SAMPLING ALGEBRA STRUCTURES ON MINIMAL FREE RESOLUTIONS OF LENGTH 3

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Abstract. Ideals in the ring of power series in three variables over a field can be classified based on algebra structures on their minimal free resolutions. The classification is incomplete in the sense that it remains open which algebra structures actually occur; this *realizability question* was formally raised by Avramov in 2012. We discuss the outcomes of an experiment performed to shed light on Avramov's question: Using the computer algebra system *Macaulay2*, we classify a billion randomly generated ideals and build a database with examples of ideals of all classes realized in the experiment. Based on the outcomes, we discuss the status of conjectures that relate to the realizability question.

Introduction

Let R be a local ring with residue field $\Bbbk$ and $I \subset R$ a perfect ideal of grade 3. By a result of Buchsbaum and Eisenbud [3, Proposition 1.1], the minimal free resolution $F_\bullet$ of R/I over R has a differential graded (DG) algebra structure. This induces a graded $\Bbbk$-algebra structure on $\mathrm{H}(F_\bullet \otimes_R \Bbbk) = \mathrm{Tor}^R_\bullet(R/I, \Bbbk)$, and while the DG algeba structure on $F_\bullet$ is not unique, the induced $\Bbbk$-algebra structure on $\mathrm{Tor}^R_\bullet(R/I, \Bbbk)$ is unique. Results of Weyman [11, Theorem 4.1] and of Avramov, Kustin, and Miller [2, Theorem 2.1] show that this structure supports a classification scheme for grade 3 perfect ideals in R. The original application of the classification scheme was to answer a question in local algebra—on the rationality of Poincaré series—and for this it was sufficient to establish the possible structures without considering their realizability. Later, however, Avramov returned to the classification to resolve another question in local algebra—on growth patterns in minimal injective resolutions—and found it necessary to rule out the realizability of certain structures. Ideally, a classification should say exactly which structures occur, and Avramov formally stated that question in [1, Question 3.8].

Since [1], various authors have addressed what has become known as the *realizability question* in essentially two different ways: Some have ruled out the realizability of certain classes while others have provided constructions of ideals in certain, other, classes. In this paper the approach is experimental: Within certain bounds we generated random grade 3 perfect ideals and classified them. Then we compare our observations to existing bounds, both established and conjectured. In addition to the results and analysis in this paper, we provide a *GitHub* repository with the simplest examples of ideals from each class that was observed in the experiment.

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It turns out that many classes can be realized by binomial ideals and some even by monomial ideals.

* * *

Set $A_\bullet = \operatorname{Tor}_\bullet^R(R/I, \Bbbk)$. The size of the algebra $A_\bullet$ is determined by two parameters, m and n, where m is the minimal number of generators of the ideal I and n is known as the *type* of the ring R/I. It is proved in [2, Theorem 2.1] that there exists bases

$$\{\mathsf{e}_i\}_{i=1,\dots,m}, \quad \{\mathsf{f}_i\}_{i=1,\dots,m+n-1}, \quad \text{and} \quad \{\mathsf{g}_i\}_{i=1,\dots,n}$$

for A_1, A_2 and A_3, respectively, such that the multiplication on $A_\bullet$ is one of the following:

$\mathbf{C}(3)$	$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3$, $\mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1$, $\mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2$	$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1$	for $1 \le i \le 3$
$\mathbf{T}$	$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3$, $\mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1$, $\mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2$		
$\mathbf{B}$	$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3$	$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1$	for $1 \le i \le 2$
$\mathbf{G}(r)$		$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1$	for $1 \le i \le r$
$\mathbf{H}(p,q)$	$\mathsf{e}_i\mathsf{e}_{p+1} = \mathsf{f}_i$ for $1 \le i \le p$	$\mathsf{e}_{p+1}\mathsf{f}_{p+j} = \mathsf{g}_j$	for $1 \le j \le q$,

The products not listed are either zero or can be deduced from the ones listed by graded-commutativity. Set

$$m = \operatorname{rank} A_1, \quad n = \operatorname{rank} A_3, \quad p = \operatorname{rank} A_1A_1, \quad q = \operatorname{rank} A_1A_2 \text{ and } r = \operatorname{rank} \delta$$

where δ is the canonical homomorphism $A_2 \to \operatorname{Hom}_\Bbbk(A_1, A_3)$ that maps an element a to the multiplication map $A_1 \xrightarrow{a\cdot} A_3$.

The realizability question for grade 3 perfect ideals can now be phrased as follows: Given m and n, which of the classes above can be realized? For small values of m and n the answer is known: Ideals of class $\mathbf{C}(3)$, i.e. grade 3 complete intersection ideals, only exist for $(m,n) = (3,1)$. For (m,n) with $m \le 4$ or $n = 1$ there is at most one permissible class and every permissible class can be realized; see Avramov [1, 1.4.2, 3.4.1(a), 3.4.2, 3.9.1]. For this reason, we only recorded ideals with $m \ge 5$ and $n \ge 2$. For practical reasons, we also only recorded ideals with $m \le 12$ and $n \le 10$. The results of our experiment indicate that there are no further restrictions on the realizability of classes $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{T}$ beyond those proved in [1, Theorem 3.1] and [6, Theorem 1.1], but they do not rule out that there may be restrictions on the existence of class $\mathbf{G}$ ideals beyond what is known or conjectured in the literature.

1. Materials and Methods

To shed light on the realizability question, we developed an algorithm that randomly generates homogeneous grade 3 perfect ideals in the trivariate polynomial ring $\Bbbk[x,y,z]$ and classifies the corresponding ideal in the local ring $\Bbbk[\![x,y,z]\!]$ using the *Macaulay2* package *TorAlgebra* [4, 5]. To be precise, given an ideal $I = (f_1, \dots, f_m)$ in the polynomial algebra, the quotient $R = \Bbbk[x,y,z]/I$ is an Artinian local ring isomorphic to $\Bbbk[\![x,y,z]\!]/I'$, where I' denotes the codepth 3 perfect ideal of the power series algebra $\Bbbk[\![x,y,z]\!]$ generated by $f_1, \dots, f_m$. The multiplicative structure on the free resolution of the quotient ring R is encoded in homological invariants of that ring. As the classification algorithm from [5] implemented in *TorAlgebra* classifies the local ring R based on these intrinsic properties, it is irrelevant that it is obtained as a quotient of the (nonlocal) polynomial algebra rather than the local ring $\Bbbk[\![x, y, z]\!]$. The algorithm maintains a running tally of the classified ideals, along with a running list of the "shortest" minimal generating set observed for ideals of each class. This section describes the algorithm in detail, highlighting the user-readable output on screen and in the recorded data structure.

The entry point into the algorithm is a function called main, which takes one positional parameter and eleven optional parameters. The positional parameter is a positive integer which controls the number of attempts to classify ideals. The optional parameters are detailed below.

Parameter	Data Type	Default Value
fieldChar	0 or a prime integer	3
checkIn	nonnegative integer	0
degSeq	(0) or a sequence of positive integers	(0)
lowDeg	positive integer	2
highDeg	positive integer	8
numTerms	nonnegative integer	0
mn	positive integer	5
useN	boolean	false
maxTries	positive integer	10
strictTerms	boolean	false
maxM	positive integer	12
maxN	positive integer	10
logging	boolean	false

When the main function is called, it executes the following steps in order:

1.1. **Set the Polynomial Ring.** The experiment takes place within an ambient polynomial ring, $R = \Bbbk[x, y, z]$, where $\Bbbk$ is a field. If fieldChar is a prime p, then $\Bbbk$ is set to $\mathbb{Z}/p\mathbb{Z}$. If fieldChar $= 0$, then $\Bbbk$ is set to $\mathbb{Q}$. Otherwise, the function prints the statement

Error: bad field.

1.2. **Load Pre-existing Data.** The function searches the current working directory for a `data` folder from a previous execution. If it finds one, then the function loads data from that folder into memory. Otherwise, the function creates an empty directory, `data`.

1.3. **Print Start Message.** The following message is printed to the terminal:

> Main Routine started at current time with options:
>
> new OptionTable from {maxTries => maxTries, degSeq => degSeq,
>
> strictTerms => strictTerms, logging => logging, mn => mn,
>
> numTerms => numTerms, highDeg => highDeg, useN => useN,
>
> maxM => maxM, maxN => maxN, checkIn => checkIn,
>
> fieldChar => fieldChar, lowDeg => lowDeg}

If logging is true, then the function appends the printed statement to the end of the `log.txt` file.

Steps 1.4 and 1.5 form the loop of the algorithm.
The positional parameter controls the number of repetitions.

1.4. **Generate an Ideal.** The function checks if the repetition counter i is a multiple of checkIn. If so, then the function prints the statement

Checking in every checkIn ideals... done i so far.

