TRANSFER OF GORENSTEIN DIMENSIONS ALONG RING HOMOMORPHISMS

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Dedicated with gratitude to Hans-Bjørn Foxby, our teacher and friend

ABSTRACT. A central problem in the theory of Gorenstein dimensions over commutative noetherian rings is to find resolution-free characterizations of the modules for which these invariants are finite. Over local rings, this problem was recently solved for the Gorenstein flat and the Gorenstein projective dimensions; here we give a solution for the Gorenstein injective dimension. Moreover, we establish two formulas for the Gorenstein injective dimension of modules in terms of the depth invariant; they extend formulas for the injective dimension due to Bass and Chouinard.

INTRODUCTION

Gorenstein dimensions are homological invariants that are useful for identifying modules and ring homomorphisms with good homological properties. This paper is concerned with the Gorenstein injective dimension and the Gorenstein flat dimension, denoted Gid and Gfd, respectively. These invariants are defined in terms of resolutions by modules from certain classes, the Gorenstein injective and the Gorenstein flat modules. See Section 1 for definitions.

Let R be a commutative noetherian ring. It is frequently useful to know that finiteness of the classical homological dimensions of an R-module M can be detected by vanishing of (co)homology. For the injective dimension one has

$$\mathrm{id}_R M = \sup\{ j \mid \mathrm{Ext}^j_R(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \mathrm{Spec}\, R \}.$$

One of the key problems in Gorenstein homological algebra has been to find criteria for finiteness of Gorenstein dimensions that are resolution-free. See the survey [7] and the introduction in [8] for a further discussion of this issue. The problem was partly solved by Christensen, Frankild, and Holm in [8]: If R has a dualizing complex and M is an R-module, then

$$\mathrm{Gid}_R M \text{ is finite if an only if } M \text{ belongs to } \mathsf{B}(R)$$

where $\mathsf{B}(R)$ is the Bass class of R; the crucial point is that verification of membership in $\mathsf{B}(R)$ does not involve construction of a Gorenstein injective resolution. Similarly $\mathrm{Gfd}_R M$ is finite if an only if M belongs to the Auslander class $\mathsf{A}(R)$.

If R is a homomorphic image of a Gorenstein ring, then it has a dualizing complex. In particular [8] solves the problem when R is local and complete or, more generally, essentially of finite type over a complete local ring. However, non-trivial

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modules of finite Gorenstein injective dimension or finite Gorenstein flat dimension may exist over rings that are not homomorphic images of Gorenstein rings; see Example (1.6). In [13] Esmkhani and Tousi show that when R is local, but not necessarily a homomorphic image of a Gorenstein ring, an R-module M has finite Gorenstein flat dimension if and only if the module $\widehat{R} \otimes_R M$ is in the Auslander class $\mathsf{A}(\widehat{R})$ of the completion $\widehat{R}$. This solves the resolution-free characterization problem for the Gorenstein flat dimension over local rings. In a separate paper [14] the same authors give a solution for the Gorenstein injective dimension of cotorsion modules over local rings.

In this paper, we complete the solution for local rings with the special case $S = \widehat{R}$ of the next result, wherein $\mathbf{R}\mathrm{Hom}_R(S, M)$ and $S \otimes^{\mathbf{L}}_R M$ are the right derived homomorphism complex and the left derived tensor product complex. More general statements are proved in (1.7) and (1.8).

Theorem A. *Let $\varphi\colon R \to S$ be a local ring homomorphism such that S has a bounded resolution by flat R-modules when considered as an R-module via φ. For every R-module M there are inequalities*

$$\mathrm{Gid}_R\, M \geqslant \mathrm{Gid}_S\, \mathbf{R}\mathrm{Hom}_R(S, M) \quad \textit{and} \quad \mathrm{Gfd}_R\, M \geqslant \mathrm{Gfd}_S(S \otimes^{\mathbf{L}}_R M).$$

If φ is flat, then equalities hold; in particular, the respective dimensions are simultaneously finite in this case.

As noted above, Esmkhani and Tousi's [14] resolution-free characterization of finiteness of Gorenstein injective dimension only applies to cotorsion modules. The cotorsion hypothesis is quite restrictive. Indeed, work of Frankild, Sather-Wagstaff, and Wiegand [18, 19] shows that a finitely generated cotorsion R-module is complete. For Gorenstein rings, the following application of Theorem A strengthens the main result of [18]; it only assumes that the Ext-modules are finitely generated over $\widehat{R}$, not over R.

Theorem B. *Let R be a Gorenstein local ring, and let M be a finitely generated R-module. If the $\widehat{R}$-modules $\mathrm{Ext}^i_R(\widehat{R}, M)$ are finitely generated for $i = 1, \dots, \dim_R M$, then the modules $\mathrm{Ext}^i_R(\widehat{R}, M)$ vanish for $i \geqslant 1$, and M is complete.*

The hypotheses of this result are satisfied if M is complete, e.g., if M has finite length; cf. Remark (3.2). Theorem B is a special case of (3.1).

In Section 2 we consider formulas that express the Gorenstein injective dimension of an R-module in terms of the depth invariant. Our main result in this direction is Theorem C below. It extends Chouinard's [4] formula for injective dimension, and it removes the assumption about existence of a dualizing complex from [8, thm. 6.8].

