

RIGIDITY OF EXT AND TOR VIA FLAT–COTORSION THEORY

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ABSTRACT. Let \mathfrak{p} be a prime ideal in a commutative noetherian ring R and denote by $k(\mathfrak{p})$ the residue field of the local ring $R_{\mathfrak{p}}$. We prove that if an R -module M satisfies $\text{Ext}_R^n(k(\mathfrak{p}), M) = 0$ for some $n \geq \dim R$, then $\text{Ext}_R^i(k(\mathfrak{p}), M) = 0$ holds for all $i \geq n$. This improves a result of Christensen, Iyengar, and Marley by lowering the bound on n . We also improve existing results on Tor-rigidity. This progress is driven by the existence of minimal semi-flat-cotorsion replacements in the derived category as recently proved by Nakamura and Thompson.

1. INTRODUCTION

Over a commutative noetherian ring R , the injective and flat dimension of a module can be detected by vanishing of Ext and Tor with coefficients in residue fields $k(\mathfrak{p})$ at the prime ideals \mathfrak{p} of R . This drives the interest in *rigidity* properties of Ext and Tor—here rigidity refers to the phenomenon that vanishing of, say, Ext^n implies vanishing of Ext^i for all $i \geq n$. Rigidity of Ext and Tor with coefficients in residue fields was studied by Christensen, Iyengar, and Marley in [7]. Here we push the investigation further in two directions: (1) We eliminate certain asymmetries in the rigidity statements for Ext/Tor and injective/flat dimension obtained in [7]. (2) We establish and improve results on flat dimension that are conceptually dual to results on injective dimension already in the literature, including results obtained in [7].

We work with complexes of modules and our main result, found in Section 3, is:

Theorem. *Let R be a commutative noetherian ring and M an R -complex. If for an integer $n \geq \dim R + \sup H^*(M)$ one has $\text{Ext}_R^n(k(\mathfrak{p}), M) = 0$ for all prime ideals \mathfrak{p} in R , then $\text{inj.dim}_R M < n$ holds.*

This improves [7, Theorem 5.7] and aligns perfectly with [7, Theorem 4.1] on flat dimension. By a result of Christensen and Iyengar [6, Theorem 1.1], the proof reduces to show, for complexes, the statement on Ext-rigidity made in the Abstract.

In Section 4 we prove results on flat dimension of complexes which are dual to already established results on injective dimension. In one of these, we remove a boundedness assumption from [12, Theorem 4.8], which is dual to [7, Corollary 5.9]:

Theorem. *Let R be a commutative noetherian ring of finite Krull dimension and M an R -complex. If M has finite flat dimension, then the next equality holds,*

$$\text{flat dim}_R M = \sup_{\mathfrak{p} \in \text{Spec } R} \{\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, M)\}.$$

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A novel aspect of our arguments involves the notion of minimal semi-flat-cotorsion replacements. Recall first that a flat-cotorsion module is one which is both flat and right Ext^1 -orthogonal to flat modules. A semi-flat complex consisting of flat-cotorsion modules is called a semi-flat-cotorsion complex. Work of Gillespie [10] shows that every complex is isomorphic in the derived category to a semi-flat-cotorsion complex; such a complex is called a *semi-flat-cotorsion replacement*.

Minimality plays a crucial role in considerations of rigidity and homological dimensions. A complex M is minimal if every homotopy equivalence $M \rightarrow M$ is an isomorphism. Minimal semi-injective resolutions always exist, and they detect injective dimension; several proofs in [7] rely on this. On the other hand, although semi-flat resolutions always exist, they may not contain a homotopically equivalent minimal summand that detects the flat dimension, see [8, Example 3.9]. Recently, Nakamura and Thompson [12] showed that every complex over a commutative noetherian ring of finite Krull dimension has a minimal semi-flat-cotorsion replacement and that such a complex detects the flat dimension.

In this paper, we also take the opportunity to clarify a couple of statements in [7]; see Remarks 2.4 and 5.3.

* * *

In this paper R is a commutative noetherian ring. We widely adopt the notation used in [7]; in particular, we use both homological and cohomological notation.

1.1. Let M be an R -complex.

For an integer s we denote by $\Sigma^s M$ the complex with the module M_{i-s} in degree i and differential $d^{\Sigma^s M} = (-1)^s d^M$.

Set $\inf H_*(M) := \inf\{i \mid H_i(M) \neq 0\}$ and $\inf H^*(M) := \inf\{i \mid H^i(M) \neq 0\}$, and define $\sup H_*(M)$ and $\sup H^*(M)$ similarly.