If degSeq is the sequence (0), then the function changes it to a sequence of mn randomly generated integers within the interval [lowDeg, highDeg]. Next, a new sequence S of length mn is created, consisting of random homogeneous polynomials of degree d for each element d of degSeq. If numTerms > 0, then these polynomials are constructed so that each has numTerms terms (e.g. numTerms $= 1$ causes the algorithm to produce monomials). Else, i.e. if numTerms $= 0$, then the number of terms in each polynomial is random. These polynomials are then used to generate a homogeneous ideal as follows:

i. If useN is false, then the function creates an ideal I which is generated (possibly non-minimally) by the elements of S. The function checks if I is minimally generated by the mn elements of S, trying up to 10 times to do so. Each time, the function generates an altered ideal by taking the previous generating set and adding a random homogeneous form of degree randomly selected from degSeq. If after 10 tries the function has still not constructed an ideal minimally generated by mn elements, it considers the attempt a failure and restarts Step 1.4. Otherwise:
   - If $\operatorname{codim} I = 3$, then the ideal has been successfully generated, and the function moves on to Step 1.5.
   - If $\operatorname{codim} I < 3$ and numTerms $\neq 1$, then the function adds a pure power of a variable to one or more of the minimal generators to construct a new ideal, keeping the polynomial generators homogeneous and mn as the minimal number of generators. If needed, this is repeated for each variable, always starting with the minimal set of generators. This process stops when an ideal of codimension 3 and mn generators is achieved, or when the set of variables is exhausted. In the latter case, the 0 ideal is returned.
   - If $\operatorname{codim} I < 3$ and numTerms $= 1$, then the function is intended to return a monomial ideal. In this case, the above strategy to fix $\operatorname{codim} I$ is not appropriate. Therefore, the function considers this attempt a failure and restarts Step 1.4.

ii. If useN is true, then the function attempts to create an ideal with quotient of type mn. It applies the Macaulay2 function fromDual to the elements of the sequence S, which returns a set of generators of a homogeneous ideal that defines a quotient ring of type (at most) mn; see for example Meyer and Smith [9, Chapter II.2]. If the type is mn, then the algorithm has succeeded in generating an ideal with the required properties. If not, then the function considers this attempt a failure and restarts Step 1.4.

If the function considers an attempt a failure, then a variable called *numTries* (initially set to 0) is checked against the optional variable maxTries. If *numTries* $<$ maxTries, then the former is incremented before the function restarts Step 1.4. On the other hand, if *numTries* $=$ maxTries, then the function returns the 0 ideal. If the function succeeds in generating an ideal with the required properties, then *numTries* is reset to 0.

If strictTerms is true, then the function checks if the minimal generators of the ideal have the exact number of terms as given by numTerms. In addition, the function verifies that the ideal has codimension 3, that it is homogeneous, that its minimal number of generators does not exceed maxM, that the type of its quotient does not exceed maxN, and that all minimal generators are of degree at least 2. If any of these checks fail, then the ideal is not classified.

1.5. **Classify the Ideal and File Classification Data.** The function classifies the ideal using the TorAlgData command in the *TorAlgebra*[4] package, resulting in a tuple $(m, n, \text{Class}, p, q, r)$. If they do not already exist, then the function creates the following files with the `data` folder: `classDat.txt` and `class.txt`, the former being *Macaulay2* readable and the latter intended to be human readable. If numTerms $= 0$, then the function edits files in the subfolder `data/0`—creating it if necessary. Otherwise, the function computes the maximum number of terms of a minimal generator of the ideal and edits files in a subfolder—creating it if necessary—called `1`, `2`, `3`, or `4` within the `data` folder corresponding to monomial, binomial, trinomial, or generators with four or more terms.

i. If the class has not been seen before, then the function creates a `.txt` file named `m-n-Class-p-q-r` with a *Macaulay2* readable matrix containing the minimal generators of the ideal. For example, the file `5-2-B-1-1-2.txt` in `data/2` could have the contents:

```
matrix{{y*z,x*z,y^2+z^2,x*y+z^2,x^2+z^2}}
```

In the `data` folder, the function adds the class to the `classDat.txt` file, recording the tuple $(m, n, \text{Class}, p, q, r)$, the *Macaulay2* readable matrix containing the minimal generators of the ideal, and a count corresponding to the number of times the class has been observed. In the running example, the corresponding entry in the file `classDat.txt` would be

```
((5,2,B,1,1,2),(matrix{{y*z,x*z,y^2+z^2,x*y+z^2,x^2+z^2}},1))
```

Finally, in the `data` folder, the function adds a row to the `class.txt` file with the same information, replacing the *Macaulay2* readable matrix containing the minimal generators of the ideal with a human readable list. Note that the `class.txt` file is ordered numerically according to the value of m, then ordered numerically according to the value of n, then ordered alphabetically according to Class, then ordered numerically according to the value of p, then q, then r. In the running example, the first entry in the `class.txt` file would be

```
| 5 2 B 1 1 2 1 | yz xz y2+z2 xy+z2 x2+z2 |
```

ii. If the class has been seen before, then the function opens the previously created `m-n-Class-p-q-r.txt` file and compares the length of the minimal generators of the current ideal to the length of the previously recorded minimal generators. Here "length" simply refers to the length of the text string. If the length of the minimal generators of the current ideal is shorter than the length of the previously recorded minimal generators, then a *Macaulay2* readable matrix containing the minimal generators of the current ideal is appended to the end of the file. For example, an updated `5-2-B-1-1-2.txt` file in `data/2` could have the contents:

```
matrix{{y*z,x*z,y^2+z^2,x*y+z^2,x^2+z^2}}
matrix{{z^2,y*z,x*z,x*y,x^2-y^2}}
```

Additionally, the function increases the count of the class in the `classDat.txt` and `class.txt` files by 1 and replaces the previously recorded minimal generators of the ideal with the minimal generators of the current ideal, provided their length is shorter. In the running example, the updated entries in `classDat.txt` and `class.txt` would be

```
((5,2,B,1,1,2),(matrix{{z^2,y*z,x*z,x*y,x^2-y^2}},2))
```

and

```
| 5 2 B 1 1 2 2 | z2 yz xz xy x2-y2 |
```

The function repeats Steps 1.4 and 1.5 according to the positional parameter entered by the user when the main function was called.

1.6. **Print Summary.** The function prints the following information:

Main Routine finished:
at current time
ran for # seconds,
classified # ideals,
generated # distinct classes,
discovered # new classes

If new classes were discovered, then the function also prints a list with entries

$$\{(m, n, \text{Class}, p, q, r), \dots\}$$

If the user set logging as true, then the printed statements above are appended at the end of the `log.txt` file.

2. Results

2.1. **Realized Classes.** The total number of ideals classified in our experiment is just above 10^9. After some initial experimentation with the field characteristic, we chose to work in characteristic 3: For computational economy the characteristic needs to be low, and compared to characteristic 2, ideals generated in characteristic 3 realize a wider variety of classes, presumably because of the existence of a sign. It turns out that, in most cases, a set of homogeneous polynomials with coefficients ± 1 generate ideals of the same class in $\mathbb{Z}_3[x,y,z]$ and $\mathbb{Q}[x,y,z]$. Now this is only in about 85 % of cases; for example, the six polynomials

$$\begin{gathered} xy^2 + y^2z + xz^2 - z^3, \\ x^2z^2 + xyz^2 - y^2z^2 - yz^3 - z^4, \\ y^3z + xyz^2 + y^2z^2 + xz^3 - yz^3, \\ x^2yz + xz^3 + z^4, \\ x^3z + xz^3 + z^4, \text{ and} \\ x^4 + y^4 + y^2z^2 + xz^3 + z^4 \end{gathered}$$

generate an ideal in $\mathbb{Z}_3[x,y,z]$ described by the tuple $(6,4,\mathbf{H},1,2,2)$ but in $\mathbb{Q}[x,y,z]$ an ideal described by $(6,6,\mathbf{H},1,1,1)$.

In the experiment, the optional parameters ranged as follows:

Parameter	Values
fieldChar	3
degSeq	$(2,\dots,2,2),(2,\dots,2,3),\dots,(10,\dots,10,10)$
lowDeg	2–6
highDeg	(lowDeg $+0$)–(lowDeg $+9$)
numTerms	0–12

The variation of the parameters was informed by the outcomes: For example, when a new class was observed, a minimal set of generators for the ideal that realized the class was recorded. These generating sets were analyzed for patterns in their degrees and number of terms, and the parameters were adjusted accordingly with the goal of observing as large a variety of classes as possible. In addition, we switched useN between true and false to observe classes of a specific value of n or m, respectively.

We visualize the observed classes in six tables:

Table 2.1.1	Ideals of class **B**, **G**, and **T**
Table 2.1.2	Monomial ideals of class **B**, **G**, and **T**
Table 2.1.3	Binomial ideals of class **B**, **G**, and **T**
Table 2.1.4	Ideals of class **H**
Table 2.1.5	Monomial ideals of class **H**
Table 2.1.6	Binomial ideals of class **H**

For reasons of space, these tables are limited to the ranges $5 \leq m \leq 9$ and $2 \leq n \leq 9$. Full tables with ranges $5 \leq m \leq 12$ and $2 \leq n \leq 10$ are available online, see 2.3. For a fixed pair (m,n), call the collection of all classes with these m and n values the (m,n)-box. Within each (m,n)-box, the tables have either (p,r)-cells (Tables 2.1.1–2.1.3) or (p,q)-cells (Tables 2.1.4–2.1.6), according to the possible values of p, q, and r, which are known to be bounded by functions of m and n; see [6, Theorem 1.1] for class **H** and [1, Theorem 3.1] for class **G**. Dotted cells represent classes that are known to be unrealizable; they are separated from cells representing permissible classes with a black border. Of the permissible classes, those that have not been observed are represented by white boxes. Classes that have been observed in the experiment have cells colored a shade of gray (Tables 2.1.1 and 2.1.4) or black (Tables 2.1.2, 2.1.3, 2.1.5, and 2.1.6). Tables 2.1.1 and 2.1.4 are colored according to the frequency with which each class was observed in the experiments. The darker the coloring, the more frequently the class was observed. These tables are colored using the same scale.