Theorem C. *For every R-module M of finite Gorenstein injective dimension there is an equality*

$$\mathrm{Gid}_R\, M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

For certain modules this formula has already been established by Khatami, Tousi, and Yassemi [24, 25]. Actually, we prove Theorems A and C for R-complexes, and the latter yields a Bass formula for homologically finite R-complexes; see (2.3). Such a formula was established for modules in [24].

1. Finiteness and descent of Gorenstein homological dimensions

Throughout this paper R and S are commutative noetherian rings. Complexes of R-modules, *R-complexes* for short, are indexed homologically: the ith differential of an R-complex M is written $\partial_i^M\colon M_i \to M_{i-1}$. We proceed by recalling the definitions of Gorenstein injective and Gorenstein flat modules from [11, 12].

(1.1) An R-module J is said to be *Gorenstein injective* if there is an exact complex I of injective R-modules such that $J \cong \operatorname{Ker} \partial_0^I$ and the complex $\operatorname{Hom}_R(E,I)$ is exact for every injective R-module E.

An R-module G is *Gorenstein flat* if there is an exact complex F of flat R-modules such that $G \cong \operatorname{Im} \partial_0^F$ and $E \otimes_R F$ is exact for every injective R-module E.

The first step toward Theorem A is to notice that the central arguments in the works of Esmkhani and Tousi [13, 14] apply to any faithfully flat ring homomorphism, not just to the map $R \to \widehat{R}$; see (1.3). To this end the next fact is key.

(1.2) **Lemma.** *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism. If E is an injective R-module, then it is a direct summand (as an R-module) of the injective S-module $\operatorname{Hom}_R(S,E)$.*

Proof. Let E be an injective R-module. It is well-known, and straightforward to show, that $\operatorname{Hom}_R(S,E)$ is an injective S-module. Because φ is faithful, it is a pure monomorphism of R-modules, cf. [26, thm. 7.5]. This implies that S/R is a flat R-module, so $\operatorname{Hom}_R(S/R,E)$ is injective. Now apply the exact functor $\operatorname{Hom}_R(-,E)$ to $0 \to R \to S \to S/R \to 0$ to obtain a split exact sequence of injective modules. □

(1.3) **Lemma.** *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism.*

- (a) *Assume that $\dim S$ is finite. An R-module M is Gorenstein injective if and only if $\operatorname{Hom}_R(S,M)$ is a Gorenstein injective S-module and $\operatorname{Ext}^i_R(F,M) = 0$ for every flat R-module F and all $i \geqslant 1$.*[‡]
- (b) *An R-module M is Gorenstein flat if and only if $S \otimes_R M$ is a Gorenstein flat S-module and $\operatorname{Tor}_i^R(E,M) = 0$ for every injective R-module E and all $i \geqslant 1$.*

Proof. Argue as in the proofs of [13, thm. 2.5] and [14, thm. 2.5], but use the injective S-module $\operatorname{Hom}_R(S,E)$ from (1.2) in place of the double Matlis dual $E^{\vee\vee}$. □

For the proofs that follow, we need some terminology.

(1.4) Let M be an R-complex; it is said to be *bounded above* if $M_i = 0$ for $i \gg 0$, *bounded below* if $M_i = 0$ for $i \ll 0$, and *bounded* if $M_i = 0$ for $|i| \gg 0$. If the homology complex $\operatorname{H}(M)$ is bounded, then M is called *homologically bounded*. If $\operatorname{H}(M)$ is finitely generated, then M is said to be *homologically finite*. The notations $\inf M$ and $\sup M$ stand for the infimum and supremum of the set $\{i \in \mathbb{Z} \mid \operatorname{H}_i(M) \neq 0\}$, with the convention that $\inf M = \infty$ and $\sup M = -\infty$ if $\operatorname{H}(M) = 0$.

From this point, we work in the derived categories $\mathsf{D}(R)$ and $\mathsf{D}(S)$; see e.g. [20]. Given two R-complexes M and N, their left derived tensor product complex and right derived homomorphism complex are denoted $M \otimes^{\mathbf{L}}_R N$ and $\mathbf{R}\operatorname{Hom}_R(M,N)$. The symbol '$\simeq$' is used to identify isomorphisms in derived categories.

[‡] The vanishing of $\operatorname{Ext}^i_R(F,M)$ for every flat R-module F and for all $i \geq 1$ means exactly that M is *cotorsion*. It is straightforward to show that this is equivalent to the standard definition of cotorsion which only requires $\operatorname{Ext}^1_R(F,M) = 0$ for every flat R-module F.

(1.5) The *Gorenstein injective dimension* of a homologically bounded R-complex M is defined as follows

$$\operatorname{Gid}_R M = \inf\left\{ \sup\{i \in \mathbb{Z} \mid J_{-i} \neq 0\} \;\middle|\; \begin{array}{l} J \text{ is a bounded above complex} \\ \text{of Gorenstein injective modules} \\ \text{and isomorphic to } M \text{ in } \mathsf{D}(R) \end{array} \right\}.$$

The *Gorenstein flat dimension* is defined similarly in terms of bounded below complexes of Gorenstein flat modules; see [5, (5.2.3)].

