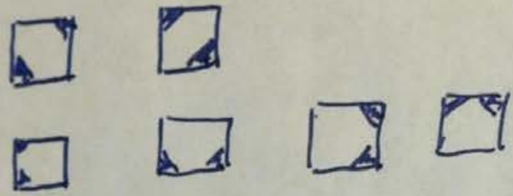
4/1/20

(1)

X



Total of 6 designs

g is rotation $90°$ counter clockwise

$g \in D_4$

$G = \langle g \rangle = \{g^0, g, g^2, g^3\}$ acts on the set X of 6 designs

<u>Def^n</u> Let G be a group acting on X

For $g \in G$ set $X_g = \{x \in X \mid g \cdot x = x\}$

(Compare to $G_x = \{g \in G \mid g \cdot x = x\}$)

<u>Burnside's theorem</u>. Let G be a finite group acting on a finite set X. The number N of orbits of the action is

$$N = \frac{1}{|G|} \sum_{g \in G} |X_g|$$

— x —

Ex $|G| = 4$, $|X_{g^0}| = 6$, $|X_g| = 0$, $|X_{g^2}| = 2$

$|X_{g^3}| = 0$. $N = \frac{1}{4}(6 + 0 + 2 + 0) = 2$.

②

4/1/20

$$N = \frac{1}{|G|}\sum_{g\in G}|X_g|$$

Proof:

$$n = \left|\{(g,x)\in G\times X \mid g\cdot x = x\}\right|$$

First: $(g,-)$ $(g,x): |X_g|$

$$n = \sum_{g\in G}|X_g|$$

$$n = \sum_{x\in X}|G_x|$$

$$|O_x| = [G:G_x] = \frac{|G|}{|G_x|} \Rightarrow \cancel{|G_x| = |O_x|}$$

$$|G_x| = \frac{|G|}{|O_x|}$$

$$\therefore\ n = |G|\sum_{x\in X}\frac{1}{|O_x|}$$

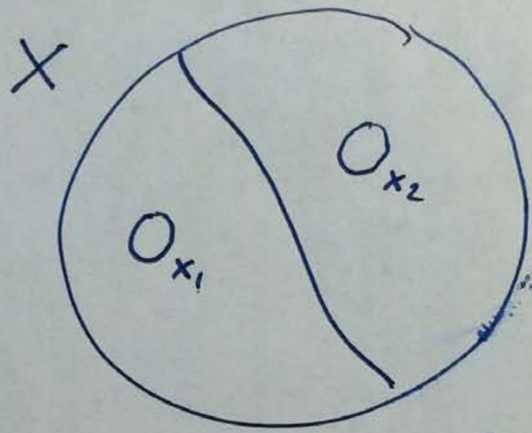
$$\sum_{x\in X}\frac{1}{|O_x|} = ?\quad \sum_{y\in O_x}\frac{1}{|O_x|} = |O_x|\left(\frac{1}{|O_x|}\right) = \underline{1}$$

$$\sum_{x\in X}\frac{1}{|O_x|} = N$$

(3) 4/1/20

$$\sum_{g\in G} |X_g| = n = |G| \sum_{x\in X} \frac{1}{|O_x|} = |G| N$$

$$\therefore \quad N = \frac{1}{|G|} \sum_{\substack{\cancel{x\in X} \\ g\in G}} |X_g| .$$



Every $x \in X$ is in O_{x_1} or O_{x_2}

For $x \in O_{x_1}$: $O_x = O_{x_1}$

$\therefore$

$$\sum_{x\in X} \frac{1}{|O_x|} = \sum_{x\in O_{x_1}} \frac{1}{|O_x|} + \sum_{x\in O_{x_2}} \frac{1}{|O_x|}$$

$$= \sum_{x\in O_{x_1}} \frac{1}{|O_{x_1}|} + \sum_{x\in O_{x_2}} \frac{1}{|O_{x_2}|}$$

$$= |O_{x_1}| \frac{1}{|O_{x_1}|} + |O_{x_2}| \frac{1}{|O_{x_2}|}$$

$$= 1 + 1 = 2$$

4 4/1/20

<u>Example</u>

6 choices of face to mark
5 -
4
3 -
2 -
1 -

$6! = 720$ designs.

6 choices of face to put on table
4 possible rotations

24 choices/actions.

$|X| = 720$

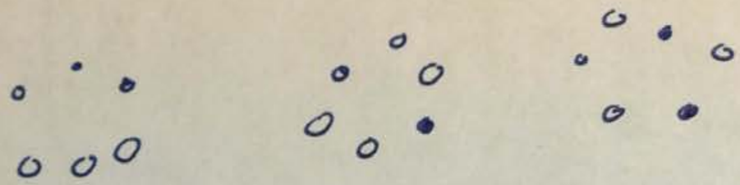
$|G| = 24$

$|X_g|$

$|X_{identity}| = 720$

$N = \frac{1}{24}(720) = 30$

4/1/20



Conjugation: The def:

$$g \cdot x = g x g^{-1}$$

gives an action of ~~X &~~ G on itself.

$$e \cdot x = e x e^{-1} = e x e = x$$

$$\begin{aligned} g_2 \cdot (g_1 \cdot x) &= g_2 \cdot (g_1 x g_1^{-1}) \\ &= g_2 (g_1 x g_1^{-1}) g_2^{-1} \\ &= (g_2 g_1) x (g_1^{-1} g_2^{-1}) \\ &= (g_2 g_1) x (g_2 g_1)^{-1} \\ &= (g_2 g_1) \cdot x. \end{aligned}$$