

DESCENT OF SEMIDUALIZING COMPLEXES FOR RINGS WITH THE APPROXIMATION PROPERTY

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Dedicated to Paul C. Roberts on the occasion of his sixtieth birthday

ABSTRACT. Let R be a commutative noetherian local ring with completion \widehat{R} . When R has the approximation property, we prove an approximation result for complexes with finitely generated homology. This is used to investigate descent of semidualizing complexes from \widehat{R} to R . We show that, if R has the approximation property, then there is a bijective correspondence between semidualizing \widehat{R} -complexes and semidualizing R -complexes. In particular, we recover a result of Hinich and Rotthaus stating that every ring with the approximation property has a dualizing complex.

As an application of the descent theorem, we prove a new version of a classical result on uniform annihilation of homology modules of perfect complexes. Finally, we resolve the finiteness question for the set of isomorphism classes of semidualizing R -modules, when R is Cohen–Macaulay and equicharacteristic.

1. INTRODUCTION

Let (R, \mathfrak{m}) be a commutative noetherian local ring with \mathfrak{m} -adic completion \widehat{R} . Grothendieck and Hartshorne [18, 19] introduced the notion of a dualizing complex as a tool for studying cohomology theories. Not every local ring has a dualizing complex. In fact, a local ring R admits a dualizing complex if and only if it is a homomorphic image of a Gorenstein ring [24]. In particular, Cohen’s Structure Theorem implies that \widehat{R} admits a dualizing complex.

One may not need to go all the way to \widehat{R} to find a flat R -algebra with a dualizing complex. For instance, if R is excellent, then there is a dualizing complex for R^h , the Henselization of R . Indeed, the ring R^h has the approximation property, so the conclusion follows from a result of Hinich [21]; see also Rotthaus [29].

The notion of a semidualizing R -complex generalizes that of a dualizing R -complex; see (4.1) for definitions. Semidualizing complexes arise naturally in the study of local ring homomorphisms and exhibit properties similar to those of dualizing complexes [5, 10]. For example, semidualizing complexes ascend along the completion map. In fact, an R -complex X is (semi)dualizing for R if and only if $\widehat{R} \otimes_R X$ is (semi)dualizing for \widehat{R} .

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This paper is concerned with the following descent question: Given a semidualizing \widehat{R} -complex C , when does there exist a semidualizing R -complex B such that $\widehat{R} \otimes_R B$ is isomorphic to C in the derived category $D(\widehat{R})$? That the answer is not “always” follows from the discussion above: If R is without a dualizing complex, then there is no semidualizing R -complex B such that $\widehat{R} \otimes_R B$ is dualizing for \widehat{R} . See also Example (5.1). Searching for settings where this descent question has a positive answer, the aforementioned work of Hinich and Rotthaus pointed us to rings with the approximation property. One main results of this paper is the following:

(1.1) **Theorem.** *Let R be a local ring with the approximation property. There is a bijective correspondence, given by $\widehat{R} \otimes_R -$, between the sets of (shift-)isomorphism classes of semidualizing complexes in the derived categories $D(R)$ and $D(\widehat{R})$.*

The proof is given (5.2), and the remainder of Section 5 is devoted to consequences of the theorem. In Corollary (5.4) we recover the fact that any local ring with the approximation property has a dualizing complex. In Corollary (5.8) the ring R is only assumed to be excellent: We prove that all semidualizing \widehat{R} -complexes descend to the Henselization R^h and, moreover, that any finite collection of semidualizing \widehat{R} -complexes descends to a flat R -algebra essentially of finite type, in fact, to a pointed étale neighborhood of R . This allows us to prove a new version of a theorem due to Roberts [27] on uniform annihilation of homology modules of perfect complexes; see Corollary (5.9).

If the set of shift-isomorphism classes of semidualizing \widehat{R} -complexes is finite, then the entire set descends to some pointed étale neighborhood of R . The finiteness of this set has been verified for certain classes of rings, see [30], and we add to this with the next result which is contained in Theorem (6.2). The formulation uses the fact that, over a Cohen–Macaulay local ring, every semidualizing complex is isomorphic in the derived category to a module.

(1.2) **Theorem.** *An equicharacteristic Cohen–Macaulay local ring has only finitely many semidualizing modules up to isomorphism.*

Our proof of Theorem (1.1) fuses together techniques employed by Hinich and Rotthaus. As in [21] we utilize the fact that Koszul complexes K^R and $K^{\widehat{R}}$ have differential graded algebra structures, but we replace the decent-datum technology with explicit equations that can be solved using the approximation property as in [29]. Consult Section 2 for definitions and notations. Our approximation argument gives rise to a result of independent interest.

(1.3) **Theorem.** *Let R be a local ring with the approximation property. Let C be an \widehat{R} -complex whose total homology $H(C)$ is finitely generated as an \widehat{R} -module. There exists a complex B of finitely generated free R -modules such that $K^{\widehat{R}} \otimes_R B$ is isomorphic to $K^{\widehat{R}} \otimes_{\widehat{R}} C$ in the derived category $D(K^{\widehat{R}})$.*

In (5.1) we provide an example to show that this theorem may fail in absence of the approximation property.

As to the organization of the paper, Section 3 is devoted to the proof of Theorem (1.3); background material is given in Section 2. The proof of Theorem (1.1), given in Section 5, relies on the discussion of semidualizing objects in Section 4.

2. ALGEBRA AND MODULE STRUCTURES

Throughout this paper, let (R, \mathfrak{m}, k) be a commutative noetherian local ring with \mathfrak{m} -adic completion $(\widehat{R}, \widehat{\mathfrak{m}}, k)$. Set $e = \text{edim } R$, and let K^R denote the Koszul complex on a minimal set of generators for \mathfrak{m} .

(2.1) **Complexes.** We employ homological grading for complexes of R -modules

$$X = \cdots \xrightarrow{\partial_{n+2}^X} X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots$$

and call them R -complexes for short. Let X be an R -complex and m an integer. The m -fold shift of X is denoted $\Sigma^m X$; it is the complex with $(\Sigma^m X)_n = X_{n-m}$ and differential $\partial_n^{\Sigma^m X} = (-1)^m \partial_{n-m}^X$. The hard right truncation of X at m , denoted $X_{\geq m}$, is given by

$$(X_{\geq m})_n = \begin{cases} X_n & \text{if } n \geq m \\ 0 & \text{if } n < m \end{cases} \quad \text{and} \quad \partial_n^{X_{\geq m}} = \begin{cases} \partial_n^X & \text{if } n > m \\ 0 & \text{if } n \leq m. \end{cases}$$

Hard left truncation is defined similarly.

A complex X is *bounded* if $X_n = 0$ when $|n| \gg 0$. The quantities $\sup X$ and $\inf X$ are the supremum and infimum, respectively, of the set $\{n \in \mathbb{Z} \mid H_n(X) \neq 0\}$. We say that X is *homologically bounded* if $H(X)$ is bounded and *homologically degreewise finite* if each module $H_n(X)$ is finitely generated. A complex is *homologically finite* if it is homologically both bounded and degreewise finite.

Isomorphisms in the category of R -complexes are identified by the symbol \cong . Isomorphisms in $D(R)$, the derived category of the category of R -modules, are identified by the symbol \simeq . A morphism α between R -complexes corresponds to an isomorphism in $D(R)$ if and only if the induced morphism $H(\alpha)$ in homology is an isomorphism or, equivalently, the mapping cone $\text{Cone } \alpha$ is exact; when these conditions are satisfied, α is called a *quasiisomorphism*.

R -complexes X and Y admit semifree and semiinjective resolutions $P \xrightarrow{\simeq} X$ and $Y \xrightarrow{\simeq} I$, respectively; see [3, sec. 1]¹ and [6, sec. 7.3, 8.4, and 9.4]. Such resolutions allow definition of derived tensor product and Hom functors $- \otimes_R^L -$ and $\mathbf{R}\text{Hom}_R(-, -)$, as the functors $P \otimes_R -$, $\text{Hom}_R(P, -)$, and $\text{Hom}_R(-, I)$ preserve quasiisomorphisms of R -complexes.