In case R is local with maximal ideal \mathfrak{m} , the right derived \mathfrak{m} -torsion functor is denoted $\text{R}\Gamma_{\mathfrak{m}}$ and $\text{L}\Lambda^{\mathfrak{m}}$ is the left derived \mathfrak{m} -completion functor. The corresponding local (co)homology modules are denoted $H_{\mathfrak{m}}^*(M)$ and $H_{\mathfrak{m}}^{\mathfrak{m}}(M)$. As always,

$$\text{depth}_R M = \inf \text{Ext}_R^*(k, M) \quad \text{and} \quad \text{width}_R M = \inf \text{Tor}_*^R(k, M)$$

where k denotes the residue field R/\mathfrak{m} .

We recall from [12] some properties of minimal semi-flat-cotorsion complexes that will be used frequently.

1.2. Assume that R has finite Krull dimension and let M be an R -complex. By [12, Theorem 3.4] there exists a minimal semi-flat-cotorsion complex F isomorphic to M in the derived category: a minimal semi-flat-cotorsion replacement of M .

If M has finite flat dimension, then it follows from [12, Lemma 4.1] that $F_i = 0$ holds for $i > \text{flat dim}_R M$.

If $H_i(M) = 0$ holds for $i \ll 0$, then $F_i = 0$ for $i \ll 0$ by [12, Lemma 4.1].

For every prime ideal \mathfrak{p} in R the complex $\text{Hom}_R(R_{\mathfrak{p}}, F)$ is isomorphic in the derived category to $\text{RHom}_R(R_{\mathfrak{p}}, M)$, see [12, (A.1)]. Further, it consists by [15, Lemma 2.2] of flat-cotorsion R -modules, so if $F_i = 0$ for $i \ll 0$, then $\text{Hom}_R(R_{\mathfrak{p}}, F)$ is a semi-flat-cotorsion replacement of $\text{RHom}_R(R_{\mathfrak{p}}, M)$ over both R and $R_{\mathfrak{p}}$.

2. RIGIDITY OF TOR

The main result of this section, Theorem 2.2 below, removes a boundedness condition on $H_*(M)$ from [7, Proposition 3.3(1)] and aligns perfectly—see Remark 2.3 below—with [7, Proposition 3.2] on rigidity of Ext.

Lemma 2.1. *Let (R, \mathfrak{m}, k) be a local ring and F a minimal semi-flat-cotorsion R -complex. There is an isomorphism of R -complexes $k \otimes_R F \cong k \otimes_R \Lambda^{\mathfrak{m}}(F)$, and both complexes have zero differential.*

Proof. The canonical map $F \rightarrow \Lambda^{\mathfrak{m}}(F)$ induces per [14, Theorem 2.2.2] an isomorphism of complexes $k \otimes_R F \rightarrow k \otimes_R \Lambda^{\mathfrak{m}}(F)$. Minimality of F implies per [12, Theorem 2.3] that $k \otimes_R F$, and hence also $k \otimes_R \Lambda^{\mathfrak{m}}(F)$, has zero differential. \square

Theorem 2.2. *Let (R, \mathfrak{m}, k) be a local ring and M an R -complex. If one has $\mathrm{Tor}_{n+1}^R(k, M) = 0$ for an integer $n \geq \sup H_*^{\mathfrak{m}}(M)$, then $\mathrm{Tor}_i^R(k, M) = 0$ holds for all $i > n$ and the next equality holds,*

$$\sup \mathrm{Tor}_*^R(k, M) = \mathrm{depth} R - \mathrm{depth}_R M.$$

Proof. If the complex $k \otimes_R^{\mathfrak{L}} M$ is acyclic, then one has $\sup \mathrm{Tor}_*^R(k, M) = -\infty$, so the assertion of the theorem is trivial, and so is the equality since $\mathrm{depth}_R M = \infty$ holds in this case, see [9, Definitions 2.3 and 4.3]. Hence we may assume that $k \otimes_R^{\mathfrak{L}} M$ is not acyclic. From [7, (2.6)] it follows that $\mathbf{L}\Lambda^{\mathfrak{m}}(M)$ is not acyclic. If $\sup H_*^{\mathfrak{m}}(M) = \infty$ holds, then the statement is vacuously true. Thus, we may assume that $\sup H_*^{\mathfrak{m}}(M) < \infty$ holds and set $s := \sup H_*^{\mathfrak{m}}(M)$.

