A COHEN-MACAULAY ALGEBRA HAS ONLY FINITELY MANY SEMIDUALIZING MODULES

LARS WINTHER CHRISTENSEN AND SEAN SATHER-WAGSTAFF

Abstract. We prove the result stated in the title, which answers the equicharacteristic case of a question of Vasconcelos.

In this paper, R is a commutative noetherian local ring. A finitely generated R-module C is *semidualizing* if the homothety morphism $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism and $\operatorname{Ext}_R^{\geq 1}(C,C) = 0$. Examples include R itself and (if one exists) a dualizing R-module in the sense of Grothendieck. Semidualizing modules have properties similar to those of dualizing modules [2] and arise in several contexts.

Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R-modules. Vasconcelos, calling these modules "spherical," asked whether $\mathfrak{S}_0(R)$ is finite when R is Cohen-Macaulay and whether it has even cardinality when it contains more than one element [8, p. 97]. In [7] affirmative answers to these questions are given, e.g., for certain determinantal rings. Here we prove:

(1) **Theorem.** *If R is Cohen-Macaulay and equicharacteristic, then $\mathfrak{S}_0(R)$ is finite.*

(2) **Remark.** This result also yields an answer to the parity part of Vasconcelos' question for certain Cohen-Macaulay rings. Let R be as in Theorem (1). If R is Gorenstein, then $\mathfrak{S}_0(R)$ has exactly one element, namely the isomorphism class of R; see [2, cor. (8.6)]. On the other hand, if R has a dualizing module and is not Gorenstein, then $\mathfrak{S}_0(R)$ has even cardinality. Indeed, in the derived category of R, every semidualizing R-complex is isomorphic to a shift of a semidualizing R-module; see [4, cor. 3.4]. Thus, the set of shift-isomorphism classes of semidualizing R-complexes is finite, and so [2, prop. (3.7)] shows that it has even cardinality.

In preparation for the proof of Theorem (1), we recall some recent results that allow us to reduce it to a problem in representation theory over artinian k-algebras.

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(3) **Fact.** Let $\varphi\colon R \to S$ be a local ring homomorphism of finite flat dimension, i.e., such that S has finite flat dimension as an R-module via φ. If C is a semidualizing R-module, then $S \otimes_R C$ is a semidualizing S-module by [2, prop. (5.7)]. If C and C' are semidualizing R-modules such that $S \otimes_R C$ and $S \otimes_R C'$ are isomorphic as S-modules, then C and C' are isomorphic as R-modules; see [3, thm. 4.5 and 4.9]. Thus, the functor $S \otimes_R -$ induces an injective map $\mathfrak{S}_0(\varphi)\colon \mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(S)$.

(4) **Lemma.** *Assume there are local ring homomorphisms of finite flat dimension*

$$R \xrightarrow{\varphi} R' \xleftarrow{\rho} Q \xrightarrow{\tau} Q'$$

such that ρ is surjective with kernel generated by a Q-regular sequence. Then there are inequalities of cardinalities $|\mathfrak{S}_0(R)| \leq |\mathfrak{S}_0(\widehat{Q})| \leq |\mathfrak{S}_0(\widehat{Q'})|$.

Proof. The completion morphism $\epsilon\colon R \to \widehat{R}$ conspires with the completions of the given maps to yield the following

$$\mathfrak{S}_0(R) \xhookrightarrow{\mathfrak{S}_0(\hat{\varphi}\epsilon)} \mathfrak{S}_0\big(\widehat{R'}\big) \twoheadleftarrow\!\!\!\xleftarrow{\mathfrak{S}_0(\hat{\rho})}\!\!\!\hookleftarrow \mathfrak{S}_0\big(\widehat{Q}\big) \xhookrightarrow{\mathfrak{S}_0(\hat{\tau})} \mathfrak{S}_0\big(\widehat{Q'}\big).$$

The injectivity of the induced maps is justified in Fact (3); the surjectivity of $\mathfrak{S}_0(\hat{\rho})$ follows from [3, prop. 4.2] because $\widehat{Q}$ is complete and $\hat{\rho}$ is surjective with kernel generated by a $\widehat{Q}$-regular sequence. The desired inequalities now follow. □

(5) **Proof of (1).** Let $\boldsymbol{x}$ be a system of parameters for R and set $R' = R/(\boldsymbol{x})$ with $\varphi\colon R \to R'$ the natural surjection. Using Lemma (4) with $Q' = Q = R' \cong \widehat{R'}$, we may replace R with R' in order to assume that R is artinian.

There is a flat homomorphism of artinian local rings $R \to R''$, such that R'' has algebraically closed residue field; see [5, prop. 0.(10.3.1)]. By Fact (3) we may replace R by R'' to assume that its residue field k is algebraically closed. As R is equicharacteristic and artinian, Cohen's structure theorem implies R is a k-algebra.

For an R-module M, let $\nu(M)$ denote the minimal number of generators of M. Let C be a semidualizing R-module and let E be the injective hull of k. Hom-evaluation [1, prop. 5.3] and the homothety map yield a sequence of isomorphisms

$$C \otimes_R \operatorname{Hom}_R(C,E) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C,C),E) \cong \operatorname{Hom}_R(R,E) \cong E.$$

Hence, there is an inequality $\nu(C) \leq \nu(E)$. This gives the second inequality below; the first is from a surjection $R^{\nu(C)} \twoheadrightarrow C$.

$$\operatorname{length}_R C \leq \nu(C) \cdot \operatorname{length} R \leq \nu(E) \cdot \operatorname{length} R$$

By [6, proof of first prop. in sec. 3] there are only finitely many isomorphism classes of R-modules M with $\operatorname{Ext}_R^{\geq 1}(M,M) = 0$ and $\operatorname{length}_R M \leq \nu(E) \cdot \operatorname{length} R$. From the displayed inequalities it follows that $\mathfrak{S}_0(R)$ is a finite set. □

Finally we illustrate how (1) and (4) apply to answer Vasconcelos' finiteness question for certain rings of mixed characteristic.

(6) **Example.** Let Q be a complete Cohen-Macaulay local ring with residue field of characteristic $p > 0$. If p is Q-regular, then $\mathfrak{S}_0(R)$ is finite for every local ring R such that $\widehat{R} \cong Q$ or $\widehat{R} \cong Q/(p^n)$ for some $n \geq 2$: Theorem (1) implies that $\mathfrak{S}_0(Q/(p))$ is finite, and Lemma (4) applies to the diagram $R \to \widehat{R} \leftarrow Q \to Q/(p)$.

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Department of Math. and Stat., Texas Tech University, Lubbock, TX 79409, U.S.A.
E-mail address: lars.w.christensen@ttu.edu
URL: http://www.math.ttu.edu/~lchriste

Department of Mathematics, 300 Minard Hall, North Dakota State University, Fargo, ND 58105-5075, U.S.A.
E-mail address: Sean.Sather-Wagstaff@ndsu.edu
URL: http://math.ndsu.nodak.edu/faculty/ssatherw/