The Koszul complex K^R is a bounded complex of finitely generated free R -modules, in particular, it is semifree and so the functors $K^R \otimes_R -$ and $K^R \otimes_R^L -$ are naturally isomorphic. This fact will be used without further mention. If X is homologically degreewise finite, then [11, 1.3] provides the (in)equalities

$$(2.1.1) \quad \inf K^R \otimes_R X = \inf X \quad \text{and} \quad \sup X \leq \sup K^R \otimes_R X \leq e + \sup X.$$

Hence, the complexes X and $K^R \otimes_R X$ are simultaneously homologically bounded.

It is straightforward to verify the following special case of tensor-evaluation [3, lem. 4.4(F)]. For R -complexes X and Y there is an isomorphism in $D(R)$

$$(2.1.2) \quad K^R \otimes_R \mathbf{R}\text{Hom}_R(X, Y) \xrightarrow{\simeq} \mathbf{R}\text{Hom}_R(X, K^R \otimes_R Y).$$

¹ Where semiinjective resolutions are called DG-injective.

(2.2) **DG modules over Koszul complexes.** The Koszul complex K^R can be realized as an exterior algebra, and the wedge product endows it with a differential graded (DG, for short) algebra structure that is commutative; see e.g. [8, prop. 1.6.2]. That is, the product is unitary and associative, and it satisfies

$$uv = (-1)^{|u||v|}vu \quad \text{and} \quad u^2 = 0 \text{ when } |u| \text{ is odd}$$

$$\partial^{K^R}(uv) = \partial^{K^R}(u)v + (-1)^{|u|}u\partial^{K^R}(v)$$

for all u, v in K^R , where $|u|$ denotes the degree of u .

A DG K^R -module is an R -complex M equipped with a K^R -multiplication: a morphism of R -complexes $K^R \otimes_R M \rightarrow M$, written $u \otimes m \mapsto um$, that is unitary and associative and satisfies the Leibniz rule

$$\partial^M(um) = \partial^{K^R}(u)m + (-1)^{|u|}u\partial^M(m)$$

for all $u \in K^R$ and $m \in M$. A DG K^R -module M is *homologically finite* if the homology module $H(M)$ is finitely generated over $H_0(K^R) \cong k$, equivalently if M is homologically finite as an R -complex.

One source for DG K^R -modules was mentioned in (2.1): If X is an R -complex, then $K^R \otimes_R X$ has a DG K^R -module structure given by $u(v \otimes x) = (uv) \otimes x$. Moreover, if X is a homologically finite R -complex, then $K^R \otimes_R X$ is a homologically finite DG K^R -module by (2.1.1) and [5, (1.2.1)].

A morphism of DG K^R -modules is a morphism of R -complexes that is also K^R -linear. Isomorphisms in the category of DG K^R -modules are identified by the symbol \cong . The derived category of the category of DG K^R -modules is denoted $D(K^R)$; isomorphisms in this category are identified by the symbol \simeq . A morphism of DG K^R -modules corresponds to an isomorphism in $D(K^R)$ if and only if it does so in $D(R)$ and is then called a *quasiisomorphism*.

Any DG K^R -modules M and N admit semifree and semiinjective resolutions $P \xrightarrow{\cong} M$ and $N \xrightarrow{\cong} I$, respectively; see [6, sec. 7.3, 8.4, and 9.4]. Such resolutions allow definition of derived tensor product and Hom functors, $- \otimes_{K^R}^L -$ and $\mathbf{R}\mathrm{Hom}_{K^R}(-, -)$, as the functors $P \otimes_{K^R} -, \mathrm{Hom}_{K^R}(P, -)$, and $\mathrm{Hom}_{K^R}(-, I)$ preserve quasiisomorphisms of DG K^R -modules.

(2.3) **Local homomorphisms and Koszul complexes.** A homomorphism of local rings $\vartheta: (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ is said to be *local* if $\vartheta(\mathfrak{m}) \subseteq \mathfrak{n}$. The flat dimension of ϑ is the flat dimension of the R -module S .

Let $\vartheta: (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$ be a flat local ring homomorphism such that $\mathfrak{m}S = \mathfrak{n}$ and the induced map $k \rightarrow l$ is bijective, e.g. the completion map $R \rightarrow \hat{R}$. A minimal set of generators for \mathfrak{m} maps to a minimal set of generators for \mathfrak{n} under ϑ , and this induces an isomorphism of S -complexes and of DG K^R -modules

$$(2.3.1) \quad S \otimes_R K^R \cong K^S.$$

Furthermore, ϑ induces a quasiisomorphism of R -complexes

$$(2.3.2) \quad K^R \xrightarrow{\cong} K^S$$

which also respects the DG algebra structures. In particular, every (homologically finite) DG K^S -module is a (homologically finite) DG K^R -module. Moreover, $K^S \otimes_{K^R} -$ is an equivalence between $D(K^R)$ and $D(K^S)$; it conspires with (2.3.2)

to yield an isomorphism in $D(K^S)$

$$(2.3.3) \quad K^S \xrightarrow{\simeq} K^S \otimes_{K^R} K^S.$$

(2.4) **The approximation property.** The ring R is said to have the *approximation property* when it satisfies the following: Given any finite system of polynomial equations with coefficients in R and any positive integer t , if the system has a solution in \widehat{R} then it also has a solution in R , and the solutions are congruent modulo $\widehat{\mathfrak{m}}^t$. By work of Popescu [26, thm. (1.3)], Rotthaus [28, thm. 1], and Spivakovsky [31, thm. 11.3], the ring R has the approximation property if and only if it is excellent and Henselian.

3. PROOF OF THEOREM (1.3)

The proof of Theorem (1.3) is given in (3.2); it relies on the following lemma. Recall that a complex P of free R -modules is *minimal* if $\partial(P) \subseteq \mathfrak{m}P$.

(3.1) **Lemma.** *Let R be a local ring with the approximation property. Let m be a positive integer and P a minimal bounded complex of finitely generated free \widehat{R} -modules such that $P_n = 0$ when $n < 0$ or $n > m$. There exists a minimal complex A of finitely generated free R -modules such that $A_n = 0$ when $n < 0$ or $n > m$, and $K^{\widehat{R}} \otimes_R A \simeq K^{\widehat{R}} \otimes_{\widehat{R}} P$ in $D(K^{\widehat{R}})$.*

Proof. We describe a finite system of polynomial equations with coefficients in R such that the complex P gives rise to a solution in \widehat{R} . The approximation property guarantees the existence of a solution in R , which in turn describes a complex A with the desired properties. The proof is broken into steps (3.1.1) to (3.1.11).

(3.1.1) **DG structures on K^R and $K^{\widehat{R}}$.** Fix a basis $\varepsilon_1, \dots, \varepsilon_{2^e}$ for K^R over R . For each $i = 1, \dots, 2^e$ set $\widehat{\varepsilon}_i = 1 \otimes \varepsilon_i \in K^{\widehat{R}}$, cf. (2.3.1). The elements $\widehat{\varepsilon}_1, \dots, \widehat{\varepsilon}_{2^e}$ form a basis for $K^{\widehat{R}}$ over \widehat{R} . The differential $\partial_n^{K^R}$ is given by a matrix of size $\binom{e}{n-1} \times \binom{e}{n}$ with entries in R

$$K_n^R \xrightarrow{[d_{nij}]} K_{n-1}^R.$$

Note that $\partial_n^{K^R} = [d_{nij}] = 0$ when $n < 1$ or $n > e$. By (2.3.1) the matrix $[d_{nij}]$ also describes the n th differential on $K^{\widehat{R}}$.

Multiplication on the degree n component of K^R by a basis vector ε_h is given by a matrix of size $\binom{e}{n+|\varepsilon_h|} \times \binom{e}{n}$ with entries in R

$$K_n^R \xrightarrow{[t_{nij}^h]} K_{n+|\varepsilon_h|}^R.$$

By (2.3.1) the matrices $[t_{nij}^h]$ also describe multiplication by $\widehat{\varepsilon}_h$ on $K^{\widehat{R}}$.