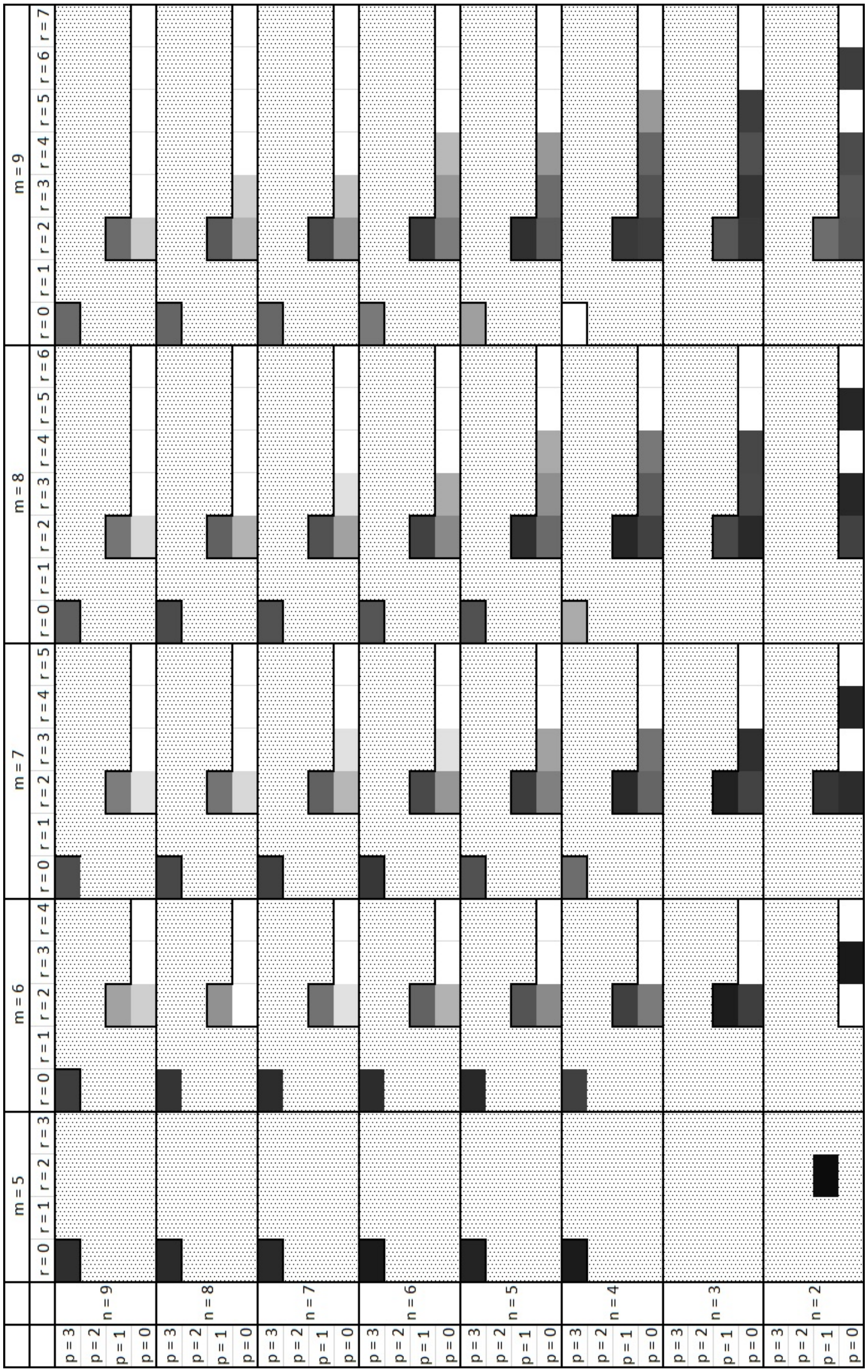

Table 2.1.1: Observed ideals of class **B**, **G**, and **T**.

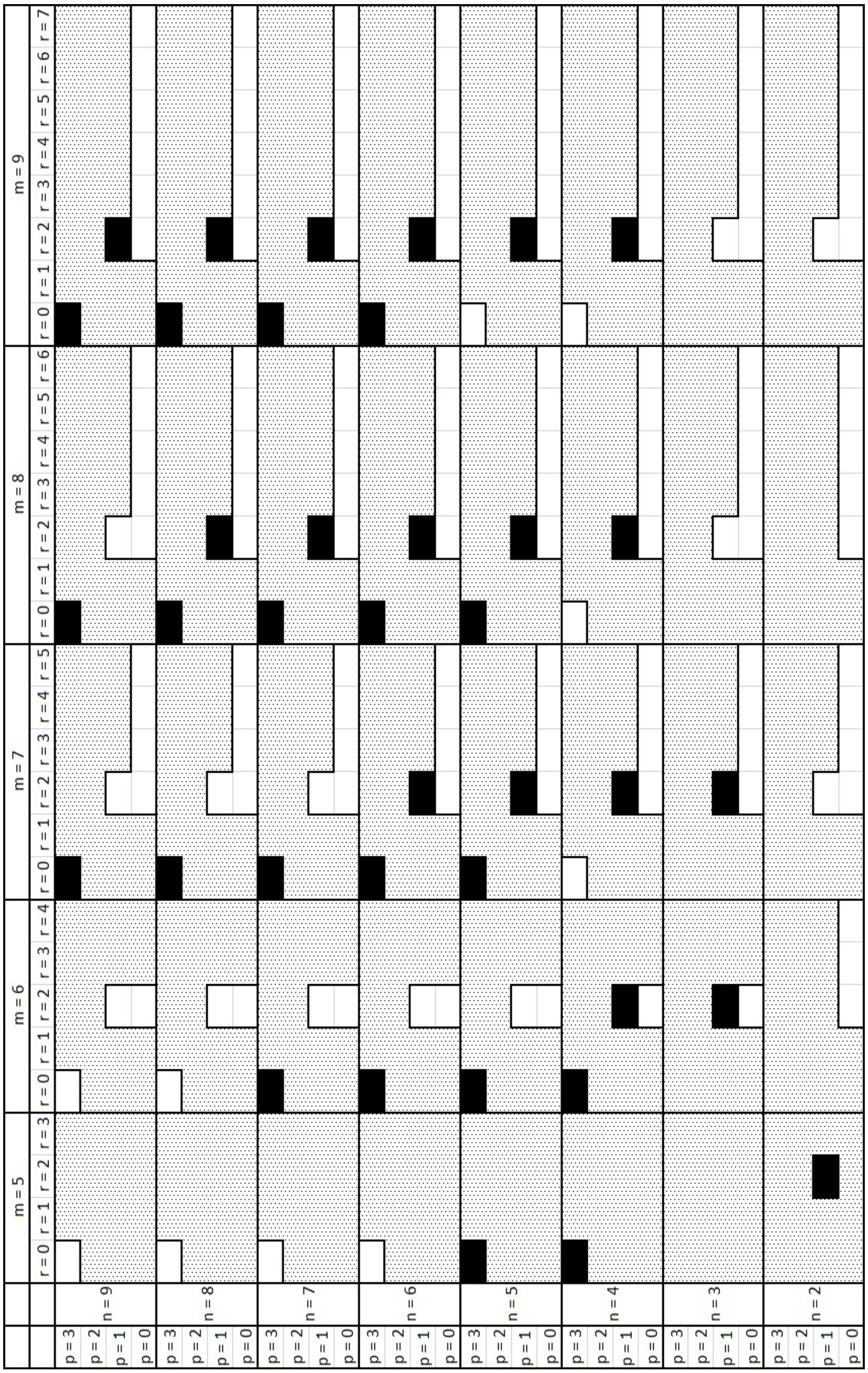

Table 2.1.2: Observed monomial ideals of class **B**, **G**, and **T**.

