① 4/24/2020

<u>Recall:</u> An nontrivial proper ideal I
- is <u>prime</u> if $ab \in I \Rightarrow a \in I$ or $b \in I$.
- is max'l if no proper ideal strictly contains it $I \subseteq J \subseteq R \Rightarrow J = I$ or $J = R$

<u>Thm</u> Let $I \subseteq R$ be an ideal and commutative with 1.

(1) I is prime $\Leftrightarrow$ R/I is integral domain

(2) I is ~~a field~~ max'l $\Leftrightarrow$ R/I is a field.

<u>Pf</u> (1)

$$(a+I)(b+I) = 0 + I = I$$
$$\Updownarrow$$
$$ab + I = I$$
$$\Updownarrow$$
$$ab \in I$$

If I is prime and $ab \in I$, then $a \in I$ or $b \in I$ so $a + I = I$ or $b + I = I$. I.e. R/I domain.

If R/I is domain and $ab \in I$ then $a + I = I$ or $b + I$ is I so $a \in I$ or $b \in I$.

②

(2)

$$\begin{array}{ccc} R & & R/J \\ \Big| J & \text{———} & J/I \\ I & & 0 = I/I \end{array}$$

If R/I is field it has only the trivial ideals, so there are no ideals between I and R.

If I max'l R/I has only the trivial ideals, so R/I a field.

$$\mathbb{Z} \xrightarrow{x \mapsto x/1} \mathbb{Q}$$

domain
not field

On $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ an equivalence relation $\sim$ is given by

$$(a,b) \sim (c,d) \Leftrightarrow ad = bc$$

The elt's $a/b \in \mathbb{Q}$ are the equivalence classes $a/b = [(a,b)]$

$$1/2 = 3/6 \; : \; 1(6) = (2)3$$
$$[(1,2)] = [(3,6)]$$

(3)

$[7]_5 = [2]_5$

$3/6 = 1/2$

Let D be a domain. On

$S = \{(a,b) \mid a \in D, b \in D, b \neq 0\}$

define $(a,b) \sim (c,d)$ if $ad = bc$

Claim: $\sim$ is equivalence relation

Reflexive: $(a,b) \overset{?}{\sim} (a,b)$

$ab = ba$ ✓ b/c D commutative.

Symmetric: $(a,b) \sim (c,d)$

$\therefore$ $ad = bc$ ← D commut.

$\therefore$ $da = cb$

$\therefore$ $cb = da$ i.e. $(c,d) \sim (a,b)$

Transitive: $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$

$\therefore$ $ad = \cancel{cd}\ bc$ and $cf = de$

Thus: $adf = bcf = bde$

$\therefore$ $daf = dbe$

$\therefore$ $d(af - be) = 0$

$\therefore$ $af - be = 0$ D domain

(4)

<u>Def</u> Denote the equivalence class of (a,b) in S by a/b or $\frac{a}{b}$

<u>Lemma</u> Addition and mult:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

and $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

are welldefined operations on S.

<u>Proof</u>: Let $\frac{a}{b} = \frac{a'}{b'}$ $(ab' = ba')$

$\frac{c}{d} = \frac{c'}{d'}$ $(cd' = dc')$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \; ; \; \frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd'+b'c'}{b'd'}$$

$$\begin{aligned} b'd'(ad+bc) &= b'd'ad + b'd'bc \\ &= b'ad'd + \cancel{b'} \cdot d'cb'b \\ &= ba'd'd + dc'b'b \\ &= bd(a'd' + c'b') \end{aligned}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \; ; \; \frac{a'}{b'} \cdot \frac{c'}{d'} = \frac{a'c'}{b'd'}$$

$$acb'd' = ab'dc' = ba'dc' = bda'c'$$

(5)

Denote by F the set of equ. classes in S with + and · as defined.

<u>claim</u> F is a field.

\+ commutative: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

$$= \frac{da+cb}{db} = \frac{cb+da}{db}$$

$$= \frac{c}{d} + \frac{a}{b}$$

$$\frac{0}{1} + \frac{a}{b} = \frac{0b+1a}{b} = \frac{a}{b}$$

$$\frac{0}{1} = \frac{0}{c} \quad \cancel{\forall \neq} \ \forall c \neq 0$$

identity for mult is $\frac{1}{1}$:

$$\frac{1}{1}\frac{a}{b} = \frac{a}{b}$$

For $\frac{a}{b}$ with $a \neq 0$

$$\frac{a}{b}\frac{b}{a} = \frac{ab}{ab} = \frac{1}{1}$$

⑥

Lemma The subset $R := \{ \frac{a}{1} \mid a \in D \}$ of F is a subring isomorphic to D.

Pf

$\cancel{\varphi : R \longrightarrow}$

$\varphi : D \longrightarrow F$

$\varphi(a) = \frac{a}{1}$

φ is ring homo'sm

$$\varphi(a+b) = \frac{a+b}{1} = \frac{a1 + b1}{1 \cdot 1}$$

$$= \frac{a}{1} + \frac{b}{1} = \varphi(a) + \varphi(b)$$

$$\varphi(ab) = \frac{ab}{1} = \frac{a}{1} \cdot \frac{b}{1} = \varphi(a)\varphi(b)$$

The image φ is a subring

$\varphi(D) = R$

$\varphi : D \to R$ is surjective

and 1-to-1:

$$\varphi(a) = \varphi(b) \Rightarrow \frac{a}{1} = \frac{b}{1}$$

$$\Rightarrow a(1) = (1)b$$

$$\Rightarrow a = b$$