(3.1.2) **A resolution of $K^{\widehat{R}} \otimes_{\widehat{R}} P$ over K^R .** The DG $K^{\widehat{R}}$ -module $K^{\widehat{R}} \otimes_{\widehat{R}} P$ is a homologically finite DG K^R -module through (2.3.2). By [1, prop. 2], see also [6, thm. 12.5.2], there exists a K^R -semifree resolution $F \xrightarrow{\simeq} K^{\widehat{R}} \otimes_{\widehat{R}} P$, such that the differential of $k \otimes_{K^R} F$ is 0 and $F^{\natural} = \coprod_{i \geq 0} \Sigma^i ((K^R)^{\natural})^{\beta_i}$ with $\beta_i \in \mathbb{N}_0$. (Here F^{\natural}

denotes the graded R -module underlying the DG K^R -module F .) Applying the functor $K^{\widehat{R}} \otimes_{K^R} -$ yields a $K^{\widehat{R}}$ -semifree resolution

$$K^{\widehat{R}} \otimes_{K^R} F \xrightarrow{\simeq} K^{\widehat{R}} \otimes_{K^R} K^{\widehat{R}} \otimes_{\widehat{R}} P,$$

and (2.3.3) induces a quasiisomorphism of DG $K^{\widehat{R}}$ -modules

$$K^{\widehat{R}} \otimes_{\widehat{R}} P \xrightarrow{\simeq} K^{\widehat{R}} \otimes_{K^R} K^{\widehat{R}} \otimes_{\widehat{R}} P.$$

As $K^{\widehat{R}} \otimes_{K^R} F$ is $K^{\widehat{R}}$ -semifree, [6, thm. 8.5.1, 8.6.1] provides a $K^{\widehat{R}}$ -morphism

$$\varphi: K^{\widehat{R}} \otimes_{K^R} F \xrightarrow{\simeq} K^{\widehat{R}} \otimes_{\widehat{R}} P.$$

The proof will be complete once we construct a minimal complex A of finitely generated free R -modules such that $K^R \otimes_R A \simeq F$ in $\mathbf{D}(K^R)$ and $A_n = 0$ when $n < 0$ or $n > m$.

Since φ is a map between semifree DG $K^{\widehat{R}}$ -modules, the induced map

$$k \otimes_{K^{\widehat{R}}} \varphi: k \otimes_{K^{\widehat{R}}} (K^{\widehat{R}} \otimes_{K^R} F) \longrightarrow k \otimes_{K^{\widehat{R}}} (K^{\widehat{R}} \otimes_{\widehat{R}} P)$$

is a quasi-isomorphism; see [6, thm. 4.2 and cor. 5.3]. Each of these complexes has zero differential (P is minimal), so $k \otimes_{K^{\widehat{R}}} \varphi$ is an isomorphism, hence the underlying graded k -vector spaces have the same rank. The semifreeness of $K^{\widehat{R}} \otimes_{K^R} F$ and $K^{\widehat{R}} \otimes_{\widehat{R}} P$ over $K^{\widehat{R}}$ implies

$$(3.1.2.1) \quad \text{rank}_R F_n = \text{rank}_{\widehat{R}}(K^{\widehat{R}} \otimes_{K^R} F)_n = \text{rank}_{\widehat{R}}(K^{\widehat{R}} \otimes_{\widehat{R}} P)_n$$

for all n . In particular, $F_n = 0$ when $n < 0$ or $n > m + e$.

(3.1.3) **DG structures on F and $K^{\widehat{R}} \otimes_{K^R} F$.** For each integer n , set $r_n = \text{rank}_R F_n$ and fix an R -basis for F_n . The differential ∂_n^F is given by an $r_{n-1} \times r_n$ matrix

$$F_n \xrightarrow{[u_{nij}]} F_{n-1}$$

with entries in R . Note that $[u_{nij}] = 0$ when $n < 1$ or $n > m + e$. There is an isomorphism of DG $K^{\widehat{R}}$ -modules

$$(3.1.3.1) \quad K^{\widehat{R}} \otimes_{K^R} F \cong \widehat{R} \otimes_R F$$

cf. (2.3.1). In the \widehat{R} -basis induced by the R -basis for F , the n th differential on $K^{\widehat{R}} \otimes_{K^R} F$ is also given by the matrix $[u_{nij}]$.

For each basis vector $\varepsilon_h \in K^R$, cf. (3.1.1), multiplication by ε_h on F is given by matrices with entries in R

$$F_n \xrightarrow{[v_{nij}^h]} F_{n+|\varepsilon_h|}.$$

By (3.1.3.1) the matrices $[v_{nij}^h]$ also describe multiplication by $\widehat{\varepsilon}_h$ on $K^{\widehat{R}} \otimes_{K^R} F$.

(3.1.4) **DG structure on $K^{\widehat{R}} \otimes_{\widehat{R}} P$.** For each integer n , fix a basis for the free \widehat{R} -module P_n and set $s_n = \text{rank}_{\widehat{R}} P_n$. The n th differential of P is then given by an $s_{n-1} \times s_n$ matrix with entries in \widehat{R}

$$P_n \xrightarrow{[x_{nij}]} P_{n-1}$$

which is zero when $n > m$ or $n < 1$. In the basis on $K^{\widehat{R}} \otimes_{\widehat{R}} P$, coming from the bases chosen for $K^{\widehat{R}}$ and the modules P_0, \dots, P_m , the differential

$$\partial_n^{K^{\widehat{R}} \otimes_{\widehat{R}} P} : \bigoplus_{p=0}^m K_{n-p}^{\widehat{R}} \otimes_{\widehat{R}} P_p \longrightarrow \bigoplus_{q=0}^m K_{n-1-q}^{\widehat{R}} \otimes_{\widehat{R}} P_q$$

is given by a block matrix

$$(3.1.4.1) \quad \partial_n^{K^{\widehat{R}} \otimes_{\widehat{R}} P} = \begin{pmatrix} [b_{nij}^{00}] & \cdots & [b_{nij}^{0m}] \\ \vdots & & \vdots \\ [b_{nij}^{m0}] & \cdots & [b_{nij}^{mm}] \end{pmatrix} = [b_{nij}^{qp}]$$

where

$$(3.1.4.2) \quad b_{nij}^{qp} = \begin{cases} d_{(n-p)ij} & \text{if } p = q \\ (-1)^{n-p} x_{pij} & \text{if } p = q + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\partial_n^{K^{\widehat{R}} \otimes_{\widehat{R}} P} = 0$ when $n < 1$ or $n > m + e$.

For each basis vector $\hat{\varepsilon}_h \in K^{\widehat{R}}$, cf. (3.1.1), multiplication by $\hat{\varepsilon}_h$ on $K^{\widehat{R}} \otimes_{\widehat{R}} P$ is given by the formula $\hat{\varepsilon}_h(f \otimes g) = (\hat{\varepsilon}_h f) \otimes g$; see (2.2). In the chosen basis for $K^{\widehat{R}} \otimes_{\widehat{R}} P$, multiplication by $\hat{\varepsilon}_h$ on the degree n component is given by

$$\bigoplus_{p=0}^m [t_{(n-p)ij}^h] \otimes_{\widehat{R}} P_p : \bigoplus_{p=0}^m K_{n-p}^{\widehat{R}} \otimes_{\widehat{R}} P_p \longrightarrow \bigoplus_{p=0}^m K_{n-p+|\varepsilon_h|}^{\widehat{R}} \otimes_{\widehat{R}} P_p.$$

Hence, multiplication by $\hat{\varepsilon}_h$ on $K^{\widehat{R}} \otimes_{\widehat{R}} P$ is given by matrices with entries in R

$$(K^{\widehat{R}} \otimes_{\widehat{R}} P)_n \xrightarrow{[w_{nij}^h]} (K^{\widehat{R}} \otimes_{\widehat{R}} P)_{n+|\varepsilon_h|}.$$

(3.1.5) **First set of variables.** We introduce a finite set of variables

$$\{X_{nij} \mid n = 1, \dots, m; i = 1, \dots, s_{n-1}; j = 1, \dots, s_n\}.$$

The equality $\partial_n^P \partial_{n+1}^P = 0$ says that the elements $x_{nij} \in \widehat{R}$ satisfy a system of quadratic equations in the variables X_{nij} with coefficients 1 and 0, namely the matrix equations

$$(3.1.5.1) \quad [X_{nij}][X_{(n+1)ij}] = [0] \quad \text{for } n = 1, \dots, m-1.$$

For later reference, define $[B_{nij}^{qp}]$ for $p, q = 0, \dots, m$ and $n = 1, \dots, m$ to be the block matrix described by

$$B_{nij}^{qp} = \begin{cases} d_{(n-p)ij} & \text{if } p = q \\ (-1)^{n-p} X_{pij} & \text{if } p = q + 1 \\ 0 & \text{otherwise} \end{cases}$$

cf. (3.1.4.1) and (3.1.4.2).