When R has a dualizing complex D, Avramov and Foxby [2] define two full subcategories $\mathsf{A}(R)$ and $\mathsf{B}(R)$ of $\mathsf{D}(R)$. The objects in the *Auslander class* $\mathsf{A}(R)$ are the homologically bounded R-complexes M such that $D \otimes^{\mathbf{L}}_R M$ is homologically bounded and the natural morphism $M \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R M)$ is an isomorphism in $\mathsf{D}(R)$. The objects in the *Bass class* $\mathsf{B}(R)$ are the homologically bounded R-complexes M such that $\mathbf{R}\mathrm{Hom}_R(D, M)$ is homologically bounded and the natural morphism $D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, M) \to M$ is an isomorphism in $\mathsf{D}(R)$.

For a homologically bounded R-complex M, the main results in [8] state

(1.5.1) $\operatorname{Gfd}_R M$ is finite if and only if M belongs to $\mathsf{A}(R)$; and

(1.5.2) $\operatorname{Gid}_R M$ is finite if and only if M belongs to $\mathsf{B}(R)$.

Before proving Theorem A, we recall an elementary construction of rings that admit non-trivial modules of finite Gorenstein dimensions.

(1.6) **Example.** Let Q be a commutative noetherian ring and consider the ring of dual numbers $R = Q[X]/(X^2)$. It is routine to show that the cyclic R-module $R/(X)$ is Gorenstein flat and not flat. Hence, for every faithfully injective R-module E the module $\mathrm{Hom}_R(R/(X), E)$ is Gorenstein injective and not injective; see [5, thm. (6.4.2)]. Furthermore, if Q is not a homomorphic image of a Gorenstein ring, then neither is R.

The next result contains half of Theorem A from the introduction. Recall that a ring homomorphism $\varphi\colon R \to S$ has *finite flat dimension* when S, considered as an R-module via φ, has a bounded resolution by flat R-modules.

(1.7) **Theorem.** *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension, and assume that $\dim R$ is finite. For every homologically bounded R-complex M there is an inequality*

$$\operatorname{Gid}_R M \geqslant \operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S, M).$$

If φ is faithfully flat and $\dim S$ is finite, then equality holds; in particular, the dimensions are simultaneously finite in this case.

Proof. Assume that M has finite Gorenstein injective dimension, and fix a bounded complex J of Gorenstein injective R-modules such that there is an isomorphism $M \simeq J$ in $\mathsf{D}(R)$. As an R-module, S has projective dimension at most $\dim R < \infty$; see [27, II. thm. (3.2.6)] and [23, prop. 6]. Therefore, by [8, cor. 2.12] there is an isomorphism $\mathbf{R}\mathrm{Hom}_R(S, M) \simeq \mathrm{Hom}_R(S, J)$ in $\mathsf{D}(S)$, and the right-hand complex is a bounded one of Gorenstein injective S-modules; see [9, Ascent table II (h)]. In particular, there is an inequality $\operatorname{Gid}_R M \geqslant \operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S, M)$.

Assume now that φ is faithfully flat and that $d := \dim S$ and $\operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S, M)$ are finite. Recall the inequalities $\mathrm{pd}_R S \leqslant \dim R \leqslant d$. Consider a resolution $M \xrightarrow{\simeq} I$ by injective R-modules. The complex $\mathrm{Hom}_R(S, I) \simeq \mathbf{R}\mathrm{Hom}_R(S, M)$ is one of injective S-modules, and one has $\mathrm{H}_i(\mathrm{Hom}_R(S,I)) = 0$ for all $i < \inf M - d$, as $\mathrm{pd}_R\, S$ is at most d. Left-exactness of the functor $\mathrm{Hom}_R(S,-)$ yields an isomorphism

$$\mathrm{Ker}\, \partial_{n-1}^{\mathrm{Hom}_R(S,I)} \cong \mathrm{Hom}_R(S, \mathrm{Ker}\, \partial_{n-1}^{I})$$

for each n. It follows that $\mathbf{R}\mathrm{Hom}_R(S,M)$ is isomorphic in $\mathsf{D}(S)$ to the complex

$$0 \to \mathrm{Hom}_R(S,I_0) \to \cdots \to \mathrm{Hom}_R(S,I_n) \to \mathrm{Hom}_R(S, \mathrm{Ker}\, \partial_{n-1}^{I}) \to 0$$

for $n < \inf M - d$. Set $K = \mathrm{Ker}\, \partial^I_{\inf M - 2d - 1}$. Since the S-complex $\mathbf{R}\mathrm{Hom}_R(S,M)$ has finite Gorenstein injective dimension, the S-module $\mathrm{Hom}_R(S,K)$ is Gorenstein injective; see [8, thm. 3.3]. To show that $\mathrm{Gid}_R\, M$ is finite, we use Lemma (1.3)(a) to prove that K is Gorenstein injective over R: For every flat R-module F, one has $\mathrm{pd}_R\, F \leqslant d$, and for every $i \leqslant 1$ dimension shifting yields

$$\mathrm{Ext}^i_R(F,K) \cong \mathrm{Ext}^{i+d}_R(F, \mathrm{Ker}\, \partial^I_{\inf M - d - 1}) = 0.$$