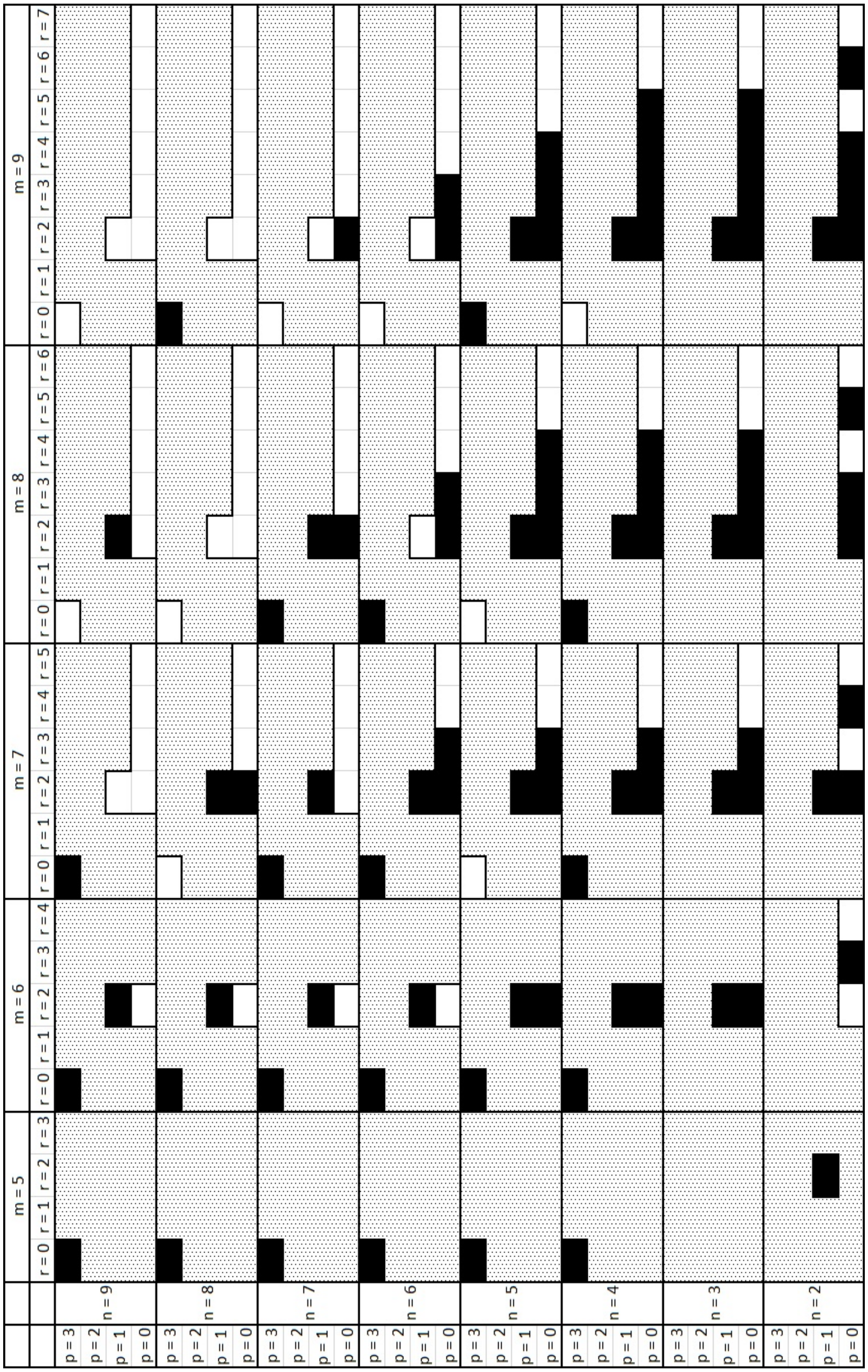

Table 2.1.3: Observed binomial ideals of class **B**, **G**, and **T**.

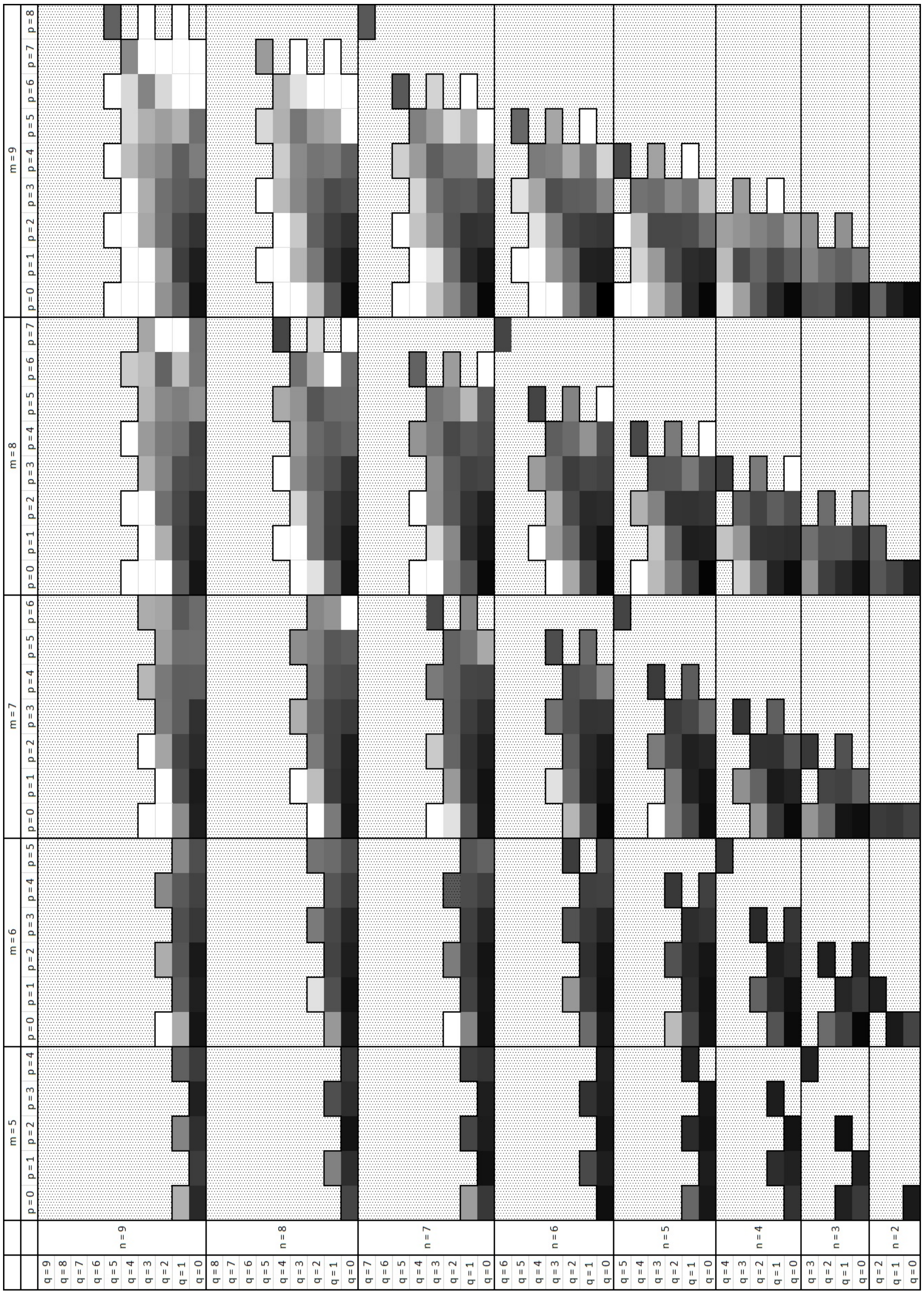

Table 2.1.4: Observed ideals of class **H**.

