GORENSTEIN WEAK GLOBAL DIMENSION IS SYMMETRIC

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ABSTRACT. We study the Gorenstein weak global dimension of associative rings and its relation to the Gorenstein global dimension. In particular, we prove the conjecture that the Gorenstein weak global dimension is a left–right symmetric invariant—just like the (absolute) weak global dimension.

INTRODUCTION

A guiding principle in Gorenstein homological algebra is to seek analogues of results about absolute homological dimensions. For example, the Gorenstein global dimension of a ring can equally well be computed in terms of the Gorenstein projective or Gorenstein injective dimensions of its modules; this was proved by Enochs and Jenda [13] in the noetherian case and by Bennis and Mahdou [5] in general.

The notion of a weak global dimension has also been considered in Gorenstein homological algebra, for example by Emmanouil [11]. The weak global dimension of a ring is a left–right symmetric invariant; that is, a ring has finite weak global dimension on the left if and only if it enjoys the same property on the right. In Section 1 we prove the corresponding statement in Gorenstein homological algebra, thus confirming a widely held conjecture that was formally stated by Bennis [3].

In Section 2 we use this symmetry to investigate the relations between the Gorenstein global and Gorenstein weak global dimensions. The main result of this section, Theorem 2.3, shows that finite Gorenstein weak global dimension together with finite projective dimension of flat modules implies finite Gorenstein global dimension; under extra assumptions on the ring this was proved by Bennis and Mahdou [4]. The same theorem relates the Gorenstein global dimension to a new invariant: the Gorenstein flat-cotorsion dimension, which was introduced in [7]. In fact, this new invariant plays a key role already in the proof of Theorem 1.4, the main result of the first section. The invariant is built on the theory of Gorenstein flat-cotorsion modules developed in [8] as well as recent work of Šaroch and Šťovíček [20].

$$* \quad * \quad *$$

Throughout the paper, A denotes an associative ring. By an A-module we mean a left A-module, and we treat right A-modules as modules over the opposite ring $A°$. By an *A-complex* we mean a complex of A-modules. For such a complex M and an

integer n, the hard truncation of M above at n is denoted $M_{\leqslant n}$ while $M_{\geqslant n}$ denotes the hard truncation of M below at n. For $v \in \mathbb{Z}$ the cycle module in degree v, i.e. the kernel of ∂_v^M is denoted $\mathrm{Z}_v(M)$, while $\mathrm{C}_v(M)$ denotes the cokernel module in degree v, i.e. the cokernel of ∂_{v+1}^M. We say that M has *bounded* homology if $\mathrm{H}_v(M) = 0$ holds for $|v| \gg 0$.

The notation and terminology above is all standard; the only non-standard terminology applied in this paper comes from [7, 8]: An acyclic complex T of flat-cotorsion A-modules is called *totally acyclic* if $\mathrm{Hom}_A(T, F)$ is acyclic for every flat-cotorsion A-module F. A cycle in such a complex is called a *Gorenstein flat-cotorsion* module. A semi-flat complex of flat-cotorsion modules is called *semi-flat-cotorsion.* In the derived category of A, every complex M is isomorphic to a semi-flat-cotorsion complex—see for example [7, Cnstr. 2.4]; such a complex is called a semi-flat-cotorsion replacement of M. The Gorenstein flat-cotorsion dimension of an A-complex M is denoted $\mathrm{Gfcd}_A M$; it is the least $n \geqslant \sup\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\}$ such that the nth cokernel in a semi-flat-cotorsion replacement of M is Gorenstein flat-cotorsion.

For the absolute homological dimensions we use two-letter abbreviations—pd, id, and fd—and we write Gpd, Gid, and Gfd for the corresponding Gorenstein dimensions. The notation for the invariant

$$\mathrm{splf}(A) \;=\; \sup\{\mathrm{pd}_A F \mid F \text{ is a flat } A\text{-module}\}$$

is another acronym, "splf" stands for "supremum of projective lengths of flat modules." The invariants sfli, spli, silp, and silf are defined similarly; see [11, §1.2].

1. Symmetry of Gorenstein weak global dimension

Holm proves in [16, Thm. 2.6] that if A is coherent and $\mathrm{splf}(A^\circ)$ is finite, then the equality $\mathrm{Gfd}_A M = \mathrm{fd}_A M$ holds for A-modules of finite injective dimension. The key to our proof of the main result in this section is to show that this equality holds without the assumptions on A. By the work done in [7] it suffices to prove the analogous equality for the Gorenstein flat-cotorsion dimension, and since this is a result of independent interest, we prove it for complexes.

