(1) 4/22/2020

$I \subseteq R$ an ideal, the set of cosets $r+I$ form a ring with operations:

$$(r+I)(r'+I) = (rr'+I)$$
$$(r+I)+(r'+I) = (r+r')+I$$

R/I is called a quotient ring.

<u>Prop.</u> If R is commutative with mult. identity 1, then R/I is commutative with mult. identity $1+I$.

<u>Pf</u>:
$$(a+I)(b+I) = ab+I = ba+I = (b+I)(a+I)$$

$$(1+I)(a+I) = (1a)+I = a+I.$$

<u>Ex</u> $\mathbb{Z}$ is an integral domain, but $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}_6$ is <u>not</u>!

<u>First Iso. Thm.</u> Let $\varphi : R \to S$ be a ring hom'sm. There is an iso. of rings:

$$R/\ker\varphi \cong \varphi(R)$$

<u>Pf</u> ~~χ~~: $\ker\varphi =: K$

$$\begin{aligned}\bar{\varphi} : R/K &\longrightarrow \varphi(R)\\ r+K &\longmapsto \varphi(r)\end{aligned}$$

$\bar{\varphi}$ is a group iso.

$$\begin{aligned}&\bar{\varphi}((r+K)\cdot(r'+K)) =\\ &\bar{\varphi}(rr'+K) = \varphi(rr') = \varphi(r)\varphi(r')\\ &= \bar{\varphi}(r+K)\,\bar{\varphi}(r'+K) \qquad \square\end{aligned}$$

<u>Thm</u> For every ideal $K \subseteq R$ there is a ring homomorphism χ ~~th~~: $R \to R/K$ with $\ker \cancel{A}\,\chi = K$.

<u>Pf</u> $\chi : R \to R/K$ given by $\chi(r) = r+K$ is a group hom'sm with $\ker\chi = K$.

$$\begin{aligned}\chi(rr') = rr'+K &= (r+K)(r'+K)\\ &= \chi(r)\,\chi(r') \qquad \square\end{aligned}$$

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Remark Given

$$\chi : R \to R/K$$

there is a 1-to-1 correspondance

$$\text{ideals } I \supseteq K \longleftrightarrow \text{ideals in } R/K$$

$$I \subseteq R \longrightarrow \{i+k \mid i \in I\} \subseteq R/K$$

See 7.2.19

Def A non-trivial ~~ide~~ proper ideal $I \subset R$ is called prime if for all $a, b \in R$ one has

$$ab \in I \text{ only if } a \in R \text{ or } b \in R.$$

Ex: The prime ideals in $\mathbb{Z}$ are $p\mathbb{Z} = \langle p \rangle$ for prime numbers p. Indeed:

If p is prime and $ab \in \langle p \rangle$, then $ab = pk$, i.e. $p \mid ab$ so $p \mid a$ or $p \mid b$. I.e. $a \in \langle p \rangle$ or $b \in \langle p \rangle$.

If $n\mathbb{Z} = \langle n \rangle$ ~~is prime~~ and $n = uv$ where $\pm u \neq 1 \neq \pm v$, then $uv \in \langle n \rangle$ but u or v does not.

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Ex $6\mathbb{Z} \ni 6 = 2\cdot 3$, $2 \notin 6\mathbb{Z}$, $3 \notin 6\mathbb{Z}$

Def^n A non-trivial proper ideal I in R is called maximal if R is the only ideal that properly contains it.

$$I \subseteq J \subseteq R \Rightarrow J = I \text{ or } J = R$$

Ex The max'l ideals of $\mathbb{Z}$:

If $\langle p \rangle$ is a prime ideal

and $\langle p \rangle \subseteq \langle x \rangle \subseteq \mathbb{Z}$

$p \in \langle p \rangle \subseteq \langle x \rangle = \{ rx \mid r \in \mathbb{Z} \}$

$\therefore$ $p = rx$

$\therefore$ r or x is ± 1 — $\langle p \rangle = \langle x \rangle$

" " " is $\pm p$ — $\langle x \rangle = \langle 1 \rangle = \mathbb{Z}$

$\langle n \rangle$ for $\langle n \rangle$ not prime.

$\langle n \rangle = \langle uv \rangle \subseteq \langle u \rangle$, $\langle n \rangle \subset \langle u \rangle$

if $v = \pm 1$.

Ex $\langle 2 \rangle \supseteq \langle 6 \rangle \subseteq \langle 3 \rangle$

$\therefore$ In $\mathbb{Z}$ prime ideals are the same as max'l ideals.

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In $\mathbb{Z}\times\mathbb{Z}$ the ideal $\mathbb{Z}\times\langle 0\rangle$ is prime: $(a_1,a_2)(b_1,b_2) = (a_1b_1, a_2b_2)$ is in $\mathbb{Z}\times\langle 0\rangle$, then $a_2=0$ or $b_2=0$ as $\mathbb{Z}$ is an int. domain.

so $(a_1,a_2)\in\mathbb{Z}\times\langle 0\rangle$ or $(b_1,b_2)\in\mathbb{Z}\times\langle 0\rangle$

$\therefore$ $\mathbb{Z}\times\langle 0\rangle$ is prime, but

$$\mathbb{Z}\times\langle 0\rangle\subset\mathbb{Z}\times\langle 5\rangle\subset\mathbb{Z}\times\mathbb{Z}$$

<u>Thm</u> Let R be a comm. ring with 1

(a) An ideal $I\subseteq R$ is prime iff R/I is a domain.

(b) R/I is a field iff I is max'l.