

If Γ is not Dynkin, then there are infinitely many friezes of type Γ.

Theorem [Gunawan-M, 2022; M, 2023]

There are finitely many friezes of each Dynkin type.

Two proofs

Two uniform proofs of finiteness in each Dynkin type:

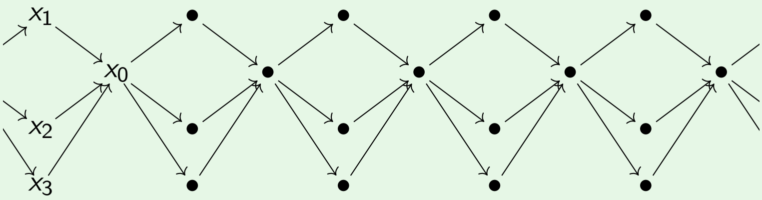
- A non-constructive proof via the geometry of cluster algebras.
- An astronomical bound on the entries via Cartan matrices

Proof via superunitary regions

Generalizing our computation for A_2

Consider the frieze of type Γ with initial variables on the vertices in a 'slice', and other values determined by the frieze condition.

Example (Type D_4)



The expressions that occur are **cluster variables**; there are finitely many, and they are all positive integral Laurent polynomials.

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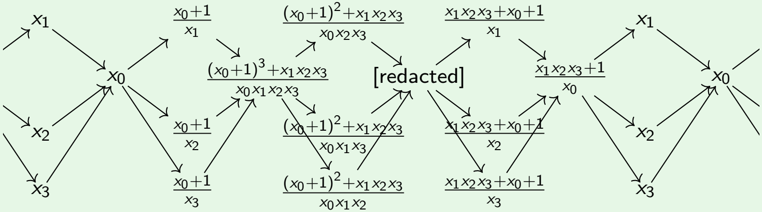


The expressions that occur are **cluster variables**; there are finitely many, and they are all positive integral Laurent polynomials.

A familiar observation

A frieze of type Γ is given by a choice of initial values which makes the cluster variables positive integers.

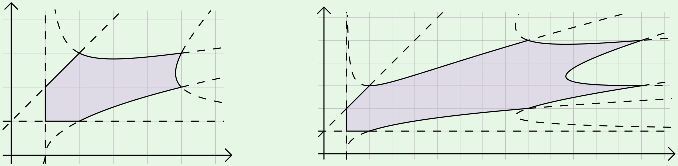
As before, we can weaken 'positive integer' to ≥ 1.

Definition: The superunitary region

The **superunitary region (of type Γ)** is the set of initial values which make the cluster variables ≥ 1.

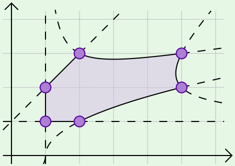
This is a subset of $\mathbb{R}^n$, where n is the number of vertices in Γ.

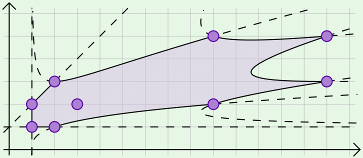
Superunitary regions in Dynkin types B_2/C_2 and G_2



These regions are compact, so they give finitely many possible a, b.

Superunitary regions in Dynkin types B_2/C_2 and G_2





These regions are compact, so they give finitely many possible a, b.
Checking these points yields **6** and **9** points determining friezes.

Theorem [Gunawan-M, 2022]

The superunitary region of each Dynkin type has a face-preserving homeomorphism to a polytope (the generalized associahedron).

Since polytopes are compact, and the friezes correspond to certain integer-valued points (a discrete subset), a topology exercise says...

Corollary (First proof of finiteness)

There are finitely many friezes of each Dynkin type.

Proof via Cartan matrices

Average logarithms

Given a Dynkin frieze, compute the average $\log_2$ of each row.