To prove the equality of Gorenstein injective dimensions, choose an injective R-module E such that $\mathrm{Gid}_R\, M = -\inf \mathbf{R}\mathrm{Hom}_R(E,M)$; cf. [8, thm. 3.3]. The module E is a direct summand of an injective S-module $\widetilde{E}$ by Lemma (1.2), hence the third step in the next sequence

$$\begin{aligned}
\mathrm{Gid}_S\, \mathbf{R}\mathrm{Hom}_R(S,M) &\geqslant -\inf \mathbf{R}\mathrm{Hom}_S(\widetilde{E}, \mathbf{R}\mathrm{Hom}_R(S,M)) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(\widetilde{E}, M) \\
&\geqslant -\inf \mathbf{R}\mathrm{Hom}_R(E,M) \\
&= \mathrm{Gid}_R\, M.
\end{aligned}$$

The first step is by [8, thm. 3.3], the second one is from Hom-tensor adjointness, and the last one comes from the choice of E. The opposite inequality was proved in the first paragraph of this proof. □

The next result contains the other half of Theorem A, and it gives a partial answer to [22, quest. 8.10]; see also Proposition (1.9). Its proof is similar to, but simpler than, the proof of Theorem (1.7). Note that (1.8) has no assumptions on the Krull dimension of R or S.

(1.8) **Theorem.** *Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. For every homologically bounded R-complex M there is an inequality*

$$\mathrm{Gfd}_R\, M \geqslant \mathrm{Gfd}_S(S \otimes^{\mathbf{L}}_R M).$$

If φ is faithfully flat, then equality holds; in particular, the dimensions are simultaneously finite in this case. □

Equality can fail in Theorems (1.7) and (1.8) if φ is not flat, even if R is local and φ is surjective. See (3.3) for an example.

We conclude this section with an application of Theorem (1.7) which, in particular, answers [22, quest. 8.10] for local ring homomorphisms.

(1.9) **Proposition.** *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism, and assume that R is semi-local. For every homologically bounded R-complex M, there are equalities*

$$\mathrm{Gfd}_S(S \otimes_R M) = \mathrm{Gfd}_R(S \otimes_R M) = \mathrm{Gfd}_R\, M.$$

Proof. If $\operatorname{Gfd}_R M$ is finite, then the desired equalities hold by [22, cor. 8.9]. Theorem (1.8) says that $\operatorname{Gfd}_S(S \otimes_R M)$ and $\operatorname{Gfd}_R M$ are simultaneously finite. Hence, it remains to assume that $\operatorname{Gfd}_R(S \otimes_R M)$ is finite and prove that $\operatorname{Gfd}_R M$ is finite.

The completion $\widehat{R}$ of R (with respect to its Jacobson radical) has a dualizing complex. By Theorem (1.8) the finiteness of $\operatorname{Gfd}_R(S \otimes_R M)$ implies that $\operatorname{Gfd}_{\widehat{R}}(\widehat{R} \otimes_R (S \otimes_R M))$ is finite, so the complex

$$\widehat{R} \otimes_R (S \otimes_R M) \simeq (\widehat{R} \otimes_R M) \otimes_R S \simeq (\widehat{R} \otimes_R M) \otimes_{\widehat{R}} (\widehat{R} \otimes_R S)$$

is in the Auslander class $\mathsf{A}(\widehat{R})$ by (1.5.1). As S is faithfully flat over R, the module $\widehat{R} \otimes_R S$ is faithfully flat over $\widehat{R}$, and it follows that $\widehat{R} \otimes_R M$ is in $\mathsf{A}(\widehat{R})$, cf. [21, rmk. 4]. Thus, $\operatorname{Gfd}_{\widehat{R}}(\widehat{R} \otimes_R M)$ is finite by (1.5.1), and Theorem (1.8) implies that $\operatorname{Gfd}_R M$ is finite. □

2. A Chouinard formula for Gorenstein injective dimension

The *width* of a complex M over a local ring R with residue field k is defined as:

$$\operatorname{width}_R M = \inf (k \otimes^{\mathbf{L}}_R M).$$

There is an inequality $\operatorname{width}_R M \geqslant \inf M$, and equality holds if M is homologically finite, by Nakayama's lemma. Let N be another R-complex; a standard application of the Künneth formula yields

$$\operatorname{width}_R(M \otimes^{\mathbf{L}}_R N) = \operatorname{width}_R M + \operatorname{width}_R N. \tag{2.0.1}$$

If M is homologically bounded and of finite projective dimension, and if $\mathrm{H}(N)$ is bounded above, then there is an equality [6, thm. (4.14)(a) and (1.6)(b)]:

$$\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(M, N) = \operatorname{depth}_R M + \operatorname{width}_R N - \operatorname{depth} R. \tag{2.0.2}$$

Foxby [15] defines the *small support* of a complex M over a noetherian ring R, denoted $\operatorname{supp}_R M$, as the set of prime ideals $\mathfrak{p}$ in R such that the complex $M_\mathfrak{p}$ has finite width over $R_\mathfrak{p}$.

(2.1) **Lemma.** *Let J be a Gorenstein injective R-module. Then one has*

$$\operatorname{depth} R_\mathfrak{p} \leqslant \operatorname{width}_{R_\mathfrak{p}} J_\mathfrak{p}$$

for every $\mathfrak{p}$ in $\operatorname{Spec} R$, and equality holds if $\mathfrak{p}$ is a maximal element in $\operatorname{supp}_R J$.

