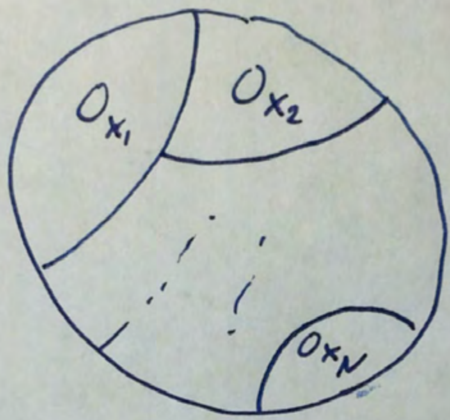
① 4/3/2016

G - a group acts on X - a set

$X_g = \{x \in X \mid g \cdot x = x\}$

$G_x = \{g \in G \mid g \cdot x = x\}$

$G_x \leq G$



N = # orbits

G finite: $N = \frac{1}{|G|} \sum_{g \in G} |X_g|$ $\qquad |O_x| = \frac{|G|}{|G_x|}$

— x —

<u>Ex</u>: $G = S_3$ acts on $X = S_3$ by conjugation: $g \cdot x = gxg^{-1}$

$g \cdot g = g$

$g^{-1} \cdot g = g$

② 4/3/2020

$O_{(1)} = \{(1)\}$

$O_{(12)} = \{(12), (23), (13)\}$

$G_{(12)}$ contains $(1), (12)$

$3 = |O_{(12)}| = \frac{|G|}{|G_{(12)}|} = \frac{6}{2}$

$O_{(123)} = \{(123), (132)\}$

$|G_{(123)}| = \frac{|G|}{|O_{(123)}|} = \frac{6}{2} = 3$

$G_{(123)} = \{(1), (123), (132)\}$

$G_{(12)} = \{(1), (12)\}$

$(13)(12)(13)^{-1}$
$= (13)(12)(13)$
$= (1)(23)$

$(23)(12)(23)$
$= (13)$

$(12)(123)(12) =$
(132)

3

4/3/2020

Def^n Let G act on G by conjugation.
The orbit O_a for $a \in G$ is called
the conjugacy class of a and
written $K_G(a)$ (or just $K(a)$).
Elements $b \in K_G(a)$ are called
conjugates of a.

$b \in K_G(a)$

Also, the stabilizer

$$G_a = \{g \in G \mid gag^{-1} = a\}$$
$$= \{g \in G \mid ga = ag\} = C_G(a)$$

is the centralizer

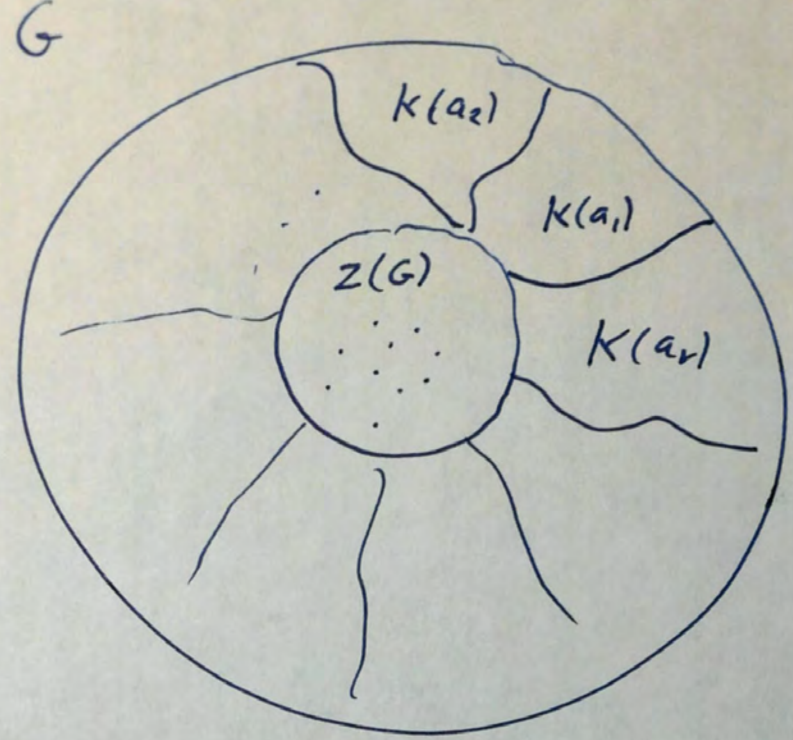
Prop: Let G be a finite group acting
on itself by conjugation. For $a \in G$:

$$|K(a)| = \frac{|G|}{|C(a)|}$$

Pf: $K(a) = O_a$
$C(a) = G_a$

$$\therefore \frac{|G|}{|C(a)|} = [G : G_a] = |O_a|$$

4/3/2020



For $z \in Z(G)$: $gzg^{-1} = zgg^{-1} = z$

$|K(z)| = \underline{1}$

<u>Thm</u>: Let G be a finite group. Let $a_1, \dots, a_r$ be a complete set of representatives for the conjugacy classes of elt's in $G \setminus Z(G)$. One has

$$|G| = |Z(G)| + \sum_{i=1}^{r} |K(a_i)|$$
$$= |Z(G)| + \sum_{i=1}^{r} [G : C_G(a_i)]$$

4/3/2020

<u>Thm</u> Every p-Group has a non-trivial center. Indeed, if $|G| = p^n$, then $|Z(G)| = p^k$, $1 \leq k \leq n$.

<u>Proof</u>

$$p^n = |G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(a_i)]$$

$$[G : C_G(a_i)] = \frac{|G|}{|C_G(a_i)|} = p^s$$

$\therefore \quad p \mid |G|$

$p \mid \sum [G : C_G(a_i)]$

$\therefore \quad p \mid Z(G).$

<u>Corollary</u> Let G be a group. If $|G| = p^2$ for some prime p, then $G \cong \mathbb{Z}_{p^2}$ or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$. In particular, G is abelian.

<u>Pf</u> $|Z(G)| \in \{p, p^2\}$

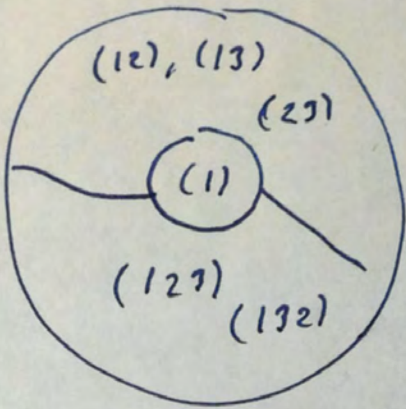
If $|Z(G)| = p$, then $|G/Z(G)| = p$

so $G/Z(G)$ is cyclic, whence

G abelian $\lightning$.

⑥ 4/3/2020

Ex: $G = S_3$



<u>Fact</u>: For $x \in K_G(a)$ one has $|x| = |a|$

<u>Pf</u>: $|a| = n$

$\cancel{xg}$ $x^n = (gag^{-1})^n$

$= (gag^{-1})(gag^{-1})\cdots(gag^{-1})$

$= ga^n g^{-1}$

$= geg^{-1} = e$

$|x| \mid n$

$x^m = e : \overset{e=}{(gag^{-1})^m} = ga^m g^{-1}$

$\Rightarrow g^{-1}eg = a^m$

$\therefore e = a^m$

$\therefore |a| \mid m$