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4/17/2020

$$R \xrightarrow{\varphi} S$$
$$+_R \ \cdot_R \qquad +_S \ \cdot_S$$

A function $\varphi : R \to S$ is a ring homomorphism if:

$$\varphi(x +_R y) = \varphi(x) +_S \varphi(y)$$
$$\varphi(x \cdot_R y) = \varphi(x) \cdot_S \varphi(y)$$

<u>Ex</u> $\varphi : \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(x) = 2x$

$$\varphi(x+y) = 2(x+y) = 2x + 2y = \varphi(x) + \varphi(y)$$

$\therefore$ φ is a group homo'sm.

$$\cancel{\varphi(*)} \quad \varphi(1) = 2$$
$$\varphi(1 \cdot 1) = \varphi(1)\varphi(1) = 2(2) = 4$$

$\therefore$ φ is not a ring homo'sm

A ring homo'sm $\mathbb{Z} \to \mathbb{Z}$ satisfies

$$\varphi(n+m) = \varphi(n) + \varphi(m)$$
$$\varphi(nm) = \varphi(n)\varphi(m)$$

(2) φ is determined by $\varphi(1)$

$$\therefore \quad \varphi(n) = \varphi(\underbrace{1+1+\dots+1}_{n}) = \underbrace{\varphi(1)+\varphi(1)+\dots+\varphi(1)}_{n}$$

$$\varphi(1) = \varphi(1\cdot 1) = \varphi(1)\varphi(1) = \varphi(1)^2$$

$$\therefore \quad \varphi(1) = \begin{cases} 1 \\ 0 \end{cases}$$

__Ex__ $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_{12}$

φ is given by $\varphi(1)$

$|\varphi(1)| \mid |1|$

$\therefore \ \varphi(1) \mid 6$

$\varphi(1) = a \in \mathbb{Z}_{12}$

$|a| \in \{1, 2, 3, 6\}$

$a \in \{0, 6, 4, 8, 2, 10\}$

Also: $\varphi(1) = \varphi(1)^2$

in $\mathbb{Z}_{12}$:

$0^2 = 0$ ✓

$2^2 = 4$ ÷

$4^2 = 4$ ✓

$6^2 = 0$ ÷

$8^2 = 4$ ÷

$10^2 = 4$

$\varphi(1) = 4$ or

$\varphi(1) = 0$

<u>Def</u>: Let $\varphi: R \to S$ be a ring homo'sm. The kernel of φ is

$$\ker \varphi = \{ r \in R \mid \varphi(r) = 0 \}$$

Prop. 7.1.9.

Properties of ring homo'sm $\varphi: R \to S$

Group homo'sm:

(a) $\varphi(0) = 0$

(b) $\varphi(-a) = -\varphi(a) \quad \forall a \in R$

(c) $\varphi(na) = n\varphi(a), \ \forall n \in \mathbb{Z} \quad \forall a \in R$

(d) φ is 1-to-1 if and only if $\ker \varphi = \{0\}$

(e) $\varphi(a^n) = \varphi(a)^n, \ \forall a \in R, \ \forall n \in \mathbb{N}$

(f) If $R' \subseteq R$ is a subring, then $\varphi(R') = \{ \varphi(a) \mid a \in R' \} \subseteq S$ is a subring

(g) If $S' \subseteq S$ is a subring, then $\varphi^{-1}(S') = \{ r \in R \mid \varphi(r) \in S' \}$ is a subring of R.

(h) $\ker \varphi$ is a subring of R

(i) If R has 1 (id. for mult) then $\varphi(1)$ is an identity for mult. in ~~S~~ $\varphi(R)$

(j) If R is commutative, then $\varphi(R)$ is commutative.

(k) If R has 1, then for $a \in U(R)$ $\varphi(a^n) =$ $= \varphi(a)^n$ $\forall n \in$

(4) Ex.

$$2x^3 - 5x^2 + 7x - 8 = 0$$

Does this eq. have a solution in $\mathbb{Z}$?

if $a \in \mathbb{Z}$ is a solution, then $[a] \in \mathbb{Z}_3$ solves the eq. in $\mathbb{Z}_3$.

$$\therefore \quad 2a^3 + a^2 + a + 1 = 0 \quad \text{in } \mathbb{Z}_3$$

Indeed: $\varphi : \mathbb{Z} \to \mathbb{Z}_3$ given by
$$x \mapsto [x]$$

is a ring homomorphism and

$\varphi(0) = 0$

$$\varphi(2a^3 - 5a^2 + 7a - 8) = 2a^3 + a^2 + a + 1$$

$a = 0$ not a solution

$a = 1$ —

$a = 2$ —

<u>Def^n</u> A ring homo'sm that is bijective is called an isomorphism of rings.

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$\langle (123) \rangle \leq S_3$

$\langle [4] \rangle \leq \mathbb{Z}_{12}$

<u>Prop.</u> If R and S are isomorphic rings, then

(1) R has an identity 1 for mult iff S has an identity for mult

(2) R commutative iff S commutative

(3) R integral domain iff S is an integral domain.

(4) S is a field iff R is a field.

<u>Ex</u> (From homework)

$$\left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \middle| (x,y) \in \mathbb{R}\times\mathbb{R} \right\} \cong \mathbb{C}$$

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \longleftrightarrow x+iy$$

— x —

proof of prop. (i).

$1 \in R$. $x \in \varphi(R) \Rightarrow x = \varphi(r)$

$\cancel{\varphi(1)\varphi(}$ $\varphi(1)x = \varphi(1)\varphi(r) = \varphi(1r)$

$= \varphi(r) = x$

$x\varphi(1) = \varphi(r)\varphi(1) = \varphi(r1) = x$