Proof. Let $\mathfrak{p}$ be given, and let T be an $R_\mathfrak{p}$-module of finite projective dimension. Because there is an exact sequence

$$\cdots \to A_2 \to A_1 \to A_0 \to J_\mathfrak{p} \to 0$$

where each A_i is an injective $R_\mathfrak{p}$-module, a standard dimension shifting argument shows that $\operatorname{Ext}^i_{R_\mathfrak{p}}(T, J_\mathfrak{p}) = 0$ for all $i > 0$.

Set $d = \operatorname{depth} R_\mathfrak{p}$, and choose a maximal $R_\mathfrak{p}$-regular sequence $\mathbf{x}$. Because The $R_\mathfrak{p}$-module $R_\mathfrak{p}/(\mathbf{x})$ has finite projective dimension, the previous paragraph provides the first inequality in the next display

$$0 \leqslant \inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/(\mathbf{x}), J_\mathfrak{p}) \leqslant \operatorname{width}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/(\mathbf{x}), J_\mathfrak{p}) = \operatorname{width}_{R_\mathfrak{p}} J_\mathfrak{p} - d$$

where the equality follows from (2.0.2). This proves the desired inequality.

Let $\mathfrak{p}$ be maximal in $\operatorname{supp}_R J$, and let I be the minimal injective resolution of J. For prime ideals $\mathfrak{q}$ that strictly contain $\mathfrak{p}$, the indecomposable module $\mathrm{E}_R(R/\mathfrak{q})$ is not a direct summand of any module I_j in I; see [15, rmk. 2.9]. It follows that

$I_j \cong (I_j)_{\mathfrak{p}} \oplus I'_j$ where I'_j is a direct sum of injective hulls of the form $\mathrm{E}_R(R/\mathfrak{q})$ such that $\mathfrak{p} \not\subseteq \mathfrak{q}$. Recall that for each such $\mathfrak{q}$ we have $\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), \mathrm{E}_R(R/\mathfrak{q})) = 0$, and so $\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), I'_j) = 0$. In conclusion, there are isomorphisms

$$\mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), I_j) \cong \mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), (I_j)_{\mathfrak{p}} \oplus I'_j) \cong \mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), (I_j)_{\mathfrak{p}}).$$

This explains the last isomorphism below; the first one is Hom-tensor adjointness

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(\mathrm{E}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}), I_{\mathfrak{p}}) \cong \mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), I_{\mathfrak{p}}) \cong \mathrm{Hom}_R(\mathrm{E}_R(R/\mathfrak{p}), I).$$

It follows that the modules $\mathrm{Ext}^i_{R_{\mathfrak{p}}}(\mathrm{E}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}), J_{\mathfrak{p}})$ vanish for $i > 0$.

Set $S = R_{\mathfrak{p}}$; it is a local ring with depth d, maximal ideal $\mathfrak{n} := \mathfrak{p}R_{\mathfrak{p}}$ and residue field $l := R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. The S-module $B := J_{\mathfrak{p}}$ has minimal injective resolution $H := I_{\mathfrak{p}}$. One has $\mathfrak{n} \in \mathrm{supp}_S B$ and

$$(1) \qquad \begin{aligned} &\mathrm{Ext}^i_S(T, B) = 0 \text{ for all } i > 0 \text{ and every } S\text{-module } T \text{ with } \mathrm{pd}_S T \text{ finite} \\ &\mathrm{Ext}^i_S(E, B) = 0 \text{ for all } i > 0, \text{ where } E \text{ is the injective envelope of } l. \end{aligned}$$

To prove the desired equality $\mathrm{width}_R B = d$, we adapt the proof of [9, cor. 6.5]. Let K denote the Koszul complex on a system of generators for $\mathfrak{n}$, and note that $K \otimes_S E$ and $\mathrm{Hom}_S(K, E)$ are isomorphic up to a shift. The total homology module $\mathrm{H}(\mathrm{Hom}_S(K, E))$ has finite length. In particular $K \otimes_S E$ is homologically finite. Fix a resolution by finitely generated free S-modules

$$(2) \qquad L \xrightarrow{\simeq} K \otimes_S E.$$

Then there are (quasi)isomorphisms:

$$(3) \qquad K \otimes_S (E \otimes^{\mathbf{L}}_S \mathrm{Hom}_S(E, B)) \simeq L \otimes_S \mathrm{Hom}_S(E, B) \cong \mathrm{Hom}_S(\mathrm{Hom}_S(L, E), B)$$

the last one is Hom-evaluation [1, lem. 4.4]. The resolution (2) induces a quasiisomorphism α from the complex $\mathrm{Hom}_S(K \otimes_S E, E) \cong \mathrm{Hom}_S(K, \widehat{S})$ to $\mathrm{Hom}_S(L, E)$. The mapping cone C of α is a bounded complex of direct sums of $\widehat{S}$ and E. By (1) the modules C_j are Ext-orthogonal to B, that is, we have $\mathrm{Ext}^i_R(C_j, B) = 0$ for all $i \geq 1$ and all j. Hence, an application of $\mathrm{Hom}_S(-, B)$ yields a quasiisomorphism

$$(4) \qquad \mathrm{Hom}_S(\mathrm{Hom}_S(L, E), B) \xrightarrow[\simeq]{\mathrm{Hom}(\alpha, B)} \mathrm{Hom}_S(\mathrm{Hom}_S(K, \widehat{S}), B).$$