(3.1.6) **The map φ .** Since φ is a morphism, it satisfies the equation

$$(3.1.6.1) \quad \varphi_{n-1} \partial_n^{K^{\widehat{R}} \otimes_{KR} F} - \partial_n^{K^{\widehat{R}} \otimes_{\widehat{R}} P} \varphi_n = 0$$

for each n . The $K^{\widehat{R}}$ -linearity of φ is tantamount to

$$\varphi(\hat{\varepsilon}_h f) = \hat{\varepsilon}_h \varphi(f)$$

for all $f \in K^{\widehat{R}} \otimes_{K^R} F$ and $h = 1, \dots, 2^e$. For each n , the map φ_n is between free \widehat{R} -modules of rank r_n , cf. (3.1.2.1); hence, it is given by an $r_n \times r_n$ matrix with entries in \widehat{R}

$$(K^{\widehat{R}} \otimes_{K^R} F)_n \xrightarrow{[y_{nij}]} (K^{\widehat{R}} \otimes_{\widehat{R}} P)_n$$

which is 0 when $n < 0$ or $n > m + e$. The $K^{\widehat{R}}$ -linearity of φ can, therefore, be expressed by commutativity of diagrams

$$(3.1.6.2) \quad \begin{array}{ccc} (K^{\widehat{R}} \otimes_{K^R} F)_n & \xrightarrow{[y_{nij}]} & (K^{\widehat{R}} \otimes_{\widehat{R}} P)_n \\ \downarrow [v_{nij}^h] & & \downarrow [w_{nij}^h] \\ (K^{\widehat{R}} \otimes_{K^R} F)_{n+|\varepsilon_h|} & \xrightarrow{[y_{(n+|\varepsilon_h|)ij}]} & (K^{\widehat{R}} \otimes_{\widehat{R}} P)_{n+|\varepsilon_h|} \end{array}$$

for $n = 0, \dots, m + e$ and $h = 1, \dots, 2^e$.

(3.1.7) **Second set of variables.** We introduce another finite set of variables

$$\{Y_{nij} \mid n = 0, \dots, m + e; i = 1, \dots, r_n; j = 1, \dots, r_n\}.$$

By (3.1.6.1) the elements $x_{nij}, y_{nij} \in \widehat{R}$ satisfy a system of equations in the variables X_{nij}, Y_{nij} with coefficients in R . Namely, the matrix equations

$$(3.1.7.1) \quad [Y_{(n-1)ij}][u_{nij}] - [B_{nij}^{qp}][Y_{nij}] = [0] \quad \text{for } n = 1, \dots, m + e.$$

By (3.1.6.2) the elements y_{nij} satisfy a second system of equations in Y_{nij} with coefficients in R , namely the matrix equations

$$(3.1.7.2) \quad [Y_{(n+|\varepsilon_h|)ij}][v_{nij}^h] - [w_{nij}^h][Y_{nij}] = [0]$$

for $h = 1, \dots, 2^e$ and $n = 0, \dots, m + e - |\varepsilon_h|$.

(3.1.8) **The mapping cone of φ .** The complex $\text{Cone } \varphi$ consists of finitely generated free \widehat{R} -modules. In the basis for $\text{Cone } \varphi$ coming from the bases chosen for $K^{\widehat{R}} \otimes_{K^R} F$ and $K^{\widehat{R}} \otimes_{\widehat{R}} P$, the n th differential is given by the block matrix

$$\partial_n^{\text{Cone } \varphi} = \begin{pmatrix} [b_{nij}^{qp}] & [y_{(n-1)ij}] \\ [0] & -[u_{(n-1)ij}] \end{pmatrix}$$

which is zero when $n < 1$ or $n > m + e + 1$. Since φ is a quasiisomorphism, the mapping cone is an exact complex of free \widehat{R} -modules and bounded (below). Hence, there exists a homotopy between 0 and the identity on $\text{Cone } \varphi$, i.e. a degree 1 homomorphism σ on $\text{Cone } \varphi$ such that

$$(3.1.8.1) \quad \sigma_{n-1} \partial_n^{\text{Cone } \varphi} + \partial_{n+1}^{\text{Cone } \varphi} \sigma_n = 1_n^{\text{Cone } \varphi}$$

for every n . Each map σ_n is given by a matrix of size $(r_{n+1} + r_n) \times (r_n + r_{n-1})$ with entries in \widehat{R}

$$(\text{Cone } \varphi)_n \xrightarrow{[z_{nij}]} (\text{Cone } \varphi)_{n+1}$$

which is 0 when $n < 0$ or $n > m + e$.

(3.1.9) **Third set of variables.** We introduce a third finite set of variables

$$\{Z_{nij} \mid n = 0, \dots, m + e; i = 1, \dots, r_{n+1} + r_n; j = 1, \dots, r_n + r_{n-1}\}.$$

The equation (3.1.8.1) means that the elements x_{nij} , y_{nij} , and z_{nij} satisfy a system of equations in X_{nij} , Y_{nij} , and Z_{nij} with coefficients in R , namely the matrix equations

$$(3.1.9.1) \quad [Z_{(n-1)ij}] \begin{pmatrix} [B_{nij}^{qp}] & [Y_{(n-1)ij}] \\ [0] & -[u_{(n-1)ij}] \end{pmatrix} + \begin{pmatrix} [B_{(n+1)ij}^{qp}] & [Y_{nij}] \\ [0] & -[u_{nij}] \end{pmatrix} [Z_{nij}] = [\delta_{ij}]$$

for $n = 0, \dots, m + e + 1$, where δ_{ij} is the Kronecker delta.

(3.1.10) **Comment.** The equations (3.1.5.1), (3.1.7.1), (3.1.7.2), and (3.1.9.1) describe a finite system of polynomial equations in the variables X_{nij} , Y_{nij} , Z_{nij} with coefficients in R . The construction of this system of equations does not rely on R having the approximation property.

(3.1.11) **Conclusion of the proof.** As noted in (3.1.2) it suffices to construct a minimal complex A of finitely generated free R -modules such that $K^R \otimes_R A \simeq F$ in $D(K^R)$ and $A_n = 0$ when $n < 0$ or $n > m$.

The finite system of equations described by (3.1.5.1), (3.1.7.1), (3.1.7.2), and (3.1.9.1) has a solution in \widehat{R} , namely x_{nij} , y_{nij} , z_{nij} . Since R has the approximation property, the system also has a solution \tilde{x}_{nij} , \tilde{y}_{nij} , \tilde{z}_{nij} in R . Furthermore, we may assume that $\tilde{x}_{nij} \equiv x_{nij} \pmod{\widehat{\mathfrak{m}}}$. By minimality of P the elements x_{nij} are in $\widehat{\mathfrak{m}}$, so the \tilde{x}_{nij} are in \mathfrak{m} .