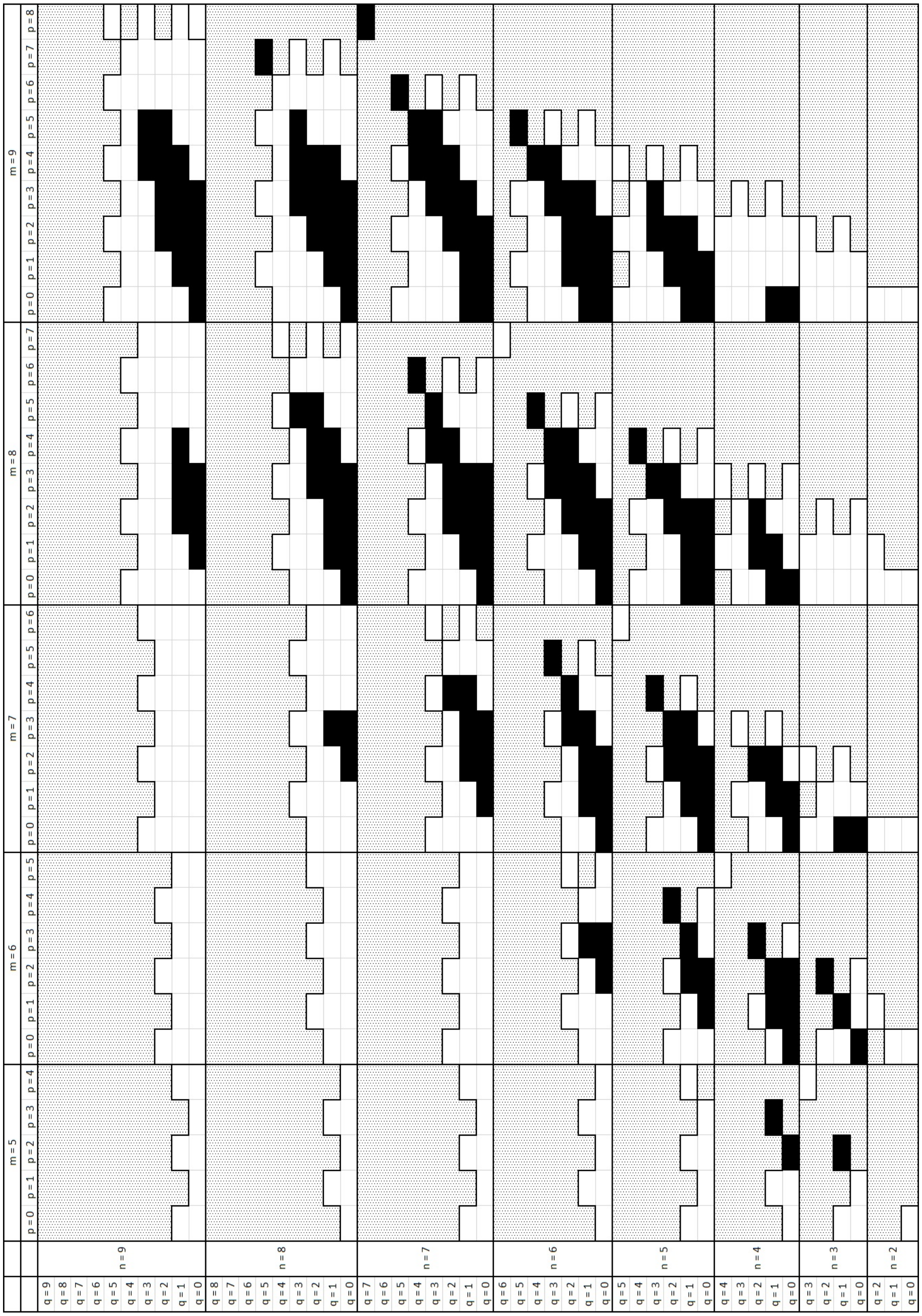

Table 2.1.5: Observed monomial ideals of class **H**.

Table 2.1.6: Observed binomial ideals of class **H**.

2.2. **Predominant Classes.** For given values of m and n, one may ask if it is possible to identify a predominant class of ideals, one that is observed more frequently than others within the experiment. A criterion is needed to determine when a certain class is observed with such prevalence as to deserve comment. In our analysis, this criterion is as follows: in order to be tagged as predominant, a specific class must have been observed at least seven times as often as every other class with the same values of m and n. Table 2.2.1 has the predominant classes for $5 \le m \le 12$ and $2 \le n \le 10$. Our criterion does not identify a predominant class for every pair (m, n). For those pairs where it fails, all classes that were observed at least one-seventh as often as the most common class are listed. In most cases these lists don't fit in the table and are replaced by labels explained below.

	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$	$m=11$	$m=12$
$n=10$	(a)	(j)	(k)	(k)	(u)	(u)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=9$	(b)	(k)	(j)	(k)	(u)	(u)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=8$	(c)	(j)	(k)	(k)	(u)	(u)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=7$	(d)	(l)	(j)	(k)	(u)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=6$	(e)	(j)	(k)	(u)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=5$	(f)	(m)	(q)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=4$	(g)	(n)	(r)	(v)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=3$	(h)	(o)	(s)	(w)	(v)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=2$	(i)	(p)	(t)	(x)	(v)	(y)	$\mathbf{G}(3)$	$\mathbf{H}(0,0)$

Table 2.2.1: Predominant classes in characteristic 3.

(a) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(3,0)$, $\mathbf{H}(4,0)$
(b) $\mathbf{H}(0,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{T}$
(c) $\mathbf{H}(2,0)$, $\mathbf{T}$
(d) $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{T}$
(e) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{H}(4,0)$, $\mathbf{T}$
(f) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(2,1)$, $\mathbf{H}(3,0)$, $\mathbf{H}(4,1)$, $\mathbf{T}$
(g) $\mathbf{H}(1,0)$, $\mathbf{H}(1,1)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,1)$, $\mathbf{T}$
(h) $\mathbf{H}(0,1)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,1)$, $\mathbf{H}(4,3)$
(i) $\mathbf{B}$, $\mathbf{H}(0,0)$
(j) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$
(k) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$
(l) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{T}$
(m) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(2,1)$, $\mathbf{T}$
(n) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,1)$
(o) $\mathbf{B}$, $\mathbf{H}(0,0)$, $\mathbf{H}(2,2)$
(p) $\mathbf{G}(3)$, $\mathbf{H}(0,1)$, $\mathbf{H}(1,2)$
(q) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(1,1)$, $\mathbf{H}(2,0)$, $\mathbf{H}(2,1)$
(r) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(1,1)$
(s) $\mathbf{B}$, $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$
(t) $\mathbf{B}$, $\mathbf{G}(2)$, $\mathbf{G}(4)$, $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$, $\mathbf{H}(0,2)$
(u) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$
(v) $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$
(w) $\mathbf{G}(2)$, $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$
(x) $\mathbf{G}(3)$, $\mathbf{G}(5)$, $\mathbf{H}(0,0)$
(y) $\mathbf{G}(2)$, $\mathbf{G}(5)$, $\mathbf{G}(7)$, $\mathbf{H}(0,1)$

The experiment was performed with changing values of numTerms and other parameters to seek out ideals of "rare" classes. To get a better feeling for the existence of predominant classes, we classified another 10^6 randomly generated ideals with $\mathsf{fieldChar} = 0$ and all other parameters set to their default values. Table 2.2.2 shows the predominant classes in the same fashion as Table 2.2.1:

	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$	$m=11$	$m=12$
$n=10$	(a)	(j)	(k)	(k)	(l)	(l)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=9$	(b)	(k)	(s)	(k)	(k)	$\mathbf{H}(0,0)$	(l)	$\mathbf{H}(0,0)$
$n=8$	(c)	(l)	(q)	(w)	(l)	(l)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=7$	(d)	(m)	$\mathbf{H}(1,0)$	(q)	$\mathbf{H}(0,0)$	(l)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=6$	(e)	(n)	$\mathbf{H}(0,0)$	$\mathbf{H}(1,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=5$	(f)	(o)	(t)	$\mathbf{H}(0,0)$	(æ)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=4$	(g)	(p)	(u)	(x)	$\mathbf{H}(0,0)$	(æ)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$
$n=3$	(h)	(q)	(v)	(y)	(z)	$\mathbf{H}(0,0)$	$\mathbf{H}(0,0)$	(l)
$n=2$	(i)	(r)	$\mathbf{G}(4)$	(z)	(ø)	(å)	none	$\mathbf{H}(0,0)$

Table 2.2.2: Predominant classes in characteristic 0.

(a) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(4,0)$
(b) $\mathbf{H}(3,0)$, $\mathbf{H}(4,0)$, $\mathbf{T}$
(c) $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{T}$
(d) $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$
(e) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(4,0)$, $\mathbf{T}$
(f) $\mathbf{H}(0,0)$, $\mathbf{H}(3,0)$, $\mathbf{H}(4,1)$
(g) $\mathbf{H}(2,0)$, $\mathbf{H}(3,1)$, $\mathbf{T}$
(h) $\mathbf{H}(0,1)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,1)$, $\mathbf{H}(4,3)$
(i) $\mathbf{B}$, $\mathbf{H}(0,0)$
(j) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{T}$
(k) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$
(l) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$
(m) $\mathbf{H}(0,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{H}(4,0)$, $\mathbf{T}$
(n) $\mathbf{H}(2,0)$, $\mathbf{H}(3,0)$, $\mathbf{H}(5,2)$, $\mathbf{T}$
(o) $\mathbf{H}(1,0)$, $\mathbf{H}(2,0)$
(p) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(3,0)$, $\mathbf{H}(5,4)$
(q) $\mathbf{H}(0,0)$, $\mathbf{H}(2,0)$
(r) $\mathbf{G}(3)$, $\mathbf{H}(0,1)$, $\mathbf{H}(1,2)$
(s) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(3,0)$, $\mathbf{T}$
(t) $\mathbf{H}(0,0)$, $\mathbf{H}(2,0)$, $\mathbf{H}(2,1)$, $\mathbf{H}(6,5)$
(u) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(1,1)$
(v) $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$, $\mathbf{H}(2,3)$
(w) $\mathbf{H}(0,0)$, $\mathbf{H}(1,0)$, $\mathbf{H}(3,0)$
(x) $\mathbf{G}(2)$, $\mathbf{H}(0,0)$, $\mathbf{H}(2,0)$
(y) $\mathbf{G}(2)$, $\mathbf{H}(0,0)$, $\mathbf{H}(0,2)$, $\mathbf{H}(1,0)$
(z) $\mathbf{G}(3)$, $\mathbf{G}(5)$, $\mathbf{H}(0,0)$
(æ) $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$, $\mathbf{H}(1,0)$
(ø) $\mathbf{H}(0,0)$, $\mathbf{H}(0,1)$
(å) $\mathbf{G}(2)$, $\mathbf{G}(7)$

2.3. **Online Repository.** A text catalogue with examples of ideals from each of the observed classes can be found on *GitHub*: `https://github.com/ogotchey/codimThreeCode`. Within the bounds $5 \leq m \leq 12$ and $2 \leq n \leq 10$ there is a `*.txt` file for each class that has been observed. The files are in the `data` folder of the repository, and they are named `m-n-class-p-q-r.txt` as discussed in Step 1.5. The lines of each text file are sorted by increasing "complexity", and furthermore the files are in *Macaulay2*-readable format. Also available in this repository are expanded versions of Tables 2.1.1–2.1.6 and the source code for the algorithm described in Section 1. For more instructions, see the `README.md` file on the repository.