The modules in the complex $\mathrm{Hom}_S(K, \widehat{S})$ are Ext-orthogonal to the modules in the mapping cone of the injective resolution $B \xrightarrow{\simeq} H$. Therefore, one has

$$(5) \qquad \mathrm{Hom}_S(\mathrm{Hom}_S(K, \widehat{S}), B) \simeq \mathrm{Hom}_S(\mathrm{Hom}_S(K, \widehat{S}), H)$$

see [8, lem. 2.4]. Now piece together (3)–(5), and use Hom-evaluation to obtain

$$(6) \qquad K \otimes_S (E \otimes^{\mathbf{L}}_S \mathrm{Hom}_R(E, B)) \simeq K \otimes_S \mathbf{R}\mathrm{Hom}_S(\widehat{S}, B).$$

By the width sensitivity of K, see [6, (4.2) and (4.11)], the complexes $\mathbf{R}\mathrm{Hom}_S(\widehat{S}, B)$ and $E \otimes^{\mathbf{L}}_S \mathrm{Hom}_S(E, B)$ have the same width. From (2.0.1) and (2.0.2) one has

$$(7) \qquad \mathrm{width}_S E + \mathrm{width}_S \mathrm{Hom}_S(E, B) = \mathrm{width}_S B.$$

The maximal ideal $\mathfrak{n}$ is in $\mathrm{supp}_S B$, so $\mathrm{width}_S B$ is finite. It follows from (7) that $\mathrm{width}_S \mathrm{Hom}_S(E, B)$ is finite; in particular, $\mathrm{Hom}_S(E, B)$ is non-zero. As every element in E is annihilated by a power of the maximal ideal $\mathfrak{n}$, it follows that $\mathfrak{n}\mathrm{Hom}_S(E, B) \neq \mathrm{Hom}_S(E, B)$. (Indeed, if $\mathrm{Hom}_S(E, B) = \mathfrak{n}\mathrm{Hom}_S(E, B)$, then $\mathrm{Hom}_S(E, B) = \mathfrak{n}^t \mathrm{Hom}_S(E, B)$ for each $t \geqslant 1$. Since $\mathrm{Hom}_S(E, B) \neq 0$, there are elements $\psi \in \mathrm{Hom}_S(E, B)$ and $e \in E$ such that $\psi(e) \neq 0$. Also, there is an integer

$t \geqslant 1$ such that $\mathfrak{n}^t e = 0$. The condition $\psi \in \mathfrak{n}^t \operatorname{Hom}_S(E, B)$ then implies $\psi(e) = 0$, a contradiction.) Thus, one has $\operatorname{width}_S \operatorname{Hom}_S(E, B) = 0$, and the desired equality follows as $\operatorname{width}_S E = d$ by [6, prop. (4.8)]. □

The next result contains Theorem C from the introduction.

(2.2) **Theorem.** *For every R-complex M of finite Gorenstein injective dimension there is an equality*

$$\operatorname{Gid}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

Proof. If $\operatorname{H}(M) = 0$ then the equality holds for trivial reasons. Assume $\operatorname{H}(M) \neq 0$; without loss of generality, assume also that $M_0 \neq 0$ and $M_i = 0$ for all $i > 0$. Set $g = \operatorname{Gid}_R M$, and notice that $g \geqslant 0$. If $g = 0$, then M is a Gorenstein injective module, and the desired equality follows immediately from Lemma (2.1).

Assume now that $g > 0$. There is an exact triangle in $\mathsf{D}(R)$

$$J \to I \to M \to \Sigma J$$

where J is a Gorenstein injective module, and I is a complex with $\operatorname{id}_R I = g$. This is dual to the special case $n = \inf N = 0$ of [10, thm. 3.1]. By the Chouinard formula for injective dimension [28, thm. 2.10], there is a prime ideal $\mathfrak{p}$ such that $\operatorname{width}_{R_{\mathfrak{p}}} I_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - g$. By Lemma (2.1) one has

$$\operatorname{width}_{R_{\mathfrak{p}}} J_{\mathfrak{p}} \geqslant \operatorname{depth} R_{\mathfrak{p}} > \operatorname{width}_{R_{\mathfrak{p}}} I_{\mathfrak{p}}$$

so from the exact sequence of homology modules

$$(1) \quad \cdots \to \operatorname{H}_{i+1}(M \otimes^{\mathbf{L}}_R R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \to \operatorname{H}_i(J \otimes^{\mathbf{L}}_R R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \to \operatorname{H}_i(I \otimes^{\mathbf{L}}_R R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \to \cdots$$

one gets the equality $\operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \operatorname{width}_{R_{\mathfrak{p}}} I_{\mathfrak{p}}$. This proves the inequality "$\leqslant$".

For the opposite inequality, let a prime $\mathfrak{q}$ be given. If $\operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \geqslant \operatorname{depth} R_{\mathfrak{q}}$, then $g > \operatorname{depth} R_{\mathfrak{q}} - \operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}}$; so assume $\operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} < \operatorname{depth} R_{\mathfrak{q}}$. Again (1) yields $\operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} = \operatorname{width}_{R_{\mathfrak{q}}} I_{\mathfrak{q}}$, as one has $\operatorname{width}_{R_{\mathfrak{q}}} J_{\mathfrak{q}} \geqslant \operatorname{depth} R_{\mathfrak{q}}$ by Lemma (2.1). Now the inequality $g \geqslant \operatorname{depth} R_{\mathfrak{q}} - \operatorname{width}_{R_{\mathfrak{q}}} M_{\mathfrak{q}}$ follows from the Chouinard formula for injective dimension. □

For modules, the Bass formula below is proved in [24, cor. 2.5]. Our argument is similar; the key tools are [16, thm. 3.6] and Theorem (2.2).