Let F be a minimal semi-flat-cotorsion replacement of M , see 1.2, and set $P = \Lambda^{\mathfrak{m}}(F)$. As F is semi-flat, one has $\mathbf{L}\Lambda^{\mathfrak{m}}(M) = P$, see for example [13, Proposition 3.6], and Lemma 2.1 yields the isomorphism $k \otimes_R^{\mathfrak{L}} M \simeq k \otimes_R P$ in the derived category. For every $i \in \mathbb{Z}$ the \mathfrak{m} -complete module $P_i = \Lambda^{\mathfrak{m}}(F_i)$ is flat-cotorsion, see [15, Lemma 2.2]. The complex $\Sigma^s(P_{\geq s})$ is a semi-flat-cotorsion replacement of the module $C = \mathrm{Coker}(P_{s+1} \rightarrow P_s)$. For every integer $i > s$ there are isomorphisms

$$\mathrm{Tor}_i^R(k, M) \cong \mathrm{H}_i(k \otimes_R P) \cong \mathrm{H}_i(k \otimes_R P_{\geq s}) \cong \mathrm{Tor}_{i-s}^R(k, C).$$

Let $n \geq s$ and assume that $\mathrm{Tor}_{n+1}^R(k, M) = 0$ holds. By the isomorphisms above one has $\mathrm{Tor}_{n+1-s}^R(k, C) = 0$. The complex P , and hence the truncated complex $P_{\geq s}$, is \mathfrak{m} -complete, so the module C is derived \mathfrak{m} -complete. It now follows from [7, Lemma 2.1] that the module C has flat dimension at most $n - s$; in particular, $\mathrm{Tor}_i^R(k, M) = \mathrm{Tor}_{i-s}^R(k, C) = 0$ holds for all $i \geq n + 1$. This proves the first claim.

To prove the asserted equality, let E be the injective envelope of k . Adjunction yields

$$\mathrm{Hom}_R(\mathrm{Tor}_i^R(k, M), E) \cong \mathrm{Ext}_R^i(k, \mathrm{Hom}_R(M, E)),$$

so $\mathrm{Ext}_R^i(k, \mathrm{Hom}_R(M, E)) = 0$ holds for $i \gg 0$. Now faithful injectivity of E , together with [7, Proposition 3.2], yields

$$\begin{aligned} \sup \mathrm{Tor}_*^R(k, M) &= \sup \mathrm{Ext}_R^*(k, \mathrm{Hom}_R(M, E)) \\ &= \mathrm{depth} R - \mathrm{width}_R \mathrm{Hom}_R(M, E) \\ &= \mathrm{depth} R - \mathrm{depth}_R M, \end{aligned}$$

where the last equality is standard, see for example [9, Proposition 4.4]. \square

Remark 2.3. The bound on n in [7, Proposition 3.2] appears to be 1 lower than the bound in Theorem 2.2, but as noted in the opening paragraph of [7, Section 3] the lower bound cannot possibly be attained. We show below that one could similarly lower the bound in Theorem 2.2 by 1 as $\mathrm{Tor}_s^R(k, M) \neq 0$ holds for $s = \sup H_*^m(M)$.

Let M be an R -complex and F a minimal semi-flat-cotorsion replacement of M . Set $n = \sup H_*^m(M)$. In degree n , the complex $\Lambda^m(F)$ is nonzero and Lemma 2.1 yields $\mathrm{Tor}_n^R(k, M) = k \otimes_R \Lambda^m(F)_n$, which is nonzero, see [14, Observation 2.1.2].

Remark 2.4. The conclusion of [7, Lemma 2.1] states that $\mathrm{Tor}_n^R(-, M) = 0$ holds if M is a derived \mathfrak{a} -complete complex with $\inf H_*(M) > -\infty$ and $\mathrm{Tor}_n^R(R/\mathfrak{p}, M) = 0$ holds for all prime ideals \mathfrak{p} that contain \mathfrak{a} . We notice here that the proof in [7] only demonstrates this for $\mathrm{Tor}_n^R(-, M)$ as a functor on the category of R -modules (not R -complexes). This is sufficient for the purposes of its use in both [7] and the proof above. To see that the conclusion fails for Tor as a functor on complexes, let (R, \mathfrak{m}, k) be a complete local ring and notice that though $\mathrm{Tor}_n^R(k, R) = 0$ holds for every $n \geq 1$ one has $\mathrm{Tor}_n^R(\Sigma^n k, R) \cong k$.

3. INJECTIVE DIMENSION

The next result improves the bound on n in [7, Proposition 5.4 and Theorem 5.7].