1.1 Theorem. *Let M be an A-complex with bounded homology. If M has finite injective dimension, then the equality $\mathrm{Gfcd}_A M = \mathrm{fd}_A M$ holds.*

Proof. The equality $\mathrm{Gfcd}_A M = \mathrm{fd}_A M$ holds trivially if M is acyclic, so assume that M is not acyclic and assume further, without loss of generality, that $\mathrm{id}_A M = 0$ holds. Set $w = \sup\{v \in \mathbb{Z} \mid \mathrm{H}_v(M) \neq 0\}$ and let $M \xrightarrow{\simeq} I$ be a semi-injective resolution with $I_v = 0$ for $v > w$ and $v < 0$. There is an exact sequence of complexes $0 \to C' \to F \to I \to 0$ with F semi-flat-cotorsion and C' an acyclic complex of cotorsion modules; this follows from work of Gillespie [14], see also [7, Fact 2.2]. Acyclicity of C' yields an exact sequence $0 \to \mathrm{Z}_0(C') \to \mathrm{Z}_0(F) \to I_0 \to 0$. Since both $\mathrm{Z}_0(C')$ and I_0 are cotorsion—for the former see Bazzoni, Cortés-Izurdiaga, and Estrada [1, Thm. 1.3]—so is $\mathrm{Z}_0(F)$. There is thus a semi-flat-cotorsion resolution $F' \to \mathrm{Z}_0(F)$ concentrated in non-negative degrees, constructed by taking successive flat covers. This complex glued together with $F_{\leqslant 0}$ is acyclic and semi-flat, see Christensen and Holm [9, 6.1], so per [9, Thm. 7.3] the module $\mathrm{Z}_0(F)$ is flat-cotorsion, and we may assume that $F_v = 0$ holds for $v < 0$.

Fix $n \geqslant \mathrm{Gfcd}_A M$; the module $\mathrm{C}_n(F)$ is Gorenstein flat-cotorsion by [7, Lem. 4.3]. We argue next that $\mathrm{Ext}_A^1(G, \mathrm{C}_n(F)) = 0$ holds for every Gorenstein flat-cotorsion module G. Fix such a module G. By definition, there is an exact sequence of A-modules, $0 \to G \to T_0 \to \cdots \to T_{-n+1} \to G' \to 0$, with each T_v flat-cotorsion and G' Gorenstein flat-cotorsion. As $\mathrm{C}_n(F)$ is cotorsion, dimension shifting yields:

$$\mathrm{Ext}_A^1(G, \mathrm{C}_n(F)) \cong \mathrm{Ext}_A^{n+1}(G', \mathrm{C}_n(F)) \,.$$

Let C be the mapping cone of the quasi-isomorphism $F \to I$; it is concentrated in non-negative degrees and consists of sums of modules that are flat-cotorsion or injective. Moreover, one has $C_0 = I_0$ as $F_v = 0$ for $v < 0$, and because $I_v = 0$ for $v > w$ and $n \geqslant w$ holds, there is an isomorphism $\mathrm{C}_{n+1}(C) \cong \mathrm{C}_n(F)$. Thus dimension shifting along

$$0 \longrightarrow \mathrm{C}_{n+1}(C) \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_1 \longrightarrow I_0 \longrightarrow 0$$

yields

$$\mathrm{Ext}_A^{n+1}(G', \mathrm{C}_n(F)) \cong \mathrm{Ext}_A^1(G', I_0) = 0 \,.$$

Combining the displayed isomorphisms one gets $\mathrm{Ext}_A^1(G, \mathrm{C}_n(F)) = 0$. Now, with $g = \mathrm{Gfcd}_A M$ and $n = g + 1$ and $G = \mathrm{C}_g(F)$ one has $\mathrm{Ext}_A^1(\mathrm{C}_g(F), \mathrm{C}_{g+1}(F)) = 0$. This means that the exact sequence $0 \to \mathrm{C}_{g+1}(F) \to F_g \to \mathrm{C}_g(F) \to 0$ splits, whence $\mathrm{C}_g(F)$ is flat-cotorsion. Thus one has $\mathrm{fd}_A M \leqslant g$, and the opposite inequality holds by [7, Lem. 5.11].   □

In particular we now have the desired strengthening of [16, Thm. 2.6].