For each n , let A_n be a free R -module of rank $s_n = \text{rank}_{\widehat{R}} P_n$ and fix an R -basis for A_n ; note that $A_n = 0$ when $n < 0$ or $n > m$. For each n let

$$\begin{aligned} \partial_n^A: A_n &\longrightarrow A_{n-1} && \text{be given by } \partial_n^A = [\tilde{x}_{nij}] \\ \tilde{\varphi}_n: F_n &\longrightarrow (K^R \otimes_R A)_n && \text{be given by } \tilde{\varphi}_n = [\tilde{y}_{nij}] \\ \tilde{\sigma}_n: (\text{Cone } \tilde{\varphi})_n &\longrightarrow (\text{Cone } \tilde{\varphi})_{n+1} && \text{be given by } \tilde{\sigma}_n = [\tilde{z}_{nij}]. \end{aligned}$$

Since the elements \tilde{x}_{nij} satisfy (3.1.5.1), one has $\partial_n^A \partial_{n+1}^A = 0$; so A is a complex. Further, A is minimal as the \tilde{x}_{nij} are in \mathfrak{m} . The elements \tilde{x}_{nij} , \tilde{y}_{nij} satisfy (3.1.7.1) and (3.1.7.2), so the map $\tilde{\varphi}$ is a K^R -linear morphism of R -complexes. Moreover, the elements \tilde{x}_{nij} , \tilde{y}_{nij} , \tilde{z}_{nij} satisfy (3.1.9.1), so the map $\tilde{\sigma}$ is a homotopy between 0 and the identity on $\text{Cone } \tilde{\varphi}$. In particular, the cone is exact, and therefore $\tilde{\varphi}$ is the desired quasiisomorphism. \square

(3.2) **Proof of Theorem (1.3).** Let C be a homologically finite \widehat{R} -complex. We identify C with its minimal semifree resolution over \widehat{R} ; see [6, thm. 12.5.1]. After a shift, we may assume that $C_n = 0$ for $n < 0$. Set $s = \sup C$ and $m = s + 2e + 1$. By Lemma (3.1) there exists a minimal complex A of finitely generated free R -modules such that $A_n = 0$ when $n < 0$ or $n > m$, and

$$(3.2.1) \quad K^{\widehat{R}} \otimes_R A \simeq K^{\widehat{R}} \otimes_{\widehat{R}} C_{\leq m}$$

in $D(K^{\widehat{R}})$. Augment A by a minimal R -free resolution of $\text{Ker } \partial_m^A$; this yields a complex B of finitely generated free R -modules with $\sup B < m$ and $B_{\leq m} \cong A$. In particular, the isomorphism (3.2.1) can be rewritten as

$$(3.2.2) \quad K^{\widehat{R}} \otimes_R B_{\leq m} \simeq K^{\widehat{R}} \otimes_{\widehat{R}} C_{\leq m}.$$

Next we show that $\sup(K^{\widehat{R}} \otimes_R B) < m$. Apply $K^{\widehat{R}} \otimes_{\widehat{R}} -$ to the triangle

$$C_{\leq m} \xrightarrow{\gamma_m^C} C \longrightarrow C_{\geq m+1} \longrightarrow \Sigma C_{\leq m}$$

and inspect the long exact homology sequence

$$\dots \longrightarrow \mathrm{H}_{i+1}(K^{\widehat{R}} \otimes_{\widehat{R}} C_{\geq m+1}) \longrightarrow \mathrm{H}_i(K^{\widehat{R}} \otimes_{\widehat{R}} C_{\leq m}) \longrightarrow \mathrm{H}_i(K^{\widehat{R}} \otimes_{\widehat{R}} C) \longrightarrow \dots$$

The module $\mathrm{H}_{i+1}(K^{\widehat{R}} \otimes_{\widehat{R}} C_{\geq m+1})$ vanishes for $i < m$ while $\mathrm{H}_i(K^{\widehat{R}} \otimes_{\widehat{R}} C)$ vanishes for $i > s + e$ by (2.1.1). Hence

$$(3.2.3) \quad \mathrm{H}_i(K^{\widehat{R}} \otimes_{\widehat{R}} C_{\leq m}) = 0 \quad \text{when } s + e < i < m.$$

The isomorphisms $(K^{\widehat{R}} \otimes_R B)_i \cong (K^{\widehat{R}} \otimes_R B_{\leq m})_i$ for $i \leq m$ yield the first isomorphism in the chain

$$\mathrm{H}_i(K^{\widehat{R}} \otimes_R B) \cong \mathrm{H}_i(K^{\widehat{R}} \otimes_R B_{\leq m}) \cong \mathrm{H}_i(K^{\widehat{R}} \otimes_{\widehat{R}} C_{\leq m}) = 0 \quad \text{when } s + e < i < m$$

while the second isomorphism is by (3.2.2), and the vanishing is by (3.2.3). Since the modules $\mathrm{H}_i(B)$ are finitely generated, Nakayama's lemma implies that

$$\mathrm{H}_i(B) = 0 \quad \text{when } s + e < i < m.$$

As $\sup B < m$ it follows that $\sup B \leq s + e$. Hence $\sup(K^{\widehat{R}} \otimes_R B) \leq s + 2e < m$.

Next we construct a commutative diagram in the category of DG $K^{\widehat{R}}$ -modules

$$(3.2.4) \quad \begin{array}{ccc} K^{\widehat{R}} \otimes_R B_{\leq m} & \xrightarrow[\cong]{\alpha} & K^{\widehat{R}} \otimes_{\widehat{R}} C_{\leq m} \\ K^{\widehat{R}} \otimes_R \gamma_m^B \downarrow & & \downarrow K^{\widehat{R}} \otimes_R \gamma_m^C \\ K^{\widehat{R}} \otimes_R B & \xrightarrow{\rho} & K^{\widehat{R}} \otimes_{\widehat{R}} C. \end{array}$$

The top horizontal map exists by (3.2.2) as $K^{\widehat{R}} \otimes_R B_{\leq m}$ is $K^{\widehat{R}}$ -semifree, see [6, thm. 8.5.1, 8.6.1]. To locate a morphism ρ making the diagram commute, consider the triangle

$$K^{\widehat{R}} \otimes_R B_{\leq m} \longrightarrow K^{\widehat{R}} \otimes_R B \longrightarrow K^{\widehat{R}} \otimes_R B_{\geq m+1} \longrightarrow \Sigma K^{\widehat{R}} \otimes_R B_{\leq m}$$

and apply the functor $\mathrm{Mor}_{\mathrm{D}(K^{\widehat{R}})}(-, K^{\widehat{R}} \otimes_{\widehat{R}} C) \cong \mathrm{H}_0(\mathbf{R}\mathrm{Hom}_{K^{\widehat{R}}}(-, K^{\widehat{R}} \otimes_{\widehat{R}} C))$ to obtain the exact sequence of abelian groups

$$\begin{aligned} \mathrm{Mor}_{\mathrm{D}(K^{\widehat{R}})}(K^{\widehat{R}} \otimes_R B, K^{\widehat{R}} \otimes_{\widehat{R}} C) &\xrightarrow{\zeta} \mathrm{Mor}_{\mathrm{D}(K^{\widehat{R}})}(K^{\widehat{R}} \otimes_R B_{\leq m}, K^{\widehat{R}} \otimes_{\widehat{R}} C) \\ &\longrightarrow \mathrm{Mor}_{\mathrm{D}(K^{\widehat{R}})}(\Sigma^{-1} K^{\widehat{R}} \otimes_R B_{\geq m+1}, K^{\widehat{R}} \otimes_{\widehat{R}} C). \end{aligned}$$

Note that $\mathrm{Mor}_{\mathrm{D}(K^{\widehat{R}})}(\Sigma^{-1} K^{\widehat{R}} \otimes_R B_{\geq m+1}, K^{\widehat{R}} \otimes_{\widehat{R}} C) = 0$ because

$$\inf(\Sigma^{-1} K^{\widehat{R}} \otimes_R B_{\geq m+1}) \geq m \quad \text{and} \quad \sup(K^{\widehat{R}} \otimes_{\widehat{R}} C) \leq s + e < m.$$

Now ρ can be chosen as any preimage of $(K^{\widehat{R}} \otimes_R \gamma_m^C) \circ \alpha$ under ζ . The map ρ is a quasiisomorphism, as the vertical maps in (3.2.4) induce isomorphisms on homology in degrees less than m and $\mathrm{H}_i(K^{\widehat{R}} \otimes_R B) = 0 = \mathrm{H}_i(K^{\widehat{R}} \otimes_{\widehat{R}} C)$ for $i \geq m$. \square

(3.3) **Remark.** By the existence of minimal semifree resolutions [6, thm. 12.5.2] and the equality of infima (2.1.1), it follows that the complex B in Theorem (1.3) can be chosen minimal and with $B_n = 0$ for $n < \inf C$.

4. SEMIDUALIZING OBJECTS

We open this section by recalling a few definitions and facts from [10]

(4.1) **Semidualizing complexes.** A homologically finite R -complex C is *semi-dualizing* if the homothety morphism $\chi_C^R: R \rightarrow \mathbf{RHom}_R(C, C)$ is an isomorphism in $D(R)$. The set of shift-isomorphism classes in $D(R)$ of semidualizing R -complexes is denoted $\mathfrak{S}(R)$.