Theorem 3.1. *Let R be a commutative noetherian ring and M an R -complex. If for an integer $n \geq \dim R + \sup H^*(M)$ one has $\mathrm{Ext}_R^n(k(\mathfrak{p}), M) = 0$ for all prime ideals \mathfrak{p} in R , then $\mathrm{inj.dim}_R M < n$ holds.*

Proof. We may assume that R has finite Krull dimension and that M is not acyclic. We may also assume that $H^i(M) = 0$ holds for $i \gg 0$, otherwise the statement is vacuous. For every prime ideal \mathfrak{p} in R one has

$$0 = \mathrm{Ext}_R^n(k(\mathfrak{p}), M) \cong \mathrm{Ext}_{R_{\mathfrak{p}}}^n(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M))$$

by Hom–tensor adjunction in the derived category. It suffices, by [6, Theorem 1.1] and [7, Proposition 3.2], to show that $\dim R + \sup H^*(M) \geq \sup H_{\mathfrak{p}}^* \mathrm{RHom}_R(R_{\mathfrak{p}}, M)$ holds. For every $R_{\mathfrak{p}}$ -complex X there is an isomorphism $\mathrm{R}\Gamma_{\mathfrak{p}} X \simeq \mathrm{R}\Gamma_{\mathfrak{p}} X$ in the derived category over $R_{\mathfrak{p}}$; this follows for example from [1, Lemma (3.2.3)] and explains the first and last isomorphisms in the next display. The second isomorphism holds by [1, Corollary (5.1.1)] while the third comes from [4, Proposition 8.3].

$$\begin{aligned} \mathrm{R}\Gamma_{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M) &\simeq \mathrm{R}\Gamma_{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M) \\ &\simeq \mathrm{R}\Gamma_{\mathfrak{p}} \mathrm{L}\Lambda^{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M) \\ (1) \quad &\simeq \mathrm{R}\Gamma_{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, \mathrm{L}\Lambda^{\mathfrak{p}} M) \\ &\simeq \mathrm{R}\Gamma_{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, \mathrm{L}\Lambda^{\mathfrak{p}} M). \end{aligned}$$

Let F be a minimal semi-flat-cotorsion replacement of M . In cohomological notation one has $F^i = 0$ for $i \gg 0$, see 1.2, so the complex $\Lambda^{\mathfrak{p}} F \simeq \mathrm{L}\Lambda^{\mathfrak{p}} M$ is again a semi-flat-cotorsion R -complex, see [15, Lemma 2.2]. In the derived category, the complex $P = \mathrm{Hom}_R(R_{\mathfrak{p}}, \Lambda^{\mathfrak{p}} F)$ is now isomorphic to $\mathrm{RHom}_R(R_{\mathfrak{p}}, \mathrm{L}\Lambda^{\mathfrak{p}} M)$. Further [12, Lemma 4.1] yields $P^i = 0$ for $i > \sup H^*(M) + \dim R/\mathfrak{p}$, which explains the second inequality in the computation below. The first equality holds by (1) and

the first inequality holds by [7, (2.7)],

$$\begin{aligned}
\sup H_{\mathfrak{p}}^*(\mathrm{RHom}_R(R_{\mathfrak{p}}, M)) &= \sup H^*(\mathrm{R}\Gamma_{\mathfrak{p}} \mathrm{RHom}_R(R_{\mathfrak{p}}, \mathrm{L}\Lambda^{\mathfrak{p}} M)) \\
&\leq \dim R_{\mathfrak{p}} + \sup H^*(\mathrm{RHom}_R(R_{\mathfrak{p}}, \mathrm{L}\Lambda^{\mathfrak{p}} M)) \\
&= \dim R_{\mathfrak{p}} + \sup H^*(P) \\
&\leq \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} + \sup H^*(M) \\
&\leq \dim R + \sup H^*(M). \quad \square
\end{aligned}$$

From the proof above one easily extracts the following rigidity result; for modules it was stated in the Abstract.

Porism 3.2. *Let \mathfrak{p} be a prime ideal in R and M an R -complex. If for an integer $n \geq \dim R + \sup H^*(M)$ one has $\mathrm{Ext}_R^n(k(\mathfrak{p}), M) = 0$, then $\mathrm{Ext}_R^i(k(\mathfrak{p}), M) = 0$ holds for all $i \geq n$.*

The bound on n in Theorem 3.1 and Porism 3.2 is sharp: Let (R, \mathfrak{m}, k) be a Cohen-Macaulay local ring that is not Gorenstein. One has

$$\inf \mathrm{Ext}_R^*(k, R) = \mathrm{depth} R = \dim R \quad \text{but} \quad \sup \mathrm{Ext}_R^*(k, R) = \mathrm{inj. dim}_R R = \infty.$$

This also shows that the bound on n in the next proposition is sharp. This statement is parallel to the first part of [7, Theorem 4.1] and could have been made in [7].

