Homework 1
Math 5316
Fall 2013 Due 30 Aug at 4 pm

1. Are the following matrices on row echelon form (Yes/No)?

$$\text{(a)}\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{(b)}\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(c)}\ \begin{pmatrix} 1 & 4 & 0 & 9 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(d)}\ \begin{pmatrix} 1 & 5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2. Write down the conjugate transpose A^* of the matrix

$$A = \begin{pmatrix} 8i & -3 & -2 & 3 \\ 4+i & -3 & -9 & 2 \\ 3 & 0 & 0 & 1 \\ 1 & -2 & 6 & 1-2i \end{pmatrix}.$$

3. Find all solutions to the equation $2x - 6y - 8z = 14$ and express them on the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \end{pmatrix} + \alpha \begin{pmatrix} \\ \\ \\ \end{pmatrix} + \beta \begin{pmatrix} \\ \\ \\ \end{pmatrix}.$$

4. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix}$$

and use it to solve the equations

$$AX = \begin{pmatrix} 8 \\ 1 \\ -1 \end{pmatrix} \qquad AX = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$$

5. Consider the matrix

$$A = \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix}.$$

   (a) Find all real numbers λ such that A is invertible.
   (b) Find all complex numbers λ such that A is invertible.

6. Decide if the following matrix is invertible and, if so, find its inverse.

$$A = \begin{pmatrix} i & 1+i \\ 2+i & 3+i \end{pmatrix}.$$

7. Recall that a matrix A is called symmetric if $A = A^T$ holds. Show that the symmetric 3×3 matrices form a subspace V of $M_{3,3}(\mathbb{R})$, the vector space of 3×3 matrices, and find a basis for V.

8. Let A and B be similar $n \times n$ matrices. Show that one has $\det A = \det B$.

9. Decide which, if any, of the following sets of vectors form bases for $\mathbb{R}^3$. You must justify your answer.

   (a) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$, and $\begin{pmatrix} 6 \\ 0 \\ -15 \end{pmatrix}$

   (d) $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

   (e) $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$, and $\begin{pmatrix} -3 \\ 8 \\ -1 \end{pmatrix}$

10. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 4 \\ 2 & 0 & 3 \end{pmatrix}.$$

   (a) Compute $\det A$.

   (b) Find the adjoint matrix $\operatorname{adj} A$.

   (c) Find the inverse matrix A^{-1}.

11. Let $\mathbb{R}_3[x]$ denote the vector space of polynomails with real coefficients and degree at most 2. Decide if the map $T\colon \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ given by $T(p(x)) = xp'(x)$ is linear, and if so find its representation with respect to the standard basis $(x^2, x, 1)$.

12. Consider the matrix

$$C = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

   (a) Find the rank of C.

   (b) Find a basis for the row space of C.

   (c) Find a basis for the column space of C.

   (d) Find a basis for the null space of C.

13. Find all solutions to the following system of equations

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1. \end{aligned}$$

14. For $n \times n$ matrices set $[A, B] = AB - BA$.

   (a) Show that $[A, I] = 0$ holds for all $n \times n$ matrices A.

   (b) Show that $[B, A] = -[A, B]$ holds for all $n \times n$ matrices A and B.

   (c) Show that $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ holds for all $n \times n$ matrices A, B, and C.