An R -complex D is *dualizing* if it is semidualizing and isomorphic in $D(R)$ to a bounded complex of injective modules; see [18, V.§2].

Let C be a semidualizing R -complex. A homologically finite R -complex X is *C -reflexive* if $\mathbf{RHom}_R(X, C)$ is homologically bounded and the biduality morphism $\delta_X^C: X \rightarrow \mathbf{RHom}_R(\mathbf{RHom}_R(X, C), C)$ is an isomorphism in $D(R)$.

(4.2) **Ascent and descent.** Let $R \rightarrow S$ be a local ring homomorphism of finite flat dimension, and let C, C' be degreewise homologically finite R -complexes. The complex $S \otimes_R^{\mathbf{L}} C$ is S -semidualizing if and only if C is R -semidualizing. Furthermore, if C and C' are semidualizing R -complexes such that $S \otimes_R^{\mathbf{L}} C \simeq S \otimes_R^{\mathbf{L}} C'$ in $D(S)$, then $C \simeq C'$ in $D(R)$; see [13, thm. 4.5 and 4.9]. In summary, there is an injective map $\mathfrak{S}(R) \rightarrow \mathfrak{S}(S)$ given by the functor $S \otimes_R^{\mathbf{L}} -$.

When $R \rightarrow S$ is flat with Gorenstein closed fiber $S/\mathfrak{m}S$, the complex C is R -dualizing if and only if $S \otimes_R C$ is S -dualizing; see [4, thm. (5.1)].

A crucial step towards Theorem (1.1) is to show that a semidualizing \widehat{R} -complex C is extended from R if the DG $K^{\widehat{R}}$ -module $K^{\widehat{R}} \otimes_R C$ is so; see the proof of Theorem (5.2). Semidualizing DG K^R -modules provide a vehicle for that argument.

(4.3) **Definition.** A *semidualizing DG K^R -module* is a homologically finite DG K^R -module M such that the homothety morphism $\chi_M^{K^R}: K^R \rightarrow \mathbf{RHom}_{K^R}(M, M)$ is an isomorphism in $D(K^R)$. The set of shift-isomorphism classes in $D(K^R)$ of semidualizing DG K^R -modules is denoted $\mathfrak{S}(K^R)$.

It is not our goal to study semidualizing DG modules for their own sake but as a tool to be used in the proof of Theorem (5.2). To this end, the next two lemmas establish the existence of an injective map $\mathfrak{S}(R) \rightarrow \mathfrak{S}(K^R)$ as in (4.2).

(4.4) **Lemma.** *Let C be a degreewise homologically finite R -complex. The DG K^R -module $K^R \otimes_R C$ is K^R -semidualizing if and only if C is R -semidualizing.*

Proof. Set $K = K^R$ and recall from (2.2) that $K \otimes_R C$ is homologically finite over K if and only if C is homologically finite over R . In the commutative diagram

$$\begin{array}{ccc} K \otimes_R R & \xrightarrow{\cong} & K \xrightarrow{\chi_{K \otimes_R C}^K} \mathbf{RHom}_K(K \otimes_R C, K \otimes_R C) \\ \downarrow K \otimes_R \chi_C^R & & \downarrow \simeq \\ K \otimes_R \mathbf{RHom}_R(C, C) & \xrightarrow{\cong} & \mathbf{RHom}_R(C, K \otimes_R C) \end{array}$$

the right-hand vertical isomorphism is by adjointness, and the lower horizontal one is tensor-evaluation (2.1.2). This shows that $\chi_{K \otimes_R C}^K$ is an isomorphism in $D(K)$ if and only if $K \otimes_R \chi_C^R$ is so. The latter is tantamount to χ_C^R being an isomorphism in $D(R)$; to see this apply (2.1.1) to $\text{Cone}(K \otimes_R \chi_C^R) \cong K \otimes_R \text{Cone} \chi_C^R$. \square

(4.5) **Lemma.** *Let C and C' be semidualizing R -complexes. If there is an isomorphism $K^R \otimes_R C \simeq K^R \otimes_R C'$ in $D(K^R)$, then $C \simeq C'$ in $D(R)$.*

Proof. Set $K = K^R$ and assume $K \otimes_R C \simeq K \otimes_R C'$ in $D(K)$. The first isomorphism in the following sequence is the homothety morphism

$$(1) \quad K \xrightarrow{\simeq} \mathbf{RHom}_K(K \otimes_R C', K \otimes_R C) \xleftarrow{\simeq} K \otimes_R \mathbf{RHom}_R(C', C)$$

while the second is the composition of adjunction and tensor-evaluation (2.1.2). It follows that $K \otimes_R \mathbf{RHom}_R(C', C)$ is homologically bounded, and hence so is $\mathbf{RHom}_R(C', C)$ by (2.1.1). In the commutative diagram

$$\begin{array}{ccc} K \otimes_R C' & \xrightarrow[\simeq]{\delta_{K \otimes_R C'}^{K \otimes_R C}} & \mathbf{RHom}_K(\mathbf{RHom}_K(K \otimes_R C', K \otimes_R C), K \otimes_R C) \\ \downarrow K \otimes_R \delta_{C'}^C & & \downarrow \simeq \\ K \otimes_R \mathbf{RHom}_R(\mathbf{RHom}_R(C', C), C) & \xrightarrow{\simeq} & \mathbf{RHom}_K(K \otimes_R \mathbf{RHom}_R(C', C), K \otimes_R C) \end{array}$$

the right-hand vertical arrow is induced by the second arrow in (1); the lower horizontal arrow arises similarly by composition of adjointness and tensor-evaluation. The diagram shows that $K \otimes_R \delta_{C'}^C$ is an isomorphism in $D(K)$. An application of (2.1.1) to $\text{Cone}(K \otimes_R \delta_{C'}^C) \cong K \otimes_R \text{Cone} \delta_{C'}^C$, implies that $\delta_{C'}^C$ is an isomorphism in $D(R)$. This proves that C' is C -reflexive. Symmetrically, C is C' -reflexive, and by [2, thm. 5.3] and (2.1.1) it then follows that $C \simeq C'$. \square

(4.6) **Remark.** We close this section by comparing our definition (4.3) of semidualizing DG K^R -modules with two notions of dualizing DG K^R -modules from the literature [12, 21]. A straightforward comparison of definitions shows that a DG K^R -module that is dualizing according to [12, def. 1.8] is semidualizing. The comparison with [21] merits a brief discussion.

Let Ω be a dualizing DG K^R -module in the sense of [21, def. (2.3.1)]. It follows from (2.3.2) that $K^{\widehat{R}} \otimes_{K^R} \Omega$ is dualizing for $K^{\widehat{R}}$, still in the sense of [op. cit.]. Let D be a dualizing complex for \widehat{R} ; by [21, prop. (2.1.1)] the DG $K^{\widehat{R}}$ -modules $K^{\widehat{R}} \otimes_{\widehat{R}} D$ and $K^{\widehat{R}} \otimes_{K^R} \Omega$ are shift-isomorphic. Since D is semidualizing for \widehat{R} , it now follows by Lemma (4.4) that $K^{\widehat{R}} \otimes_{K^R} \Omega$ is semidualizing for $K^{\widehat{R}}$. Because $K^{\widehat{R}} \otimes_{K^R} -$ is an equivalence between $D(K^R)$ and $D(K^{\widehat{R}})$, the module Ω is semidualizing for K^R .

5. PROOF AND CONSEQUENCES OF THEOREM (1.1)

As noted in the introduction, the map $\mathfrak{S}(R) \rightarrow \mathfrak{S}(\widehat{R})$ defined in (4.2) need not be surjective. Theorem (1.1) says that it is bijective when R has the approximation property. Before proceeding with the proof of (1.1) we supply an example to show how badly surjectivity can fail in absence of the approximation property.

(5.1) **Example.** Let S_0 be a field of characteristic zero. For $n > 0$ let $[V_{nij}]$ be a 2×3 matrix of indeterminants and consider the complete normal Cohen–Macaulay local domain

$$S_n = S_{n-1}[[V_{nij}]]/I_2(V_{nij}).$$

By [30, cor. 4.8(b)] there are exactly 2^n distinct shift-isomorphism classes of semidualizing complexes in $D(S_n)$; by [10, cor. (3.7)] each class contains a module.