Proposition 3.3. *Let R be a commutative noetherian ring and M an R -complex. If for a prime ideal \mathfrak{p} and an integer $n \geq \dim R_{\mathfrak{p}} + \sup H^*(\mathrm{RHom}_R(R_{\mathfrak{p}}, M))$ one has $\mathrm{Ext}_R^n(k(\mathfrak{p}), M) = 0$ then*

$$\sup \mathrm{Ext}_R^*(k(\mathfrak{p}), M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{width}_{R_{\mathfrak{p}}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M) < n.$$

Proof. Hom-tensor adjunction in the derived category yields for every integer n an isomorphism

$$\mathrm{Ext}_R^n(k(\mathfrak{p}), M) \cong \mathrm{Ext}_{R_{\mathfrak{p}}}^n(k(\mathfrak{p}), \mathrm{RHom}(R_{\mathfrak{p}}, M)).$$

Thus, the assertions follow immediately from [7, Proposition 3.2] since one has $n \geq \dim R_{\mathfrak{p}} + \sup H^*(\mathrm{RHom}_R(R_{\mathfrak{p}}, M)) \geq \sup H_{\mathfrak{p}}^*(\mathrm{RHom}(R_{\mathfrak{p}}, M))$; see [7, (2.7)]. \square

4. FLAT DIMENSION

In this section we prove three statements that are dual to statements about injective dimension in the literature. Our proofs rely on the existence and structure of minimal semi-flat-cotorsion replacements, hence the assumption that the ring has finite Krull dimension. The first result below is a counterpart to [7, Theorem 5.1], and could have been stated even in [12].

Theorem 4.1. *Let R be a commutative noetherian ring of finite Krull dimension and M an R -complex with $\inf H_*(M) > -\infty$. If for an integer $n \geq \sup H_*(M)$ one has*

$$\mathrm{Tor}_{n+1}^{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M)) = 0 \quad \text{for every prime ideal } \mathfrak{p} \text{ in } R,$$

then the flat dimension of M is at most n .

Proof. Let F be a minimal semi-flat-cotorsion replacement of M ; the assumption $\inf H_*(M) > -\infty$ guarantees that $F_i = 0$ holds for $i \ll 0$, see 1.2. Per [12, Remark 4.5], for every integer i one has

$$F_i = \prod_{\mathfrak{p} \in \mathrm{Spec} R} \Lambda^{\mathfrak{p}} \left(R_{\mathfrak{p}}^{(B_i^{\mathfrak{p}})} \right),$$

where the cardinality of $B_i^{\mathfrak{p}}$ is the $\kappa(\mathfrak{p})$ -dimension of $\mathrm{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M))$. Thus it follows from the assumption that $|B_{n+1}^{\mathfrak{p}}| = 0$ holds for all primes \mathfrak{p} . As $n \geq \sup \mathrm{H}_*(M)$ holds, the flat dimension of M is at most n . \square

The boundedness condition in the next result, which is dual to [2, Proposition 5.3.I], is necessary, without it the flat dimension may grow under colocalization; see Example 5.1.

Proposition 4.2. *Let R be a commutative noetherian ring of finite Krull dimension and M an R -complex with $\inf \mathrm{H}_*(M) > -\infty$. The next equalities hold*

$$\begin{aligned} \mathrm{flat\,dim}_R M &= \sup \{i \in \mathbb{Z} \mid \mathrm{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M)) \neq 0 \text{ for some } \mathfrak{p} \in \mathrm{Spec} R\} \\ &= \sup_{\mathfrak{p} \in \mathrm{Spec} R} \{\mathrm{flat\,dim}_{R_{\mathfrak{p}}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M)\}. \end{aligned}$$

Proof. If M is acyclic, all three quantities equal $-\infty$, and so we may assume $\mathrm{H}_*(M)$ is nonzero. Set $s := \sup \mathrm{H}_*(M)$, one has $\mathrm{Tor}_s^{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M)) \neq 0$ for some prime ideal \mathfrak{p} in R , see [12, Remark 4.5]. The inequality

$$\mathrm{flat\,dim}_R M \leq \sup \{i \in \mathbb{Z} \mid \mathrm{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M)) \neq 0 \text{ for some } \mathfrak{p} \in \mathrm{Spec} R\}$$

now follows immediately from Theorem 4.1. To verify the opposite inequality, assume that $f := \mathrm{flat\,dim}_R M$ is finite and let F be a minimal semi-flat-cotorsion replacement of M . As $\inf \mathrm{H}_*(M) > -\infty$ holds, one has $F_i = 0$ for $i > f$ and $i \ll 0$, see 1.2. For every prime ideal \mathfrak{p} in R , the $R_{\mathfrak{p}}$ -complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$ is semi-flat-cotorsion and isomorphic to $\mathrm{RHom}_R(R_{\mathfrak{p}}, M)$ in the derived category over $R_{\mathfrak{p}}$. Since $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)_i = \mathrm{Hom}_R(R_{\mathfrak{p}}, F_i) = 0$ holds for $i > f$ one has