1.2 Corollary. *Let M be an A-complex with bounded homology. If M has finite injective dimension, then the equality $\mathrm{Gfd}_A M = \mathrm{fd}_A M$ holds.*

Proof. The Gorenstein flat dimension is a refinement of the flat dimension, so if $\mathrm{Gfd}_A M = \infty$ holds, then the equality is trivial. If $\mathrm{Gfd}_A M < \infty$, then [7, Thm. 5.7] yields $\mathrm{Gfcd}_A M = \mathrm{Gfd}_A M$ and the asserted equality follows from Theorem 1.1.   □

The Gorenstein global dimension of A, denoted $\mathrm{Ggldim}(A)$, is the supremum of the Gorenstein projective dimensions (equivalently, see [5, Thm. 1.1], the Gorenstein injective dimensions) of all A-modules.

1.3 Definition. The *Gorenstein weak global dimension* of A is

$$\mathrm{Gwgldim}(A) = \sup\{\mathrm{Gfd}_A M \mid M \text{ is an } A\text{-module}\} \,.$$

This is the invariant that Bennis and Mahdou denote $l.\,\mathrm{wGgldim}(A)$ in [3, 5] and G-wdim(A) in [4]. When $\mathrm{Gwgldim}(A)$ and $\mathrm{Gwgldim}(A^\circ)$ are finite, and only then, Emmanouil [11] uses the symbol Gw. dim A for their common value.

If $\mathrm{Gwgldim}(A)$ is finite then so is $\mathrm{sfli}(A^\circ)$; this is elementary, see [11, Lem. 5.1]. On the other hand, if both $\mathrm{sfli}(A)$ and $\mathrm{sfli}(A^\circ)$ are finite, then per [11, Thm. 5.3] both $\mathrm{Gwgldim}(A)$ and $\mathrm{Gwgldim}(A^\circ)$ are finite. Thus, the key to prove symmetry of the Gorenstein weak global dimension is to see that $\mathrm{Gwgldim}(A) < \infty$ implies $\mathrm{sfli}(A) < \infty$. In our proof of Theorem 1.4 this follows from Corollary 1.2, which through [7, Thm. 5.7] relies crucially on the work of Šaroch and Šťovíček [20]. In Remark 1.6 we sketch how to obtain symmetry directly from [20]. However, there is more to Theorem 1.4: In the next section it facilitates the comparison of $\mathrm{Gwgldim}(A)$ to $\mathrm{Ggldim}(A)$, see for example Corollary 2.6.

1.4 Theorem. *The following conditions are equivalent.*

  (i) $\mathrm{Gwgldim}(A) < \infty$.
 (ii) *All A-modules have finite Gorenstein flat dimension.*
(iii) $\mathrm{sfli}(A)$ *and* $\mathrm{sfli}(A^\circ)$ *are finite.*
(iv) *All A- and A°-modules have finite Gorenstein flat dimension.*
 (v) *All A- and A°-modules have finite Gorenstein flat-cotorsion dimension.*

Proof. Conditions (i) and (ii) are equivalent as the class of Gorenstein flat modules is closed under coproducts; see for example Holm [15, Prop. 3.2]. Conditions (iii) and (iv) are equivalent by [11, Thm. 5.3], and (iv) evidently implies (ii).

   (ii)$\Longrightarrow$(iii): It follows from Corollary 1.2 that $\mathrm{sfli}(A)$ is finite and from [11, Lem. 5.1] that $\mathrm{sfli}(A^\circ)$ is finite.

   (iv)$\Longrightarrow$(v): The Gorenstein flat-cotorsion dimension of a module is bounded above by its Gorenstein flat dimension, see [7, Thm. 5.7].

   (v)$\Longrightarrow$(iii): It follows from Theorem 1.1 that $\mathrm{sfli}(A)$ and $\mathrm{sfli}(A^\circ)$ are finite. $\square$

   The next equality was formally conjectured by Bennis [3, Conj. 1.1]. For emphasis, we point out that it shows that Emmanouil's [11] definition can be relaxed: It suffices to consider finiteness of Gorenstein flat dimensions on one side of the ring.