For each $n > 0$ [20, thm. 8] guarantees the existence of a Cohen–Macaulay local unique factorization domain R_n such that $S_n \cong \widehat{R}_n$. By [30, prop. 3.4] each ring R_n has only one semidualizing complex up to shift-isomorphism. Hence, the map $\mathfrak{S}(R_n) \rightarrow \mathfrak{S}(\widehat{R}_n)$ has trivial image.

In view of Theorem (1.3), the descent theorem (1.1) stated in the introduction is a special case of (5.2) below. The example above shows that the assumption in (5.2) is not vacuous.

(5.2) Theorem. *Let R be a local ring. Assume that for every homologically finite \widehat{R} -complex C , there is a homologically finite R -complex B such that $K^{\widehat{R}} \otimes_R B \simeq K^{\widehat{R}} \otimes_{\widehat{R}} C$ in $\mathrm{D}(K^{\widehat{R}})$. The functor $\widehat{R} \otimes_R -$ gives a bijective correspondence between the sets of (shift-)isomorphism classes of semidualizing complexes in the derived categories $\mathrm{D}(R)$ and $\mathrm{D}(\widehat{R})$.*

Proof. By (4.2) it suffices to show that every \widehat{R} -semidualizing complex is isomorphic in $\mathrm{D}(\widehat{R})$ to a complex of the form $\widehat{R} \otimes_R B$ for some homologically finite R -complex B .

Let C be a semidualizing \widehat{R} -complex. Since C is homologically finite, there exists by assumption a complex B of finitely generated free R -modules such that $K^{\widehat{R}} \otimes_R B \simeq K^{\widehat{R}} \otimes_{\widehat{R}} C$ in $\mathrm{D}(K^{\widehat{R}})$. To see that this implies an isomorphism $\widehat{R} \otimes_R B \simeq C$ in $\mathrm{D}(\widehat{R})$, note that there are isomorphisms

$$K^{\widehat{R}} \otimes_{\widehat{R}} (\widehat{R} \otimes_R B) \simeq K^{\widehat{R}} \otimes_R B \simeq K^{\widehat{R}} \otimes_{\widehat{R}} C.$$

Since $K^{\widehat{R}} \otimes_{\widehat{R}} C$ is $K^{\widehat{R}}$ -semidualizing, Lemma (4.4) implies that $\widehat{R} \otimes_R B$ is \widehat{R} -semidualizing. The isomorphisms above and Lemma (4.5) yield $\widehat{R} \otimes_R B \simeq C$. \square

(5.3) Remark. Assume that R has the approximation property. Theorem (1.1) says that the map $\mathfrak{S}(R) \rightarrow \mathfrak{S}(\widehat{R})$, see (4.2), is bijective. Here we mention a few immediate consequences. Analogous statements hold for Corollaries (5.5) and (5.8)(a).

Faithful flatness of \widehat{R} yields the equality $\mathrm{amp} B = \mathrm{amp} \widehat{R} \otimes_R B$, which shows that the map $\mathfrak{S}(R) \rightarrow \mathfrak{S}(\widehat{R})$ restricts to a bijection between the sets of isomorphism classes of semidualizing modules over R and \widehat{R} .

One can employ reflexivity to endow the sets $\mathfrak{S}(R)$ and $\mathfrak{S}(\widehat{R})$ with additional structure. For instance, in [14, 1.7 and thm. 2.9] it is shown that each of these sets admits an ordering and a metric. From [13, thm. 4.8] and [14, prop. 4.2] it follows that the map $\mathfrak{S}(R) \rightarrow \mathfrak{S}(\widehat{R})$ is an isomorphism of ordered sets and an isometry.

The immediate corollary below recovers [21, thm. (4.1)]; see also [29, thm. 3.1].

(5.4) Corollary. *Any local ring with the approximation property admits a dualizing complex.* \square

We next extend part of [14, prop. 4.2].

(5.5) Corollary. *Let R be a local ring with the approximation property and $\mathbf{x} \in \mathfrak{m}$ an R -sequence. There is a bijective correspondence, given by the functor $R/(\mathbf{x}) \otimes_R^{\mathbf{L}} -$, between the sets of (shift-)isomorphism classes of semidualizing complexes in $\mathrm{D}(R)$ and $\mathrm{D}(R/(\mathbf{x}))$.*

Proof. In the following display, the first commutative diagram consists of natural local ring homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & \widehat{R} \\ \downarrow & & \downarrow \\ R/(\mathbf{x}) & \longrightarrow & \widehat{R}/(\mathbf{x}) \end{array} \qquad \begin{array}{ccc} \mathfrak{S}(R) & \xrightarrow{(1)} & \mathfrak{S}(\widehat{R}) \\ \downarrow (3) & & \downarrow (2) \\ \mathfrak{S}(R/(\mathbf{x})) & \xrightarrow{(4)} & \mathfrak{S}(\widehat{R}/(\mathbf{x})) \end{array}$$

while the second commutative diagram is induced from the first using (4.2). The maps (1) and (2) are bijective by Theorem (1.1) and [14, prop. 4.2], respectively. The maps (3) and (4) are injective by [13, thm. 4.9], and a diagram-chase shows that (3) and (4) are bijective. \square

(5.6) **Henselization.** A pointed étale neighborhood of R is a flat local homomorphism of the form $R \rightarrow R' = R[X]_{\mathfrak{n}}/(f)$, where f is a monic polynomial whose derivative is a unit in $R[X]_{\mathfrak{n}}$, and \mathfrak{n} is a prime ideal lying over \mathfrak{m} . The set of pointed étale neighborhoods of R forms a filtered direct system $\{R_{\lambda} \mid \lambda \in \Lambda\}$, and the Henselization of R is the limit $R^h = \varinjlim_{\lambda} R_{\lambda}$.

(5.7) **Remark.** By unpublished examples of Nishimura an excellent local ring need not possess a dualizing complex [25, exa. 6.1], and a ring with a dualizing complex need not be excellent [25, exa. 4.2]. The next corollary shows that when R is excellent, one need not go all the way to the completion to find a ring with a dualizing complex.

(5.8) **Corollary.** *Let R be an excellent local ring.*

- (a) *For every semidualizing \widehat{R} -complex C there exists a semidualizing R^h -complex B such that $C \simeq \widehat{R} \otimes_{R^h} B$.*
- (b) *For each list of semidualizing \widehat{R} -complexes C^1, \dots, C^n there exists a pointed étale neighborhood $R \rightarrow R'$ and semidualizing R' -complexes B^1, \dots, B^n such that each $C^i \simeq \widehat{R} \otimes_{R'} B^i$. In particular, there exists a pointed étale neighborhood $R \rightarrow R'$ such that R' admits a dualizing complex.*

Proof. (a) Because R is excellent, the Henselization R^h is excellent as well by [16, cor. (18.7.6)], and hence R^h has the approximation property; see (2.4). The conclusion now follows from Theorem (1.1).

(b) By (4.2) it suffices to verify the first statement. For each C^i adopt the conventions from the proof of (5.2). Apply the arguments in (3.1.2)–(3.1.9) to $C^i_{\leq m_i}$ in order to produce matrix equations (3.1.5.1), (3.1.7.1), (3.1.7.2), and (3.1.9.1) with coefficients in R . As commented in (3.1.10) the construction of this system does not depend on R having the approximation property. The system has a solution in R^h , because R^h has the approximation property. Therefore, there exists a pointed étale neighborhood $R \rightarrow R_{\lambda_i}$ such that R_{λ_i} contains the solution from R^h . Apply the construction in (3.1.11) and the argument in the proof of (5.2) to obtain a semidualizing R_{λ_i} -complex \widetilde{B}^i such that $C^i \simeq \widehat{R} \otimes_{R_{\lambda_i}} \widetilde{B}^i$.