$$\begin{aligned} \sup \{i \in \mathbb{Z} \mid \mathrm{Tor}_i^{R_{\mathfrak{p}}}(k(\mathfrak{p}), \mathrm{RHom}_R(R_{\mathfrak{p}}, M)) \neq 0\} &\leq \mathrm{flat\,dim}_{R_{\mathfrak{p}}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M) \\ &\leq \mathrm{flat\,dim}_R M. \end{aligned} \quad \square$$

For complexes with bounded homology the next result was proved in [12, Theorem 4.8]; it compares to [7, Proposition 5.2 and Corollary 5.9], and the proof is modeled on the proof of [7, Proposition 5.2].

Theorem 4.3. *Let R be a commutative noetherian ring of finite Krull dimension and M an R -complex. If M has finite flat dimension, then*

$$\mathrm{flat\,dim}_R M = \sup_{\mathfrak{p} \in \mathrm{Spec} R} \{\mathrm{depth}_{R_{\mathfrak{p}}} - \mathrm{depth}_{R_{\mathfrak{p}}} \mathrm{RHom}_R(R_{\mathfrak{p}}, M)\}.$$

Proof. The equality is trivial if M is acyclic, so assume that it is not and set $f := \mathrm{flat\,dim}_R M$. Let F be a semi-flat replacement of M with $F_i = 0$ for $i > f$. To prove the asserted equality it suffices to show that the inequality

$$(2) \quad \mathrm{flat\,dim}_R F \geq \mathrm{depth}_{R_{\mathfrak{p}}} - \mathrm{depth}_{R_{\mathfrak{p}}} \mathrm{RHom}_R(R_{\mathfrak{p}}, F)$$

holds for every prime \mathfrak{p} with equality for some \mathfrak{p} . For every $n \leq f$ there is an exact sequence of complexes of flat R -modules

$$0 \longrightarrow F_{\leq n-1} \longrightarrow F \longrightarrow F_{\geq n} \longrightarrow 0.$$

The complex $F_{\geq n}$ is a bounded complex of flat modules, so it is semi-flat and hence so is $F_{\leq n-1}$; see for example [5, 6.1]. Evidently one has

$$\mathrm{flat\,dim}_R F_{\leq n-1} \leq n-1 \quad \text{and} \quad \mathrm{flat\,dim}_R F_{\geq n} = f.$$

Now Proposition 4.2 and [7, (2.3)] conspire to yield

$$(3) \quad f = \sup_{\mathfrak{p} \in \text{Spec } R} \{\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\geq n})\}.$$

To prove (2) we fix a prime ideal \mathfrak{p} . Without loss of generality we can assume that $\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F)$ is finite, set $d = -\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F)$. Now one has

$$-d \geq \inf H^*(\text{RHom}_R(R_{\mathfrak{p}}, F)) \geq \inf H^*(F) = -\sup H_*(F) \geq -f$$

and, thus, $d \leq f$. As one now has

$$\begin{aligned} \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\leq d-1}) &\geq \inf H^*(\text{RHom}_R(R_{\mathfrak{p}}, F_{\leq d-1})) \\ &\geq \inf H^*(F_{\leq d-1}) \\ &\geq 1 - d \\ &= 1 + \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F), \end{aligned}$$

the depth lemma yields

$$\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F) = \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\geq d}).$$

The inequality (2) follows from (3) by taking $n = d$.

Now choose by (3) a prime ideal \mathfrak{p} such that

$$(4) \quad f = \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\geq f-1})$$

holds. The inequality (2) holds for every complex of finite flat dimension; applied to the truncated complex $F_{\leq f-2}$ it yields

$$(5) \quad f - 2 \geq \text{flat dim}_R F_{\leq f-2} \geq \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\leq f-2}).$$

Elimination of $\text{depth } R_{\mathfrak{p}}$ between (4) and (5) yields

$$\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\geq f-1}) + 2 \leq \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\leq f-2}).$$

Now apply the depth lemma to the triangle

$$\text{RHom}_R(R_{\mathfrak{p}}, F_{\leq f-2}) \longrightarrow \text{RHom}_R(R_{\mathfrak{p}}, F) \longrightarrow \text{RHom}_R(R_{\mathfrak{p}}, F_{\geq f-1}) \longrightarrow$$

to get $\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F) = \text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, F_{\geq f-1})$; substituting this into (4) yields the desired result. \square