Example

1.034	3	1	4	1	3	3	1	4
1.634	2	3	3	2	8	2	3	3
2.058	5	5	2	5	5	5	5	2
1.192	6	1	3	2	3	2	3	1
1.192	2	3	1	6	1	6	1	3

For brevity, I will assume familiarity with Cartan matrices.

Lemma [M, 2023]

If $\vec{v}$ is the vector of average logs of a Dynkin frieze, then each entry of the product with the Cartan matrix $C\vec{v}$ is between 0 and 1.

The proof is elementary and less than a page long.

Example

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1.034 \\ 1.634 \\ 2.058 \\ 1.192 \\ 1.192 \end{bmatrix} = \begin{bmatrix} 0.434 \\ 0.176 \\ 0.096 \\ 0.327 \\ 0.327 \end{bmatrix}$$

Theorem [M, 2023]

The average of the base 2 logarithms of the ith row of a frieze with Cartan matrix C is at most the sum of the ith row of C^{-1}.

Example

$$C^{-1} = \begin{bmatrix} 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 3 & \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{5}{4} & \frac{3}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} & \frac{3}{4} & \frac{5}{4} \end{bmatrix} \quad \text{row sums } \begin{bmatrix} 4 \\ 7 \\ 8 \\ 5 \\ 5 \end{bmatrix} \quad \text{ave. } \log_2 \begin{bmatrix} 1.034 \\ 1.634 \\ 2.058 \\ 1.192 \\ 1.192 \end{bmatrix}$$

Corollary (Second proof of finiteness)

An entry in the ith row of a frieze of type Γ is at most 2^{pb_i}, where p is the period and b_i is the sum of the ith row of C^{-1}.

The enumeration of Dynkin friezes

Coxeter-Conway, 1973

The number of A_n friezes is $\frac{1}{n+2}\binom{2n+2}{n+1}$.

Fontaine-Plamondon, 2016

B_n	C_n	D_n	G_2
$\sum_{m=1}^{\sqrt{n+1}} \binom{2n-m^2+1}{n}$	$\binom{2n}{n}$	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$	9

$d(m) :=$ the number of divisors of m

Cuntz-Plamondon, 2020

E_6	F_2
868	112

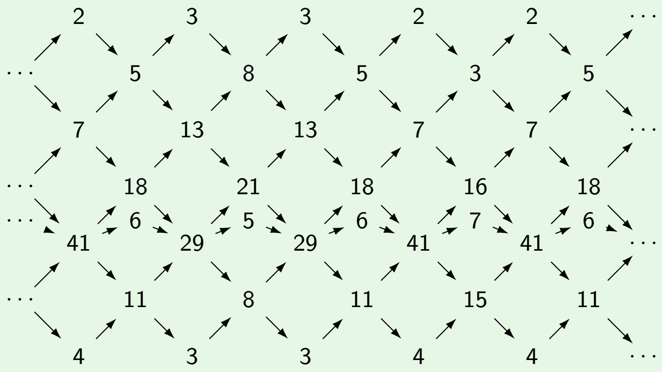
Zhang, 2025 (preprint)

E_7	E_8
4400	26952

A curiosity: Dynkin friezes without any 1s are extremely rare!

One of type G_2, one of type B_{n^2} for each n, $d(n)$-many of type D_n for each n, and one (up to translation) of type E_8.

The unique-up-to-translation frieze of type E_8 with no 1s

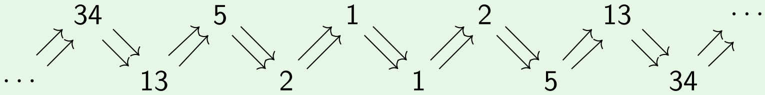


An open question

What about the finiteness of friezes...up to translation?

Friezes of type $A_1^{(1)}$

Every frieze of type $\bullet = \bullet$ is a translation of the following frieze.



Conjecture

There are finitely many friezes of type Γ up to translation if and only if Γ is **affine Dynkin type**.

Both directions of the conjecture are currently open.