# MATH 2360-D01 WEEK 8

SECTIONS 4.5 AND 4.6; PAGES 180–201

ABSTRACT. A *basis* for a vector space is a linearly independent spanning set. The *dimension* of a vector space is the number of elements in a basis.

To a matrix one associates three vector space: *row, column,* and *null space.* Existence and uniqueness of solutions to systems of linear equations can be understood in terms of the column space and the null space of the associated matrix.

## SECTION 4.5

**Reading.** Make sure that you understand the following:

(1) A *basis* for a vector space is a spanning set that is also linearly idenpendent.
(2) Given a basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ for a vector space $V$, every vector $\mathbf{v}$ in $V$ can be written as a linear combination

$$\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$$

in exactly one way. That is, the constants $c_1, \dots, c_n$ are uniquely determined.
(3) Every vector space has a basis, and for a given space $V$ every basis has the same number of elements; that number is the *dimension* of $V$.
(4) Every linearly independent set of vectors in a vector space $V$ can be supplemented to a basis for $V$.
(5) Every spanning set for a vector space $V$ can be depleted to a basis for $V$.
(6) If $V$ is a vector space of dimension $n$, then a set of $n$ vectors in $V$ is a basis if it is linearly independent (i.e. spanning comes for free).
(7) If $V$ is a vector space of dimension $n$, then a set of $n$ vectors in $V$ is a basis if it spans $V$ (i.e. linear independence comes for free).

**Suggested problems.** To verify that you have understood the material, solve the following problems at the end of the section: 1–8, 17, 23, 39, and 45.

## SECTION 4.6

**Reading.** Make sure that you understand the following:

(1) For an $m \times n$ matrix, the row space is a subspace of $\mathbb{R}^n$ and the column space is a subspace of $\mathbb{R}^m$.
(2) Performing row operations on a matrix $A$ does not change its row space. To find a basis for the row space of $A$, one can bring $A$ on (Reduced) Row Echelon Form and take the non-zero rows.
(3) Performing row operations on a matrix $A$ **does** (usually) change its column space. To find a basis for the column space of $A$, one can bring $A$ on Row Echelon Form and take the columns **in the original matrix** $A$ that correspond to columns in the REF with leading 1s.
(4) The column space of a matrix $A$ is the row space of the transpose $A^{\mathrm{T}}$.

*Date*: October 14, 2014.

(5) The row space and the column space of a matrix $A$ have the same dimension; that number is the *rank* of $A$.
(6) The *null space* of an $m \times n$ matrix $A$ is a subspace of $\mathbb{R}^n$. The dimension of the null space is the *nullity* of $A$, and one has $\operatorname{rank} A + \operatorname{null} A = n$.
(7) A system of linear equations $AX = B$ is consistent if and only if $B$ is in the column space of $A$.
(8) An $n \times n$ matrix is invertible if and only if it has rank $n$.

**Suggested problems.** To verify that you have understood the material, solve the following problems at the end of the section: 5, 7, 9, 13, 17, and 23.