(2.3) **Corollary.** *Let R be local, and let M be a homologically finite R-complex. If M has finite Gorenstein injective dimension, then there is an equality*

$$\operatorname{Gid}_R M = \operatorname{depth} R - \inf M.$$

Proof. Let $\mathfrak{p}$ be a prime ideal in R, and choose a prime ideal $\mathfrak{q}$ in $\widehat{R}$ minimal over $\mathfrak{p}\widehat{R}$. The map $R_{\mathfrak{p}} \to \widehat{R}_{\mathfrak{q}}$ is local and flat with artinian closed fiber $\widehat{R}_{\mathfrak{q}}/\mathfrak{p}\widehat{R}_{\mathfrak{q}}$. Hence one has $\operatorname{depth} R_{\mathfrak{p}} - \inf M_{\mathfrak{p}} = \operatorname{depth} \widehat{R}_{\mathfrak{q}} - \inf (\widehat{R} \otimes_R M)_{\mathfrak{q}}$, and from Theorem (2.2) follows the inequality

$$\operatorname{Gid}_R M \leqslant \operatorname{Gid}_{\widehat{R}}(\widehat{R} \otimes_R M).$$

By [16, thm. 3.6] the complex $\widehat{R} \otimes_R M$ has finite Gorenstein injective dimension over $\widehat{R}$. Since $\widehat{R}$ has a dualizing complex, one has

$$\operatorname{Gid}_{\widehat{R}}(\widehat{R} \otimes_R M) = \operatorname{depth} \widehat{R} - \inf (\widehat{R} \otimes_R M) = \operatorname{depth} R - \inf M$$

by [8, thm. 6.3]. The two displays combine to establish the inequality "$\leqslant$"; the opposite one is from Theorem (2.2). □

3. Module structures and vanishing of homology

The first result of this section contains Theorem B from the introduction. Indeed, when R is Gorenstein, every R-module has finite Gorenstein injective dimension, cf. [5, thm. (6.2.7)], and (3.1) applies to the natural map $R \to \widehat{R}$.

(3.1) **Theorem.** *Let $\varphi\colon R \to S$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module with $\operatorname{Gid}_R M$ finite. If the S-module $\operatorname{Ext}^i_R(S,M)$ is finitely generated for every $i = 1,\dots,\dim_R M$, then one has $\operatorname{Ext}^i_R(S,M) = 0$ for $i \geqslant 1$, and M has an S-module structure compatible with its R-module structure via φ.*

Proof. The module $\operatorname{Hom}_R(S,M)$ is finitely generated over R and hence over S; and the modules $\operatorname{Ext}^i_R(S,M)$ vanish for $i > \dim_R M$; see [19, cor. 1.7 and proof of thm. 2.5]. Thus, the S-complex $\mathbf{R}\operatorname{Hom}_R(S,M)$ is homologically finite. In the sequence below the first and third equalities are from Corollary (2.3)

$$\operatorname{depth} R = \operatorname{Gid}_R M = \operatorname{Gid}_S \mathbf{R}\operatorname{Hom}_R(S,M) = \operatorname{depth} S - \inf \mathbf{R}\operatorname{Hom}_R(S,M).$$

The second equality is from Theorem (1.7). The assumptions on φ imply that R and S have the same depth, whence $\inf \mathbf{R}\operatorname{Hom}_R(S,M) = 0$. This establishes the desired vanishing of Ext-modules, and the existence of the S-structure on M follows from [19, thm. 2.5]. □

(3.2) **Remark.** If R is Gorenstein, then every finitely generated complete R-module (in particular, every R-module of finite length) satisfies the hypotheses of Theorem (3.1). See [19, thm. 2.5] or [3, thm. 2.3].

The next example shows that the flatness hypothesis is necessary for the equality in Theorems (1.7) and (1.8).

(3.3) **Example.** Let R be a complete Cohen-Macaulay local ring with a non-maximal prime ideal $\mathfrak{p} \subset R$ such that $R_\mathfrak{p}$ is not Gorenstein. For example, the ring could be $R = k[\![X,Y,Z]\!]/(X^2,XY,Y^2)$ with prime ideal $\mathfrak{p} = (X,Y)R$.

As an R-module, $R_\mathfrak{p}$ has infinite Gorenstein injective dimension. Indeed, if $\operatorname{Gid}_R R_\mathfrak{p} < \infty$, then $\operatorname{Gid}_{R_\mathfrak{p}} R_\mathfrak{p}$ is finite as well by [8, prop. 5.5], and this contradicts the assumption that $R_\mathfrak{p}$ is not Gorenstein; cf. [5, thm. (6.3.2)].

