MATH 2360-012 WEEK 8

SECTIONS 4.5 AND 4.6; PAGES 186–207

Abstract. A *basis* for a vector space is a linearly independent spanning set. The *dimension* of a vector space is the number of elements in a basis.

To a matrix one associates three vector space: *row, column,* and *null space.* Existence and uniqueness of solutions to systems of linear equations can be understood in terms of the column space and the null space of the associated matrix.

Section 4.5

Reading. Make sure that you understand the following:

(1) A *basis* for a vector space is a spanning set that is also linearly independent.
(2) Given a basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ for a vector space V, every vector $\mathbf{v}$ in V can be written as a linear combination
$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$$
in exactly one way. That is, the constants $c_1, \dots, c_n$ are uniquely determined.
(3) Every vector space has a basis, and for a given space V every basis has the same number of elements; that number is the *dimension* of V.
(4) Every linearly independent set of vectors in a vector space V can be supplemented to a basis for V.
(5) Every spanning set for a vector space V can be depleted to a basis for V.
(6) If V is a vector space of dimension n, then a set of n vectors in V is a basis if it is linearly independent (i.e. spanning comes for free).
(7) If V is a vector space of dimension n, then a set of n vectors in V is a basis if it spans V (i.e. linear independence comes for free).

Suggested problems. To verify that you have understood the material, solve the following problems at the end of the section: 1–7, 9, 11, 17, 23, 39, and 45.

Section 4.6

Reading. Make sure that you understand the following:

(1) For an $m \times n$ matrix, the row space is a subspace of $\mathbb{R}^n$ and the column space is a subspace of $\mathbb{R}^m$.
(2) Performing row operations on a matrix A does not change its row space. To find a basis for the row space of A, one can bring A on (Reduced) Row Echelon Form and take the non-zero rows.
(3) Performing row operations on a matrix A **does** (usually) change its column space. To find a basis for the column space of A, one can bring A on Row Echelon Form and take the columns **in the original matrix** A that correspond to columns in the REF with leading 1s.
(4) The column space of a matrix A is the row space of the transpose A^{T}.

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(5) The row space and the column space of a matrix A have the same dimension; that number is the *rank* of A.
(6) The *null space* of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. The dimension of the null space is the *nullity* of A, and one has $\operatorname{rank} A + \operatorname{null} A = n$.
(7) A system of linear equations $AX = B$ is consistent if and only if B is in the column space of A.
(8) An $n \times n$ matrix is invertible if and only if it has rank n.

Suggested problems. To verify that you have understood the material, solve the following problems at the end of the section: 5, 7, 9, 13, 17, 23, 29, and 31.