3. Discussion

A main aspect of the realizability question involves bounds on the parameters p, q, and r in terms of m and n for ideals of class $\mathbf{G}$ and $\mathbf{H}$.

3.1. **Ideals of Class H.** The current bounds on p and q in terms of m and n for ideals of class $\mathbf{H}$ were proved in [6, Theorem 1.1]. In Tables 2.1.4–2.1.6 they are represented by the black border that separates (p,q)-cells within each (m,n)-box. For example, the bounds on p and q for $(m,n) = (8,5)$ are seen in the table below.

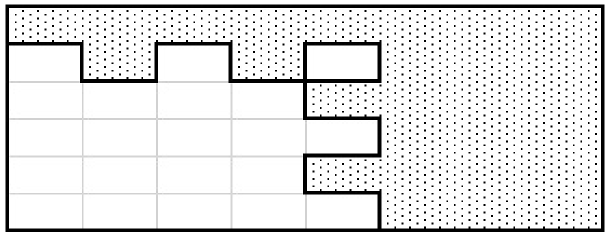

Table 3.1.1. Permissible and unrealizable classes for $(m,n) = (8,5)$.

It was conjectured in [6, Conjecture 7.4] that for ideals of class **H** with $m \geq 5$ and $n \geq 3$, the bounds established in [6, Theorem 1.1] are optimal. For low values of m and n we did, indeed, observe ideals of all possible classes within these bounds; see Table 2.1.4. For larger values of m and n this was not the case. For example, for $m \geq 8$ and $n \geq 4$, we never observed classes with $p = n - 1$ and least possible q, i.e. $q = 0$ or $q = 1$. For $m \geq 8$ and $n \geq 5$, we never observed classes with $q = m - 4$ and least possible p, i.e. $p = 0$ or $p = 1$. For $(m,n) = (8,5)$ these were in fact the only **H** classes not observed, as seen in the table below.

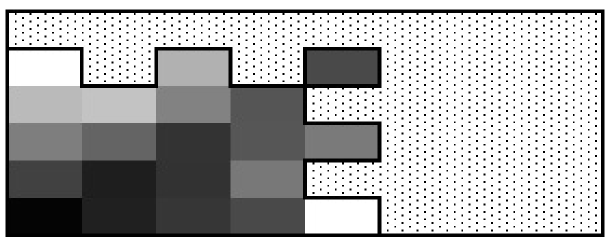

Table 3.1.2. Observed classes for $(m,n) = (8,5)$.

While not all permissible classes with $p = n - 1$ or $q = m - 4$ were observed in the experiment, it should be pointed out that they are not unrealizable. Indeed, for $m \geq 5$ and $n \geq 3$, Hardesty proves in [8, Theorem 5.2] that all these classes can be realized; the proof is based on linkage theory. Further, within the bounds $m \leq 10$ and $n \leq 12$ the experiment established the realizability of sufficiently many classes to prove, again using linkage, that all classes within the bounds from [6, Theorem 1.1] are realizable. Thus, the experiment combined with Hardesty's work provides strong evidence for [6, Conjecture 7.4] regarding ideals of class **H**.

3.2. **Ideals of Class G.** The current bounds on the r parameter for class **G**, as proved in [1, Theorem 3.1], are represented in Tables 2.1.1–2.1.3 by the black borders that separate (p,r)-cells within each (m,n)-box. It was conjectured in [6, Conjecture 7.4] that for ideals of class **G**, if $n = 2$, then $2 \leq r \leq m-5$ or $r = m-3$, and if $n \geq 3$, then $2 \leq r \leq m-4$. The observations made in our experiment provide strong evidence for the conjectured bounds in the $n = 2$ case. However, for $n \geq 3$ the conjectured bound on r seems to be too loose; the optimal bound on r seems more likely to be a function of both m and n—increasing in m and decreasing in n.

3.3. **Classes Realized by Monomial and Binomial Ideals.** As seen in Table 2.1.5, **H** classes realized monomially tend to have p and q values in close proximity. Classes that can be realized binomially, see Table 2.1.6, do not exhibit the same pattern and, actually, it seems that most **H** classes are realized binomially. One may wonder if every permissible **H** class can be realized by a binomial ideal.

No monomial ideals of class **G** were observed in the experiment. This is further evidence for [10, Conjecture 6.4] of Painter as it pertains to class **G**. (As it pertains to class **H**, it was disproved in Faucett's dissertation [7, Chapter IV].) All but a few observed **G** classes were realized binomially. Similar to class **H**, it seems feasible that all permissible **G** classes can be realized binomially.

In the experiment, monomial ideals of class **B** were only observed for (m, n) with n bounded above by m and below by an increasing function in m. On the other hand, binomial ideals of class **B** were observed for all $m \geq 5$ and low values of n.

Monomial ideals of class **T** were only observed for (m, n) with n bounded above and below by increasing functions of m. There is no apparent pattern for binomial ideals of class **T**.

3.4. **Comparing Frequency of the Parameters** p, q, r**.** From Table 2.1.4 one notices that the most frequently observed **H** classes have values of p and q in close proximity, creating a gradient that is darkest in the bottom left corner of a given (m, n)-box. That is, the lower values of p and q tend to be realized more often. This is consistent with Tables 2.2.1 and 2.2.2 which show predominant classes.

For class **G**, low values of r tend to occur more frequently than high values, see Table 2.1.1.

3.5. **Comparing Frequency to Linkage.** To facilitate this discussion, we recall that linkage is a symmetric relation on grade 3 perfect ideals and say that two classes are *directly linked* if by one application of linkage one can obtain an ideal of one class from an ideal of the other. Within the bounds $1 \leq n \leq 10$ and $3 \leq m \leq 12$ all permissible **B** classes were realized in the experiment. All but seven permissible **T** classes were realized. However, these seven **T** classes are directly linked to observed classes. In [8, 3.2], Hardesty proves that one can use linkage to realize ideals of class **T** for all $m \geq 5$ and $n \geq 4$.

Within the bounds of the experiment, 201 permissible **G** classes were not observed; of these 158 are directly linked to observed classes. These 158 directly linked classes occurred as rarely as once and as frequently as four million times. Most of these classes were observed frequently, with all but four of them occurring at least a hundred times.

A total of 639 permissible **H** classes went unobserved in the experiment. Of these, 66 are directly linked to observed classes. These 66 directly linked classes occurred as rarely as once and as frequently as forty thousand times; most of them are rare, with all but six classes occurring fewer than a hundred times. However, each of the 639 permissible **H** classes that were not observed is in the linkage class of an observed class, i.e. for each of them there is a chain of directly linked classes that connects it to an observed class.

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Appendix

For given values of m and n, it may be useful to have specific, "simple" examples of ideals of the various classes. We provide such examples below; they were first observed in characteristic 3 but have been verified to work also in characteristic 0.

m	n	class	p	q	r	minimal generators
5	2	**B**	1	1	2	z^2, xz, y^2, xy, x^2
5	2	**H**	0	0	0	$z^2, xz, x^2y, x^3+y^2z, y^4$
5	3	**H**	0	0	0	$yz^2, x^2z, y^3, x^2y+z^3, x^3$
5	3	**H**	0	1	1	$z^3, yz^2, y^3, xy^2+x^2z, x^3$
5	3	**H**	1	0	0	$xz, yz^2, y^3, xy^2, x^3+z^3$
5	3	**H**	2	1	1	xz, y^2, z^3, yz^2, x^3
5	3	**H**	4	3	3	$z^2, y^3, xy^2, x^2y, x^3$
5	4	**H**	0	0	0	$z^4, y^2z^2, xy^6-x^6z, x^7, y^8$
5	4	**H**	1	0	0	$xyz+y^2z, y^3, x^3, z^4, xz^3$
5	4	**H**	1	1	1	$x^3+xyz, z^4, y^2z^2, x^2z^2, y^5$
5	4	**H**	2	0	0	$z^2, xyz, y^3, x^3, x^2y^2$
5	4	**H**	3	1	1	$y^2, z^3, x^2z, x^3, xyz^2$
5	4	**T**	3	0	0	$z^3, y^3, x^3, xyz^2, xy^2z$
5	5	**H**	0	0	0	$xz^2+z^3, y^2z, y^3-z^3, x^2y, x^3-z^3$
5	5	**H**	0	1	1	$xy^2z, x^2yz-z^4, y^4+x^3z+y^2z^2, x^3y-y^2z^2, x^4-z^4$
5	5	**H**	1	0	0	$xz^2, y^2z+z^3, y^5+x^4z, x^6y, x^7+x^4y^3$
5	5	**H**	2	0	0	$y^2z, x^2z, y^3+z^3, x^4+xyz^2, x^3y^2$
5	5	**H**	2	1	1	$xyz-z^3, x^2y-y^2z, z^5, y^5+yz^4, x^6$
5	5	**H**	3	0	0	$xz+z^2, y^4-y^3z, xy^3-yz^3, x^4-y^2z^2, z^5$
5	5	**H**	4	1	1	$y^2+xz, xz^3+z^4, xyz^2, x^3z, x^4$
5	5	**T**	3	0	0	$z^4, xyz^2, x^2y^2z, x^5, y^6$
6	2	**G**	0	1	3	$yz, xz, y^3, xy^2+z^3, x^2y, x^3$
6	2	**H**	0	0	0	$z^3, x^2z, x^2y+y^2z, x^3, y^4, xy^3$
6	2	**H**	0	1	1	$yz, xz^2, y^3, xy^2-z^3, x^2y, x^3$
6	2	**H**	1	2	2	$xy-z^2, z^3, xz^2, y^2z, y^3, x^3$