Remark 4.4. For an R -complex M of finite flat dimension, the equality

$$\text{flat dim}_R M = \sup_{\mathfrak{p} \in \text{Spec } R} \{\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}\}$$

holds, see [7, (4.3)]. Echoing [7, Remark 5.10] we remark that we do not know how the numbers $\text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ and $\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, M)$ compare: If (R, \mathfrak{m}, k) is local of positive Krull dimension and E is the injective envelope of k , then one has $\text{depth}_{R_{\mathfrak{p}}} \text{RHom}_R(R_{\mathfrak{p}}, E) = 0$ for every prime ideal \mathfrak{p} by the isomorphisms

$$\text{RHom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), \text{RHom}_R(R_{\mathfrak{p}}, E)) \simeq \text{RHom}_R(k(\mathfrak{p}), E) \simeq \text{Hom}_R(k(\mathfrak{p}), E) \neq 0.$$

On the other hand, for $\mathfrak{p} \neq \mathfrak{m}$ the module $E_{\mathfrak{p}}$ is zero and hence of infinite depth.

5. EXAMPLES

We close with a series of examples to show that boundedness conditions, such as the one in Proposition 4.2, are necessary in certain statements about semi-flat complexes. In particular, neither colocalization nor completion need preserve finiteness of flat dimension. The examples build on [12, Example 5.11].

Example 5.1. Let k be a field and consider the local ring $R = k[[x, y]]/(x^2)$; it has only two prime ideals: $\mathfrak{p} = (x)$ and $\mathfrak{m} = (x, y)$. By [12, Example 5.11] there exists a minimal semi-flat-cotorsion R -complex Y , called Y_P in [12], such that $\mathrm{Hom}_R(R_{\mathfrak{p}}, Y)$ is not semi-flat.

Set $F = (Y)_{\leq 0}$ and $F' = (Y)_{\geq 1}$, the hard truncations of Y above at 0 and below at 1, respectively. There is an exact sequence $0 \rightarrow F \rightarrow Y \rightarrow F' \rightarrow 0$ which is degreewise split. Application of $\mathrm{Hom}_R(R_{\mathfrak{p}}, -)$ now yields the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R_{\mathfrak{p}}, F) \longrightarrow \mathrm{Hom}_R(R_{\mathfrak{p}}, Y) \longrightarrow \mathrm{Hom}_R(R_{\mathfrak{p}}, F') \longrightarrow 0.$$

Since the complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, F')$ consists of flat modules and $\mathrm{Hom}_R(R_{\mathfrak{p}}, F')_i = 0$ holds for $i \ll 0$, it is semi-flat. Since $\mathrm{Hom}_R(R_{\mathfrak{p}}, Y)$ is not semi-flat, neither is the complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$.

It follows from [12, Theorem 2.3] that F is a minimal semi-flat-cotorsion complex; as $H_0(F) \neq 0$ it has flat dimension 0. We claim, however, that $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$ has infinite flat dimension. Suppose, to the contrary, that the complex $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$ has finite flat dimension. Choose a semi-flat resolution $P \rightarrow \mathrm{Hom}_R(R_{\mathfrak{p}}, F)$ with $P_i = 0$ for $i \gg 0$ and let C be its mapping cone. Evidently, C is an acyclic complex of flat R -modules and $C_i = 0$ holds for $i \gg 0$. We argue that C is pure-acyclic; that is, all its cycle modules are flat: For every integer i , let $Z_i = \mathrm{Ker}(C_i \rightarrow C_{i-1})$ be the cycle module in degree i . There is an exact sequence $0 \rightarrow Z_i \rightarrow C_i \rightarrow Z_{i-1} \rightarrow 0$. The ring R is Gorenstein of Krull dimension 1, so by a result of Bass [3, Corollary 5.6] the finitistic flat dimension of R is 1. As C_i is flat, it follows that Z_i is flat. As i was arbitrary, this shows that C is pure-acyclic. It now follows from [5, Theorem 7.3] that C is semi-flat. As C fits in a short exact sequence with the complexes P and $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$, of which the latter is not semi-flat, this is a contradiction. It follows that $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$ has infinite flat dimension.

One can draw the same conclusion as in the previous example about the \mathfrak{m} -completion of F :

Example 5.2. Let R and F be as in Example 5.1. By [12, (1.17)] there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R_{\mathfrak{p}}, F) \longrightarrow F \longrightarrow \Lambda^{\mathfrak{m}}(F) \longrightarrow 0.$$

As F is semi-flat of finite flat dimension, equal to 0, but $\mathrm{Hom}_R(R_{\mathfrak{p}}, F)$ is not semi-flat and does not have finite flat dimension, it follows that the complex $\Lambda^{\mathfrak{m}}(F)$ is not semi-flat and does not have finite flat dimension.

