Thm

$$\mathrm{Hom}(M \otimes N, X) \xrightarrow[g]{\cong} \mathrm{Hom}(M, \mathrm{Hom}(N, X))$$

$$g(\varphi)(m)(n) = \varphi(m \otimes n)$$

$$g^{-1}(\psi)(m \otimes n) = \psi(m)(n)$$

Claim: The tensor product is right exact

$$M' \otimes X \to M \otimes X \to M'' \otimes X \to 0$$

$$0 \to \mathrm{Hom}(M'' \times X, E) \to \mathrm{Hom}(M \otimes X, E)$$
$$\to \mathrm{Hom}(M' \otimes X, E)$$

$$\mathrm{Hom}(M'', \mathrm{Hom}(X, E)) \to \cdot$$

$$0 \to (M'', E) \to (M, E) \to (M', E) \to 0$$

$$\begin{tikzcd}
R^{a_1} \arrow[r, "\alpha"] & R^{a_0} \arrow[r] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, "\gamma'"] & 0 \\
R^{b_1} \arrow[r, "\beta"] & R^{b_0} \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

$\operatorname{Im} \gamma\alpha \subseteq \operatorname{Im} \beta$

<u>Ex</u> $R = k[x, y, z]$

$$\begin{tikzcd}
R^2 \arrow[r, "{\begin{bmatrix} x & y \\ y & z \end{bmatrix}}"] & R^2 \arrow[r] \arrow[d, "\gamma"] & M \arrow[r] \arrow[d, "\gamma'"] & 0 \\
R \arrow[r, "(y)"] & R \arrow[r] & N &
\end{tikzcd}$$

$\gamma(x\varepsilon_1 + y\varepsilon_2) \subseteq \operatorname{Im}[y]$

$\gamma(y\varepsilon_1 + z\varepsilon_2) \subseteq \operatorname{Im}[y]$

$x\gamma(\varepsilon_1) + y\gamma(\varepsilon_2) \in R\langle y\rangle$

$y\gamma(\varepsilon_1) + z\gamma(\varepsilon_2) \in R\langle y\rangle$

$x\gamma(\varepsilon_1) \in R\langle y\rangle$

$x\gamma(\varepsilon_2) \in R\langle y\rangle$

$\therefore \gamma' = 0$

$$R \xrightarrow{[\quad]} R^{a_0} \otimes R^{b_0} \longrightarrow M \otimes N \longrightarrow 0$$

$a \in \alpha(R^{a_1})$

$a \otimes r \longmapsto 0$

$b \in \operatorname{im} \beta.$

$r \otimes b \longmapsto 0$