m	n	class	p	q	r	minimal generators
6	3	**B**	1	1	2	$xy, x^2, z^3, yz^2, y^3z, y^4$
6	3	**G**	0	1	2	$z^3, x^2z, y^3, xy^2, x^3 - yz^2, xyz^2$
6	3	**H**	0	0	0	$yz, xz, xy, z^3, x^3, y^4$
6	3	**H**	0	1	1	$xz^2, y^2z, x^2y, x^4, xy^4 - z^5, y^6$
6	3	**H**	0	2	2	$z^3, x^3z + y^2z^2, y^4 - x^2yz + xy^2z,$ $x^2y^2, x^3y - xy^3, x^4 + xy^3 + xy^2z$
6	3	**H**	1	0	0	$z^3, x^2z, xy^2 + yz^2, x^3, y^4z, y^5$
6	3	**H**	1	1	1	$x^2, z^3, yz^2, xyz, y^3, xy^2$
6	3	**H**	2	0	0	$y^3, x^2y - y^2z - xz^2 - z^3, x^3, z^4, xz^3, y^2z^2$
6	3	**H**	2	2	2	$y^2, z^3, xz^2, x^2z, x^2y, x^4$
6	4	**B**	1	1	2	$z^3, yz^2, x^2y, x^4, y^5, xy^4z$
6	4	**G**	0	1	2	$x^2z + y^2z, x^3 + xy^2, yz^3, y^3z,$ $y^4 + xyz^2 - y^2z^2, x^2y^2 - z^4$
6	4	**H**	0	0	0	$xz^2, y^2z, x^2y, x^3, z^4, y^4$
6	4	**H**	0	1	1	$xy^2 + y^3, x^2y, xz^3, xyz^2 + z^4, x^3z - y^2z^2, x^5$
6	4	**H**	1	0	0	$xy, z^3, y^2z, x^2z, y^4, x^4$
6	4	**H**	1	1	1	$z^3, xyz, y^3, x^2y, x^3, y^2z^2$
6	4	**H**	1	2	2	$x^2z^3 + xyz^3, x^3z^2 - x^2yz^2, xy^3z + x^2yz^2,$ $x^2y^2z + x^2yz^2, x^2y^3 + y^5, x^5 + y^5 + x^2yz^2 + z^5$
6	4	**H**	2	0	0	$z^3, y^3, xyz^2, x^2yz, x^2y^2, x^4$
6	4	**H**	2	1	1	$z^2, y^2z, x^2y, x^3, y^4, xy^3$
6	4	**H**	3	0	0	$y^3z - yz^3, xy^2z + yz^3, y^4 - z^4, xz^4 + z^5, x^2y^3 - z^5, x^5 - z^5$
6	4	**H**	3	2	2	$y^2, z^3, xz^2, x^2z, x^3y, x^4$
6	4	**H**	5	4	4	$z^2, y^4, xy^3, x^2y^2, x^3y, x^4$
6	4	**T**	3	0	0	$xy^2z, x^2yz, y^5, x^5, z^6, xyz^5$
6	5	**B**	1	1	2	$x^2z + z^3, yz^3 - z^4, xyz^2 + z^4, y^6z, x^4y^3, x^7 + xy^6 + y^7$
6	5	**H**	0	0	0	$z^3, y^2z, x^2z + yz^2, y^3, xy^2, x^3$
6	5	**H**	0	1	1	$z^3, y^3z + x^2z^2, xy^3, x^2y^2 + x^3z, x^3y + y^2z^2, x^4 + y^4$
6	5	**H**	1	0	0	$y^3, z^4, x^3z, x^2y^2, xyz^3, x^5$
6	5	**H**	1	1	1	$y^3 - xyz, xy^2, x^4z + xyz^3 - xz^4, x^5, xz^6, z^8$
6	5	**H**	2	0	0	$z^2, xy^2, x^3, y^3z, x^2yz, y^4$
6	5	**H**	2	1	1	$z^3, yz^2, x^3, y^4, xy^3, x^2y^2z$
6	5	**H**	2	2	2	$xy^2z + z^4, xy^3 - z^4, x^3y - z^4, xyz^3 + z^5, y^6, x^6 + x^3z^3$
6	5	**H**	3	0	0	$x^2y - z^3, xy^2z - z^4, x^4 + x^2z^2, yz^4, y^2z^3, y^5 + xyz^3$
6	5	**H**	3	1	1	$x^2, z^4, yz^3, xy^2z, y^4, y^3z^2$
6	5	**H**	4	0	0	$x^2y - x^2z - xyz, y^3z^2 - x^2z^3 + xyz^3 + z^5,$ $x^3z^2 - xyz^3 - z^5, xy^4 - xyz^3 + yz^4 - z^5,$ $x^5 + y^5 + x^4z - x^2z^3 + xyz^3 - z^5, yz^5 + z^6$
6	5	**H**	4	2	2	$y^2, x^3z, z^5, x^2z^3, x^5, xyz^4$
6	5	**T**	3	0	0	$xy^2z, y^4, x^2yz^2, x^5, z^6, xyz^4$
7	2	**B**	1	1	2	$z^3, xz^2, x^2z, xy^2, x^2y, x^3 - xyz, y^4$
7	2	**G**	0	1	2	$z^3, xz^2, y^2z, xyz, y^3 + x^2z, x^3y, x^5$
7	2	**G**	0	1	4	$z^3, y^2z, x^2z, y^3, xy^2, x^2y, x^3 - yz^2$
7	2	**H**	0	0	0	$xyz^2, x^3y, x^4z - y^3z^2, y^5, x^5, z^7, xz^6$
7	2	**H**	0	1	1	$z^3, y^2z, xy^2, x^2y, x^2z^2, y^4 + x^3z, x^4$
7	2	**H**	0	2	2	$x^2, xy^2, y^2z^3, yz^5, y^5z - xz^5, y^6, z^7$