Let $\mathbf{x} = x_1,\dots,x_d$ be a maximal R-regular sequence and set $S = R/(\mathbf{x})$. The surjection $R \twoheadrightarrow S$ is a homomorphism of finite flat dimension. The small supports of S and $R_\mathfrak{p}$ are disjoint, so [15, lem. 2.6 and prop. 2.7] yields $\operatorname{H}(S \otimes^{\mathbf{L}}_R R_\mathfrak{p}) = 0$. The complexes $S \otimes^{\mathbf{L}}_R R_\mathfrak{p}$ and $\mathbf{R}\operatorname{Hom}_R(S,R_\mathfrak{p})$ are isomorphic (up to a shift) in $\mathsf{D}(R)$. In particular, one has

$$\operatorname{Gfd}_S(S \otimes^{\mathbf{L}}_R R_\mathfrak{p}) = -\infty = \operatorname{Gid}_S \mathbf{R}\operatorname{Hom}_R(S,R_\mathfrak{p}),$$

but $\operatorname{Gfd}_R R_\mathfrak{p} = 0$ and $\operatorname{Gid}_R R_\mathfrak{p} = \infty$.

(3.4) **Remark.** No finitely generated R-module can take the place of $R_\mathfrak{p}$ in Example (3.3). Indeed, let $\varphi\colon R \to S$ be a local ring homomorphism, and let $M \neq 0$ be a finitely generated R-module. As $S/\mathfrak{m}S$ and $M/\mathfrak{m}M$ are not zero, then Nakayama's lemma yields $S \otimes_R M \neq 0$, whence $\operatorname{H}(S \otimes^{\mathbf{L}}_R M)$ is not zero. Assume that φ has finite flat dimension. Then (2.0.2) yields $\operatorname{H}(\mathbf{R}\operatorname{Hom}_R(S,M)) \neq 0$ because $\operatorname{depth}_R S$ and $\operatorname{width}_R M$ are both finite. Now [17, thm. 4.8] yields $\operatorname{Gfd}_S(S \otimes^{\mathbf{L}}_R M) =$

$\operatorname{Gfd}_R M$. Assuming further that φ is module-finite, the corresponding equality $\operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S,M) = \operatorname{Gid}_R M$ is proved below.

(3.5) **Proposition.** *Let $\varphi\colon R \to S$ be a module-finite local ring homomorphism of finite flat dimension, and assume that R admits a dualizing complex. For every homologically finite R-complex M one then has*

$$\operatorname{Gid}_R M = \operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S,M).$$

Proof. By (2.0.2) one has $\inf \mathbf{R}\mathrm{Hom}_R(S,M) = \operatorname{depth} S + \inf M - \operatorname{depth} R$, so by Corollary (2.3) it is sufficient to prove that $\operatorname{Gid}_R M$ is finite if and only if $\operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S,M)$ is finite. The "only if" is already known from Theorem (1.7), so assume that $\operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S,M)$ is finite.

Let D be a dualizing complex for R. Since the homomorphism φ is module-finite, the complex $\mathbf{R}\mathrm{Hom}_R(S,D)$ is dualizing for S, cf. [2, (2.12)]. By [8, cor. 6.4] the complex $\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S,M), \mathbf{R}\mathrm{Hom}_R(S,D))$ has finite Gorenstein flat dimension over S. Adjunction and Hom-evaluation [1, lem. 4.4] yield

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S,M), \mathbf{R}\mathrm{Hom}_R(S,D)) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(S,M), D)\\ &\simeq S \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,D).\end{aligned}$$

It follows from [17, thm. 4.8] that $\mathbf{R}\mathrm{Hom}_R(M,D)$ has finite Gorenstein flat dimension over R, and therefore [8, cor. (6.4)] implies that $\operatorname{Gid}_R M$ is finite. □

(3.6) **Proposition.** *Let $\varphi\colon R \to S$ be a module-finite local ring homomorphism of finite flat dimension. For every finitely generated complete R-module M one has*

$$\operatorname{Gid}_R M = \operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S,M).$$

Proof. Since M is finitely generated and complete, it follows from [19, thm. 2.5] that M is isomorphic to $\mathrm{Hom}_R(\widehat{R},M)$ and that one has $\mathrm{Ext}^i_R(\widehat{R},M) = 0$ for $i \geq 1$. In particular, the complex $\mathbf{R}\mathrm{Hom}_R(\widehat{R},M)$ is homologically finite over $\widehat{R}$.

Let $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ denote the local homomorphism induced on completions. Since S is module finite and has finite flat dimension over R, the completion $\widehat{S}$ is module finite and has finite flat dimension over $\widehat{R}$. Theorem (1.7) explains the first and fourth equalities in the next sequence:

$$\begin{aligned}\operatorname{Gid}_R M &= \operatorname{Gid}_{\widehat{R}} \mathbf{R}\mathrm{Hom}_R(\widehat{R},M)\\ &= \operatorname{Gid}_{\widehat{S}} \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{S}, \mathbf{R}\mathrm{Hom}_R(\widehat{R},M))\\ &= \operatorname{Gid}_{\widehat{S}} \mathbf{R}\mathrm{Hom}_S(\widehat{S}, \mathbf{R}\mathrm{Hom}_R(S,M))\\ &= \operatorname{Gid}_S \mathbf{R}\mathrm{Hom}_R(S,M).\end{aligned}$$

The third equality is due to the isomorphisms

$$\mathbf{R}\mathrm{Hom}_S(\widehat{S}, \mathbf{R}\mathrm{Hom}_R(S,M)) \simeq \mathbf{R}\mathrm{Hom}_R(\widehat{S},M) \simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}}(\widehat{S}, \mathbf{R}\mathrm{Hom}_R(\widehat{R},M))$$

and the second equality is from Proposition (3.5). □

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