The examples above are dual to the examples in [7, Section 6], and we take this opportunity to clarify one of the statements made there.

Remark 5.3. Let (R, \mathfrak{m}, k) be local. It is stated in [7, Remark 6.2] that the support of the product $E_R(k)^{\mathbb{N}}$ is all of $\mathrm{Spec} R$, and that is correct though the argument provided in [7] is too brief to be accurate. Here is a complete argument:

Set $E = E_R(k)$ and for every $n \in \mathbb{N}$ let E_n denote the submodule

$$(0 :_E \mathfrak{m}^n) \cong \mathrm{Hom}_R(R/\mathfrak{m}^n, E).$$

It is the injective envelope of the artinian ring R/\mathfrak{m}^n , so one has $(0 :_R E_n) = \mathfrak{m}^n$. Indeed, for x in R the isomorphism $R/\mathfrak{m}^n \cong \text{Hom}_{R/\mathfrak{m}^n}(E_n, E_n) \cong \text{Hom}_R(E_n, E_n)$ identifies the homothety $E_n \xrightarrow{x} E_n$ with the coset $x + \mathfrak{m}^n$ in R/\mathfrak{m}^n . Each submodule E_n has finite length; in particular, it is generated by elements $e_{n,1}, \dots, e_{n,m_n}$. Now let e be the family of all these generators in the countable product $\prod_{n \in \mathbb{N}} \prod_{i=1}^{m_n} E_n$. It follows from Krull's intersection theorem that the homomorphism $R \rightarrow E^{\mathbb{N}}$ given by $1 \mapsto e$ is injective, since an element in the kernel annihilates E_n for every $n \in \mathbb{N}$ and hence belongs to the intersection $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n$. As localization is exact, $R_{\mathfrak{p}}$ is now a non-zero submodule of $(E^{\mathbb{N}})_{\mathfrak{p}}$ for every prime ideal \mathfrak{p} in R .

Let R be a commutative noetherian ring, P a projective R -module, and F a semi-flat R -complex. As products of flat R -modules are flat, $\text{Hom}_R(P, F)$ is a complex of flat R -modules. If $F_i = 0$ holds for $i \ll 0$, then the complex $\text{Hom}_R(P, F)$ satisfies the same boundedness condition, whence it is semi-flat. Without this boundedness condition the conclusion may fail.

Example 5.4. Let R be the ring and F the semi-flat R -complex from Example 5.1. There exist projective R -modules P such that $\text{Hom}_R(P, F)$ is not semi-flat: As R is Gorenstein of Krull dimension 1, the finitistic projective dimension of R is 1 by [3, Corollary 5.6]. It follows that there is an exact sequence $0 \rightarrow P_1 \rightarrow P_0 \rightarrow R_{\mathfrak{p}} \rightarrow 0$ where P_1 and P_0 are projective R -modules. As F is a complex of flat-cotorsion modules, in particular modules that are Ext^1 -orthogonal to $R_{\mathfrak{p}}$, it yields an exact sequence

$$0 \longrightarrow \text{Hom}_R(R_{\mathfrak{p}}, F) \longrightarrow \text{Hom}_R(P_0, F) \longrightarrow \text{Hom}_R(P_1, F) \longrightarrow 0.$$

Assume towards a contradiction that $\text{Hom}_R(P_1, F)$ and $\text{Hom}_R(P_0, F)$ are both semi-flat R -complexes. As $F_i = 0$ holds for $i > 0$, it follows that both complexes have finite flat dimension, at most 0, and hence so has $\text{Hom}_R(R_{\mathfrak{p}}, F)$. This contradicts the conclusion in Example 5.1 that $\text{Hom}_R(R_{\mathfrak{p}}, F)$ has infinite flat dimension.

For the ring $R = k[[x, y]]/(x^2)$ with $\mathfrak{p} = (x)$ from Example 5.1 one has $R_{\mathfrak{p}} = R_y$, so it follows from [12, Example 1.6] that the modules P_0 and P_1 can be chosen as countable direct sums of copies of R . Thus, F is an example of a semi-flat complex such that the product $F^{\mathbb{N}}$ is not semi-flat. Compare this to the fact that for a semi-injective complex I the coproduct $I^{(\mathbb{N})}$ need not be semi-injective; see Iacob and Iyengar [11, Theorem 2.8].

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