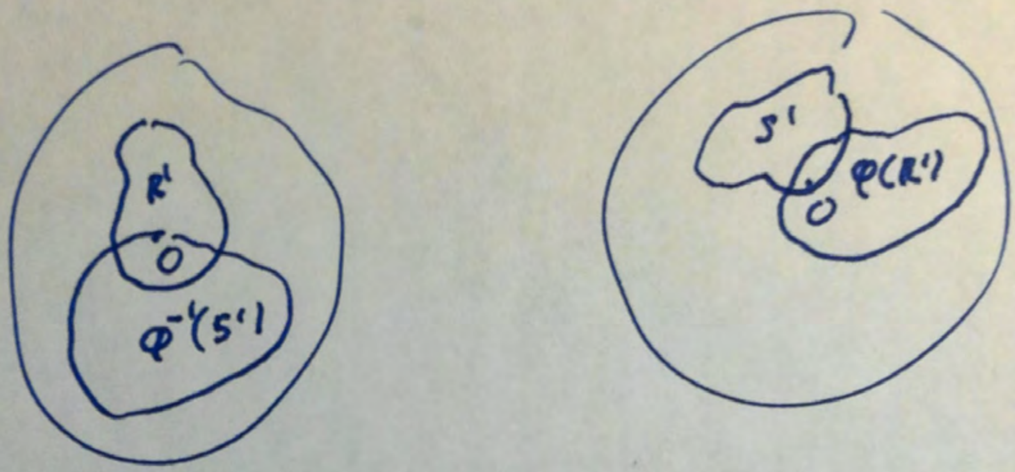
① 4/20/2020

$$R \xrightarrow{\varphi} S$$



<u>Ex</u> $R = \mathbb{Z}$ $S = \mathbb{Q}$

$\varphi : z \longmapsto z$

$\therefore$ $\mathbb{Z}$ is a sub ring of $\mathbb{Q}$

<u>Ex</u> $R = 2\mathbb{Z}$ $S = \mathbb{Z}$

$\varphi : 2z \longmapsto 2z$

Note: $\forall a \in \mathbb{Z}$ $a(2z) = 2az \in 2\mathbb{Z}$

$\frac{1}{5} \in \mathbb{Q}$, $1 \in \mathbb{Z}$ $\left(\frac{1}{5}\right)1 \notin \mathbb{Z}$

Def^n Let R be a ring. $I \subseteq R$ is an ideal if:

(1) I is a subring

(2) For all $x \in I$ and all $r \in R$
$rx \in I$ and $xr \in I$.

Aim: $R/I = \{\text{cosets of } I\}$
$a + I, \; a \in R$

Prop: Let $\varphi: R \to S$ be a ring-homom'sm. $\operatorname{Ker}\varphi = \{r \in R \mid \varphi(r) = 0\}$ is an ideal in R

Pf: $\operatorname{Ker}\varphi$ is a subgroup of $\langle R, + \rangle$.

Let $x \in \operatorname{Ker}\varphi$ and $r \in R$:

$$\varphi(rx) = \varphi(r)\varphi(x) = \varphi(r) \cdot 0 = 0$$

$$\varphi(xr) = \varphi(x)\varphi(r) = 0 \cdot \varphi(r) = 0.$$

<u>Ex</u> Ideals of $\mathbb{Z}$.

$I \subseteq \mathbb{Z}$

$I = \langle n \rangle = n\mathbb{Z}$.

Let $x = nz \in n\mathbb{Z}$

$a \in \mathbb{Z}$

$a(nz) = (nz)a = n(za) \in n\mathbb{Z}$

<u>Def</u> Let R be a commutative ring (with 1.). An ideal of the form

$$\langle a \rangle = \{ra \mid r \in R\} = Ra = aR$$

is called a principle ideal.