Choose a pointed étale neighborhood $R \rightarrow R' = R_{\lambda}$ where $\lambda \geq \lambda_1, \dots, \lambda_n$. Setting $B^i = R' \otimes_{R_{\lambda_i}} \widetilde{B}^i$ the isomorphisms

$$C^i \simeq \widehat{R} \otimes_{R_{\lambda_i}} \widetilde{B}^i \cong \widehat{R} \otimes_{R'} B^i$$

show that R' and B^1, \dots, B^n have the desired properties, cf. (4.2). \square

We close this section with an application of Corollary (5.8)(b). When R is a ring with a dualizing complex, the conclusion of the next result holds by [27, thm. 1]. In view of Remark (5.7) the hypothesis “excellent” is neither stronger nor weaker.

(5.9) **Corollary.** *Let R be an excellent local ring of Krull dimension d . There exists a chain of ideals $\mathfrak{b}_d \subseteq \dots \subseteq \mathfrak{b}_1 \subseteq \mathfrak{b}_0$ satisfying the following conditions:*

- (a) *For each $i = 0, \dots, d$ there is an inequality $\dim R/\mathfrak{b}_i \leq i$.*
- (b) *If $F = 0 \rightarrow F_r \rightarrow \dots \rightarrow F_0 \rightarrow 0$ is a complex of finitely generated free R -modules with $\text{length}_R H(F) < \infty$, then \mathfrak{b}_i annihilates $H_j(F)$ for each $j \geq r - i$.*

Proof. By Corollary (5.8)(b) there exists a pointed étale neighborhood $R \rightarrow R'$ such that R' admits a dualizing complex. Note that $\dim R' = d$. By [27, thm. 1 and preceding prop. and def.] there exists a chain of ideals $\mathfrak{b}'_d \subseteq \dots \subseteq \mathfrak{b}'_1 \subseteq \mathfrak{b}'_0$ in R' such that: (a') $\dim R'/\mathfrak{b}'_i \leq i$, and (b') if $F' = 0 \rightarrow F'_r \rightarrow \dots \rightarrow F'_0 \rightarrow 0$ is a complex of finitely generated free R' -modules with $\text{length}_{R'} H(F') < \infty$, then $\mathfrak{b}'_i H_j(F') = 0$ for $j \geq r - i$.

For each $i = 0, \dots, d$ set $\mathfrak{b}_i = \mathfrak{b}'_i \cap R$. It is not difficult to verify that $\dim R/\mathfrak{b}_i = \dim R'/\mathfrak{b}'_i \leq i$. Let F be a complex satisfying the hypothesis of (b). The complex $F' = R' \otimes_R F$ satisfies the hypothesis of (b'). Indeed, the isomorphisms $R'/\mathfrak{m}R' \cong k$ and $H_j(F') \cong H_j(F) \otimes_R R'$ guarantee $\text{length}_{R'} H(F') < \infty$. The ideal \mathfrak{b}'_i annihilates $H_j(F')$ for $j \geq r - i$ and contains \mathfrak{b}_i ; hence the R -isomorphisms $H_j(F') \cong H_j(F)$ implies that \mathfrak{b}_i annihilates $H_j(F)$ for $j \geq r - i$. \square

A classical application of Robert’s theorem [27, thm. 1] is to find uniform annihilators of local cohomology modules $H_{\mathfrak{m}}^j(R)$. For instance, Hochster and Huneke [22, 23] do this when R is an equidimensional local ring that admits a dualizing complex or is unmixed and excellent. Corollary (5.9) allows us to drop the unmixedness condition, thus recovering a recent result of Zhou [32, cor. 3.3(ii)].

(5.10) **Observation.** Let R be an equidimensional excellent local ring of Krull dimension $d > 1$. For each j the local cohomology module $H_{\mathfrak{m}}^j(R)$ is a direct limit of homology modules in degree $d - j$ of Koszul complexes on powers of a system of parameters. Thus, for each $i = 0, \dots, d$ the ideal \mathfrak{b}_i annihilates $H_{\mathfrak{m}}^j(R)$ for $j \leq i$. In particular, \mathfrak{b}_{d-1} annihilates $H_{\mathfrak{m}}^j(R)$ for $j < d$ and $\dim R/\mathfrak{b}_{d-1} \leq d - 1$. If R is equidimensional, then \mathfrak{b}_{d-1} is not contained in any minimal prime of R . In the terminology of [32], this means that \mathfrak{b}_{d-1} contains a uniform cohomological annihilator for R .

6. FINITENESS OF $\mathfrak{S}(R)$ FOR CERTAIN COHEN–MACAULAY RINGS

When R is excellent and the set $\mathfrak{S}(\widehat{R})$ is finite, Corollary (5.8)(b) implies that there exists a pointed étale neighborhood $R \rightarrow R'$ such that the map $\mathfrak{S}(R') \rightarrow \mathfrak{S}(\widehat{R})$ is bijective. The question of the finiteness of $\mathfrak{S}(R)$ has been answered affirmatively for certain classes of rings, see [30], but remains open in general. In this section we verify finiteness for a large class of Cohen–Macaulay local rings. Without further comment we use the fact that, when R is Cohen–Macaulay, every semidualizing R -complex is shift-isomorphic in $D(R)$ to a module; see [10, cor. (3.7)].

(6.1) **Lemma.** *If R is an equicharacteristic artinian local ring, then $\mathfrak{S}(R)$ is finite; i.e., there are only finitely many semidualizing R -modules up to isomorphism.*

Proof. There exists a flat map of artinian local rings $(R, \mathfrak{m}, k) \rightarrow (R', \mathfrak{m}', k')$, such that k' is algebraically closed; see [15, prop. 0.(10.3.1)]. By (4.2) we may replace R by R' to assume that k is algebraically closed.

For an R -module N , we denote the minimal number of generators by $\beta_0^R(N)$. Let C be a semidualizing R -module. Let $E_R(k)$ be the injective hull of k . Hom-evaluation [9, prop. 5.3] and the semidualizing property yield an isomorphism $E_R(k) \cong C \otimes_R \text{Hom}_R(C, E_R(k))$. Hence, there is an inequality $\beta_0^R(C) \leq \beta_0^R(E_R(k))$. This yields the second inequality below; the first is from a surjection $R^{\beta_0^R(C)} \rightarrow C$.

$$(1) \quad \text{length}_R C \leq \beta_0^R(C) \cdot \text{length } R \leq \beta_0^R(E_R(k)) \cdot \text{length } R$$

By [17, proof of first prop. in sec. 3] there are only finitely many isomorphism classes of R -modules M with $\text{Ext}_R^{i>0}(M, M) = 0$ and $\text{length}_R M \leq \beta_0^R(E_R(k)) \cdot \text{length } R$. From (1) it follows that $\mathfrak{S}(R)$ is a finite set. \square

The next result contains Theorem (1.2) from the introduction.

(6.2) **Theorem.** *Assume there are local ring homomorphisms*

$$R \xrightarrow{\varphi} R' \xleftarrow{\psi} Q \xrightarrow{\tau} Q'$$

such that Q' is Cohen–Macaulay and equicharacteristic, the maps φ and τ have finite flat dimension, and the map ψ is surjective with kernel generated by a Q -sequence. Then $\mathfrak{S}(R)$ is finite; that is, there are only finitely many semidualizing R -modules up to isomorphism. In particular, $\mathfrak{S}(R)$ is finite when R is Cohen–Macaulay and equicharacteristic.

Proof. In view of (4.2) we may, without loss of generality, assume that Q is complete. The assumptions imply that Q , R' , and R are Cohen–Macaulay, see [7, thm. (2.1)]. By factoring out a system of parameters, we may assume that Q' is artinian. By Lemma (6.1) the set $\mathfrak{S}(Q')$ is finite, and by (4.2) so is $\mathfrak{S}(Q)$. By [13, prop. 4.2] also $\mathfrak{S}(R')$ is finite, and finally $\mathfrak{S}(R)$ is finite by (4.2). \square

Theorem (6.2) applies to certain rings of mixed characteristic.

(6.3) **Example.** Set $Q = \widehat{\mathbb{Z}_{(p)}}[[X]]/I$ where X is a finite set of variables, and I is an ideal such that Q is Cohen–Macaulay and p is Q -regular. Set $Q' = Q/(p)$ and $R' = Q/(p^n)$ for some $n \geq 2$. If $\widehat{R} \cong R'$, then $\mathfrak{S}(R)$ is finite by Theorem (6.2).

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