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MATH 3360 HOMEWORK ASSIGNMENT 7
SKETCH OF SOLUTIONS

(1) Find, up to isomorphism, all abelian groups of order 84.

As $84 = 2^2 \cdot 3 \cdot 7$ there are up to isomorphism two groups

$$\underline{\mathbb{Z}_4} \times \mathbb{Z}_3 \times \mathbb{Z}_7 \qquad \text{and} \qquad \underline{\mathbb{Z}_2 \times \mathbb{Z}_2} \times \mathbb{Z}_3 \times \mathbb{Z}_7$$

(2) Let p, q, and r be primes. Find, up to isomorphism, all abelian groups of order p^3q^2r.

$$\mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$$
$$\mathbb{Z}_{p^3} \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_r$$
$$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$$
$$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_r$$
$$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{q^2} \times \mathbb{Z}_r$$
$$\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_r$$

(3) Let G be an abelian group and $\pi\colon G \to G$ a group homomorphism with $\pi^2 = \pi$.
(a) Show that $g - \pi(g)$ belongs to $\operatorname{Ker}\pi$ for every $g \in G$.
(b) Show that G is isomorphic to $\pi(G) \times \operatorname{Ker}\pi$.

(a): $\pi(g - \pi(g)) = \pi(g) - \pi^2(g) = \pi(g) - \pi(g) = 0$. (b) The map $g \mapsto (\pi(g), g - \pi(g))$ is a group homomorphism $G \to \pi(G) \times \operatorname{Ker}\pi$. If $\pi(g) = 0$, then g maps to $(0, g)$, so only 0 maps to $(0,0)$. Given $p \in \pi(G)$ and $k \in \operatorname{Ker}\pi$ the element $p + k$ maps to (p, k).

$p = \pi(g)$

$p + k \mapsto (\pi(p+k), p+k - \pi(p+k))$

$= (\pi(p), \pi(p+k) - \pi(p+k))$

$= (p, k)$

(4) Let p be a prime. Show that every group of order p^2 is abelian. (Hint: problem (2) on Homework Assignment 6.)

$|Z| \stackrel{?}{=} 1$

The order of the center $Z = Z(G)$ is p or p^2. ? The second possibility means that G is abelian, so assume towards a contradiction that the order is p. The quotient G/Z then has order p, so it is cyclic, and then G is abelian by the hint.

(5) Show that the group of inner automorphisms of S_3 is isomorphic to S_3.
The center of S_3 is trivial, so this is an application of 2.5.10.

$\operatorname{Inn} G \cong G/Z(G)$

$x \mapsto g^{-1}xg$

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Problem (3) on HW 7.

$\pi(G) \leq G$

$\ker \pi \leq G$

$$\pi : G \to G \qquad \pi(G) \subset G, \quad \ker\pi \leq G$$

Claim: $G = \pi(G) \oplus \ker \pi$

$$\forall g \in G: \quad g = \underbrace{\pi(g)}_{\pi(G)} + \underbrace{(g - \pi(g))}_{\ker \pi}$$

$x \in \pi(G) \cap \ker \pi$

$x = \pi(g)$

$\pi(x) = 0$

$$\therefore \quad 0 = \pi(x) = \pi(\pi(g)) = \pi^2(g) = \pi(g) = x.$$

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2 Recall:

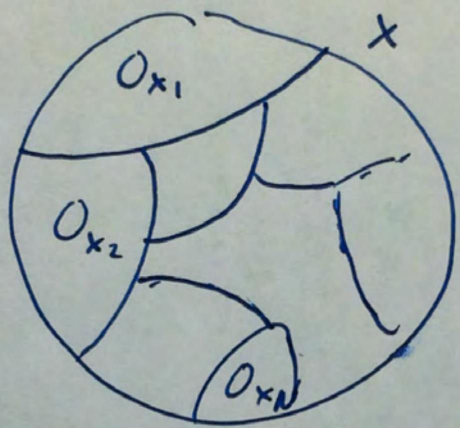
Group G acting on set X:

$$e \cdot x = x, \quad \forall x \in X$$

$$g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x, \quad \forall g_1, g_2 \in G, \ \forall x \in X$$

Stabilizer: $G_x = \{ g \in G \mid g \cdot x = x \} \leq G$

Orbit: $O_x = \{ g \cdot x \mid g \in G \}$



Thm. Let G be a group acting on a set X. For every $x \in X$:

$$[G : G_x] = |O_x|$$

In part., if G is finite, then

$$|O_x| = \frac{|G|}{|G_x|}.$$

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Ex $X = \{1, \dots, n\}$, $G = S_n$

$G_1 = ?$

$O_1 = X$

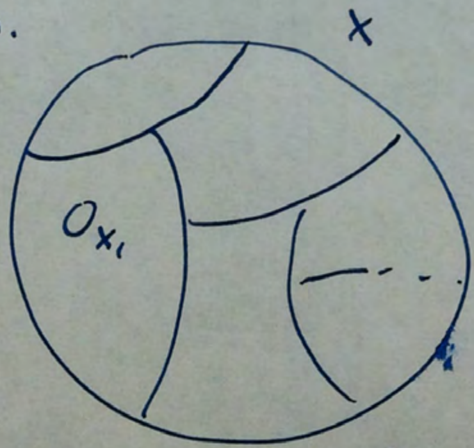
$$|O_1| = n = \frac{|G|}{|G_1|} = \frac{n!}{|G_1|}$$

$$|G_1| = (n-1)!$$

$$G_1 \cong S_{n-1}$$

Def^n A group action with only one orbit is called transitive.

Thm.



Let $x_1, \dots, x_N$ be a complete set of representatives for the orbits of G's action on X

$$|X| = \sum_{i=1}^{N} |O_{x_i}| = \sum_{i=1}^{N} [G : G_{x_i}].$$