m	n	class	p	q	r	minimal generators
7	3	**B**	1	1	2	$z^3, xz^2, y^3, xy^2, x^3z, x^3y, x^4$
7	3	**G**	0	1	2	$xz^2, xy^2 - z^3, x^2y, x^3, z^4, y^3z, y^5$
7	3	**G**	0	1	3	$yz, xy, z^4, xz^3, x^4z, y^5 + x^3z^2, x^5$
7	3	**H**	0	0	0	$z^3, xz^2, y^2z, xyz, y^3, x^2y, x^3$
7	3	**H**	0	1	1	$z^2, y^2z, x^2z, y^3, xy^2, x^2y, x^4$
7	3	**H**	0	2	2	$z^4, y^4, xy^3, x^2y^2, x^4, x^2z^3, x^3yz^2 + xy^2z^3 - y^3z^3$
7	3	**H**	0	3	3	$z^3, xy^4 - x^3yz + x^2y^2z - xy^3z - x^2yz^2,$ $x^2y^3 - x^4z - y^4z - x^3z^2, y^5 - x^4z - y^4z, x^4y,$ $x^3y^2 + x^4z - x^3z^2 - xy^2z^2, x^5 - x^4z - x^2y^2z$
7	3	**H**	1	0	0	$z^4, y^4z, x^4z, y^5 - x^3z^2, x^4y, x^2y^4, x^6 - y^3z^3$
7	3	**H**	1	1	1	$xy^2, z^4, x^2z^2, x^4z, x^4y, y^7, x^9 - y^6z^3$
7	3	**H**	1	2	2	$z^4, xz^3, x^2z^2, x^3z, y^4, x^4 + yz^3, x^2y^3$
7	3	**H**	2	1	1	$y^2z, x^3, z^4, yz^3, x^2z^2 - xyz^2, x^2yz + xz^3, y^4$
7	3	**H**	2	3	3	$yz^3, xz^3, y^2z^2, x^4, y^4z, y^5, x^3y^3z - z^7$
7	4	**B**	1	1	2	$yz^2, y^2z, x^2z, y^3, x^3, z^4, xz^3$
7	4	**G**	0	1	2	$x^2z, x^2y, y^3z, y^4, xy^3 + z^4, x^4 + yz^3, xy^2z^2$
7	4	**G**	0	1	3	$xz^3, xyz^2, xy^2z - y^3z - z^4, y^4 - x^2yz,$ $x^2y^2 - xy^3 + y^3z + z^4, x^3y, x^4 - y^3z + x^2z^2$
7	4	**H**	0	0	0	$xz, xy, yz^2, y^2z, x^3, z^4, y^4$
7	4	**H**	0	1	1	$z^3, y^3 + xz^2, xy^2, x^2y, y^2z^2, x^3z, x^4$
7	4	**H**	1	0	0	$y^3, x^3, z^4, xz^3, y^2z^2, xyz^2, x^2yz$
7	4	**H**	1	1	1	$y^2, z^3, xz^2, xyz, x^2z, x^2y, x^3$
7	4	**H**	1	2	2	$y^3, z^4, xz^3, xyz^2, x^3z - x^2yz - y^2z^2, x^4 - x^2y^2, x^2y^2z$
7	4	**H**	2	1	1	$z^3, y^3, xy^2, x^2z^2, x^3z, x^3y, x^5$
7	4	**H**	2	2	2	$x^2, xy^2, z^4, yz^3, y^2z^2, y^4z, y^5$
7	4	**H**	3	3	3	$y^2, yz^3, xz^3, x^2z^2, x^3z, x^4, z^6$
7	5	**B**	1	1	2	$y^2z, y^3, x^3z, x^4, z^5, xz^4, x^2yz^3$
7	5	**H**	0	0	0	$z^3, yz^2, x^2z, y^3, xy^2, x^3y, x^4$
7	5	**H**	0	1	1	$x^3, z^4, yz^3, y^2z^2, y^4, xy^3 - x^2yz, x^2z^3$
7	5	**H**	1	0	0	$z^2, x^2z, xy^2, y^3z, y^4, x^3y, x^4$
7	5	**H**	1	1	1	$z^3, xz^2, y^3, xy^2z, x^3y, x^4z, x^5$
7	5	**H**	2	0	0	$z^3, y^3, xyz^2, xy^2z, x^2yz, x^4, x^3z^2$
7	5	**H**	2	1	1	$z^2, xy^2, x^2y, y^3z, x^3z, y^4, x^4$
7	5	**H**	2	2	2	$z^3, x^2y^2, x^2yz^2, y^5, xy^4, x^5z, x^6$
7	5	**H**	3	2	2	$z^3, yz^2, y^2z, x^4, x^2y^3, y^6, xy^5$
7	5	**H**	4	3	3	$x^2, z^4, yz^3, y^2z^2, y^3z, y^5, xy^4$
8	2	**G**	0	1	2	$xyz, x^2z, xy^2, y^2z^2 - xz^3, x^4 + y^3z, z^5, yz^4, y^5$
8	2	**G**	0	1	3	$x^2z, x^2y, y^3z, xy^3, yz^4, x^5 - y^2z^3, z^7, y^7 + xz^6$
8	2	**G**	0	1	5	$x^2z, x^2y + xz^2, x^3, z^4, yz^3, y^3z, y^4, xy^3 + y^2z^2$
8	2	**H**	0	0	0	$yz^2, xz^2, y^2z, x^2z, y^3 + z^3, xy^2, x^2y, x^3 - z^3$
8	2	**H**	0	1	1	$xy^2, z^4, xz^3, x^2z^2 - yz^3, x^3z, y^4, x^3y, x^5 + y^3z^2$
8	2	**H**	0	2	2	$y^3, yz^3, xz^3, x^2z^2, x^3z, x^2y^2 - z^4, x^3y, x^5 - z^5$
8	2	**H**	1	2	2	$z^5, x^4z, x^3y^2, x^4y, y^5z, xy^5, y^7 + x^3z^4, x^8 + y^4z^4$
8	3	**B**	1	1	2	$z^3, y^2z^2, xy^2z, xy^5, x^4y^2, x^6, y^7, x^3y^4 + y^6z$
8	3	**G**	0	1	2	$yz^2, y^2z, xyz, x^2z, y^3, xy^2, x^2y - z^3, x^4$
8	3	**G**	0	1	3	$y^2z^3, xy^4, z^6, y^4z^2, x^5z, y^6, x^5y, x^6 - yz^5$
8	3	**G**	0	1	4	$z^3, yz^2, xyz, y^4 - x^3z, xy^3, x^2y^2, x^3y, x^4$
8	3	**H**	0	0	0	$z^3, yz^2, y^2z, x^2z, y^3 + xz^2, xy^2, x^2y, x^3$
8	3	**H**	0	1	1	$yz^2, x^2z, y^3z, xy^3, x^2y^2, x^3y + z^4, y^5, x^5$
8	3	**H**	0	2	2	$x^2, z^4, yz^3, xz^3, y^2z^2, y^3z - xyz^2, y^4, xy^3$

m	n	class	p	q	r	minimal generators
8	3	**H**	1	0	0	$y^3+z^3, x^2y, z^4, yz^3, xyz^2, xy^2z+x^2z^2, x^3z, x^4-y^2z^2$
8	3	**H**	1	1	1	$x^2y, z^5, x^2z^3-y^2z^3, y^3z^2, xy^3z, xy^4, y^7-x^6z, x^7$
8	3	**H**	1	2	2	$z^4, yz^3, xz^3, x^2z^2, y^4, x^4z, x^4y, x^5+y^3z^2$
8	3	**H**	1	3	3	$y^2, yz^5, x^5z, x^2z^5, x^6y-x^4yz^2-xz^6,$ $x^7, z^8, x^4z^4+x^3yz^4$
8	3	**H**	2	2	2	$xyz+z^3, yz^3, xy^3-y^2z^2, x^2y^2, x^3y, x^4, y^4z, y^5$
8	4	**B**	1	1	2	$z^3, yz^2, y^2z, xyz, xy^2, x^2y, x^3, y^5$
8	4	**G**	0	1	2	$y^2z, x^2z, x^2y, x^3-yz^2, z^4, xz^3, y^4, xy^3$
8	4	**G**	0	1	3	$yz^2, y^2z, x^2z^2, y^5, xy^4, x^2y^3, x^4y+z^5, x^6$
8	4	**G**	0	1	4	$xy^3+y^4-y^3z, x^2y^2-y^3z, x^2yz^2+x^2z^3, y^4z,$ $x^5-xyz^3+xz^4+z^5, y^2z^2, x^4y-x^3yz-x^2z^3+yz^4,$ $x^4z-x^3yz-x^3z^2-x^2z^3-yz^4+z^5$
8	4	**H**	0	0	0	$z^3, yz^2, xz^2, y^2z, xy^2, x^2y, x^3, y^4$
8	4	**H**	0	1	1	$z^2, y^2z, x^3z, x^2y^2, x^3y, y^5, xy^4, x^5$
8	4	**H**	0	2	2	$x^3, xz^4-yz^4, y^2z^3, y^3z^2+xyz^3, xy^2z^2+x^2z^3,$ $y^4z, y^5-xy^3z-x^2yz^2+yz^4, x^2y^3+z^5$
8	4	**H**	1	0	0	$xz^2, yz^3, y^4, x^2y^2, x^4z, x^4y, x^5+x^3yz+y^3z^2, z^6$
8	4	**H**	1	1	1	$x^3, z^4, yz^3, xz^3, x^2z^2, y^3z, x^2yz, y^4$
8	4	**H**	1	2	2	$y^3, yz^3, xz^3, x^2z^2, x^3z, x^3y, z^6, x^6$
8	4	**H**	2	1	1	$x^2z^2, x^4+y^2z^2, x^2y^2z+yz^4, z^6, y^4z^2,$ $y^5z-xz^5, xy^4z, x^3y^3+x^2y^4+xy^5+y^6$
8	4	**H**	2	2	2	$y^3, x^2z^2, x^2y^2, z^6, xz^5, x^6z, x^6y, x^8$
8	4	**H**	3	4	4	$z^3, x^3y^2, y^6z, xy^6, x^2y^5, x^6y, x^9, y^{10}+x^8z^2$
8	5	**B**	1	1	2	$z^3, yz^2, x^2yz, x^3y, x^4, xy^4, y^5z, y^6$
8	5	**G**	0	1	2	$xz^3, xy^3+z^4, x^2y^2, x^3y+x^2z^2+xyz^2,$ $x^4, y^3z^2, xy^2z^2, y^6$
8	5	**H**	0	0	0	$xy, yz^2, xz^2, y^2z, x^2z, y^3, x^3, z^4$
8	5	**H**	0	1	1	$x^3, y^2z^2, xyz^2, y^3z, xy^3, z^5, x^2z^4, y^6$
8	5	**H**	1	0	0	$z^3, xy^2, x^2z^2, y^3z, x^3z, y^4, x^3y, x^4$
8	5	**H**	1	1	1	$z^3, xyz, y^3, xy^2, x^2z^2, x^4z, x^4y, x^5$
8	5	**H**	1	2	2	$z^3, y^4, xy^3, x^2y^2, x^2yz^2,$ $x^3y-x^3z+x^2yz+y^3z, x^4z, x^5$
8	5	**H**	2	0	0	$y^3, x^3z, x^4+z^4, yz^4, y^2z^3,$ $x^2z^3+xz^4, x^2yz^2, x^2y^2z-xyz^3$
8	5	**H**	2	1	1	$z^3, yz^2, x^3, xy^2z, x^2y^2, y^4z, y^5, xy^4$
8	5	**H**	2	2	2	$z^2, y^3z, xy^3, x^2y^2, x^3y, x^5z, y^6, x^6$
8	5	**H**	3	2	2	$z^3, y^4, xy^3, x^2y^2, x^3z^2, x^3yz, x^5y, x^6$
8	5	**H**	3	3	3	$y^3, x^3y, z^5, xz^4, x^2z^3, x^3z^2, x^5z, x^6$
8	5	**H**	4	4	4	$z^2, x^2y^3, y^5z, y^6, xy^5, x^4y^2, x^5y, x^6$

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