Check that $\langle a \rangle$ is an ideal:

$ra + sa = (r+s)a \in \langle a \rangle$

$ra + (-ra) = 0$, $-ra = (-r)a \in \langle a \rangle$

Let $q \in R$: $q(ra) = (qr)a \in \langle a \rangle$

$(ra)q = q(ra)$

④

Ex Ideals of $\mathbb{Q}$: $\{0\}$, $\mathbb{Q}$

$\{0\} = \langle 0 \rangle$ $\quad \mathbb{Q} = \langle 1 \rangle$

$I \subseteq \mathbb{Q}$ take $x \in \mathbb{Q}$

If $x \neq 0$, then $x^{-1}(x) \in I$

$x^{-1}x = 1 \quad \therefore \quad I = \mathbb{Q}$.

Prop. Let R be ~~&~~ commutative with 1.

R is a field if and only if $\langle 0 \rangle$

and $\langle 1 \rangle = R$ are the only ideals in R.

Pf: "Only if": Assume R a field.

Let $I \subseteq R$ be an ideal. If $I \neq \langle 0 \rangle$,

then let $x \neq 0$ be an elt of I.

As $x^{-1} \in R$ and I an ideal $1 = x^{-1}x \in I$.

So $r1 = r \in I$ for all $r \in R$.

"If": Let $x \neq 0$ be an elt. of R.

Now $\langle x \rangle \neq \langle 0 \rangle$ so $\langle x \rangle = R$

$\therefore \quad 1 \in \langle x \rangle \quad \therefore \quad 1 = rx$ for some $r \in R$

$\therefore \quad x$ is a unit! $\quad \square$

(5) $5\mathbb{Z} = \langle 5 \rangle$ is a normal subgroup of $\mathbb{Z}$.

$$\mathbb{Z}/5\mathbb{Z} = \{a + 5\mathbb{Z} \mid a \in \mathbb{Z}\}$$

is a group under $+$.

$\mathbb{Z}/5\mathbb{Z}$ is $\mathbb{Z}_5$ as abelian group.

$$1 + 5\mathbb{Z} = \{\ldots, -9, -4, 1, 6, 11 \ldots\}$$

$\mathbb{Z}_5$ is a ring with $[x][y] = [x+y]$

$$(x + 5\mathbb{Z})(y + 5\mathbb{Z}) = xy + 5\mathbb{Z}$$

$$\parallel$$

$$xy + \underbrace{x5x + y5\mathbb{Z} + 5\mathbb{Z}\cdot 5\mathbb{Z}}_{5\mathbb{Z}}$$

<u>Lemma</u> Let I be a ~~subring~~ subring of R.
I is an ideal if and only if the opn mult. $(a+I)(b+I) = ab+I$ is a well-def. operation on cosets.

(6) Pf Let $I \subseteq R$ be an ideal.

If $a+I = a'+I$ then $a-a' =: i \in I$
If $b+I = b'+I$ then $b-b' = j \in I$

$$ab = (a'+i)(b'+j)$$
$$= a'b' + a'j + ib' + ij$$

$$\therefore \; ab - a'b' = \underbrace{a'j}_{I} + \underbrace{ib'}_{I} + \underbrace{ij}_{I} \in I$$

$$\therefore \; ab + I = a'b' + I$$

Thm. Let $I \subseteq R$ be an ideal.
The set R/I is a ring under

$$(a+I) + (b+I) = (a+b) + I$$
$$(a+I)(b+I) = ab + I$$

Pf We know that R/I is a group under this addition, and mult is well-defined.

(7)

Assoc:

$$((a+I)(b+I))(c+I) = (ab+I)(c+I)$$

$$= (ab)c + I \underset{\substack{\uparrow \\ \text{assoc in } R}}{=} a(bc) + I$$

$$= (a+I)(bc+I)$$

$$= (a+I)((b+I)(c+I))$$

Distribution:

$$(a+I)((b+I)+(c+I)) =$$

$$(a+I)((b+c)+I) =$$

$$a(b+c)+I = (ab+ac)+I$$

$$= (ab+I)+(ac+I)$$

$$= (a+I)(b+I) + (a+I)(c+I). \quad \square$$