1.5 Corollary. *One has* $\mathrm{Gwgldim}(A) = \mathrm{Gwgldim}(A^\circ)$.

Proof. The invariants $\mathrm{Gwgldim}(A)$ and $\mathrm{Gwgldim}(A^\circ)$ are simultaneously finite by Theorem 1.4, and when finite they are equal by [11, Thm. 5.3]. $\square$

1.6 Remark. That $\mathrm{Gwgldim}(A) < \infty$ implies $\mathrm{sfli}(A) < \infty$ can be deduced directly from [20]: In the notation of that paper, given a module $M \in \mathcal{PGF}^\perp$, there exists by [20, Thm. 4.9] an exact sequence, $0 \to H \to T_{n-1} \to \cdots \to T_0 \to M \to 0$, with each T_v a projective A-module and H in $\mathcal{PGF}^\perp$. If $\mathrm{Gfd}_A M \leqslant n$, then H is also Gorenstein flat, hence flat per [20, Thm. 4.11].

2. Comparing Gorenstein global dimensions

In this section, we consider relations between finiteness of the Gorenstein global dimensions. We begin with a key lemma that compares the relevant invariants at the level of (complexes of) modules.

2.1 Lemma. *For every A-complex M with $\mathrm{H}(M) \neq 0$ one has*

$$\mathrm{Gpd}_A M \leqslant \mathrm{Gfd}_A M + \mathrm{splf}(A)\,.$$

Proof. Set $n = \mathrm{splf}(A)$ and assume that it is finite. Let M be an A-complex with $\mathrm{Gfd}_A M = d$ for some integer d. Let $P \to M$ be a semi-projective resolution; the module $C = \mathrm{C}_d(P)$ is Gorenstein flat—see Christensen, Köksal, and Liang [10, Prop. 5.12][1]—and it suffices to show that $\mathrm{Gpd}_A C \leqslant n$ holds, as this implies that $\mathrm{C}_{d+n}(P)$ is Gorenstein projective. By assumption there is an acyclic complex, $0 \to C \to F_0 \to F_{-1} \to \cdots$, with each module F_v flat and each cokernel Gorenstein flat. As in Cartan and Eilenberg's [6, Chapter XVII, §1], or the proof of [11,

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[1]Every ring is GF-closed by [20, Cor. 4.12].

Lem. 5.2], construct a projective resolution of this complex in the category of A-complexes:

This induces an exact sequence

$$0 \longrightarrow \mathrm{C}_n(Q) \longrightarrow \mathrm{C}_n(Q^{(0)}) \longrightarrow \mathrm{C}_n(Q^{(-1)}) \longrightarrow \cdots .$$

The class of Gorenstein flat A-modules is resolving by [20, Cor. 4.12], so the module $\mathrm{C}_n(Q)$ is Gorenstein flat. By assumption, $\mathrm{C}_n(Q^{(i)})$ is projective for $i \leqslant 0$, and by construction the cokernels of the exact sequence are Gorenstein flat, so $\mathrm{C}_n(Q)$ is Gorenstein projective by [20, Thm. 4.4]. Thus $\mathrm{Gpd}_A C \leqslant n$ holds as desired. $\qquad\square$

Jiangsheng Hu pointed us to the following easy consequence of Lemma 2.1.

2.2 Proposition. *Let n be an integer. The following conditions are equivalent.*

(i) $\mathrm{splf}(A) \leqslant n$.

(ii) *Every Gorenstein flat A-module has Gorenstein projective dimension at most n and every flat Gorenstein projective A-module is projective.*

(iii) *Every flat A-module has Gorenstein projective dimension at most n and every flat Gorenstein projective A-module is projective.*

Proof. Evidently, (ii) implies (iii).

$(i) \Longrightarrow (ii)$: For a Gorenstein flat A-module M, Lemma 2.1 yields $\mathrm{Gpd}_A M \leqslant n$. A flat Gorenstein projective A-module G is projective as $\mathrm{Gpd}_A G = \mathrm{pd}_A G$ holds because the Gorenstein projective dimension refines the projective dimension.

$(iii) \Longrightarrow (i)$: The nth syzygy of a flat A-module is Gorenstein projective and flat, hence projective. $\qquad\square$

This brings us to the main result of this section; it compares to [4, Thm. 2.1] as does Corollary 2.5.

2.3 Theorem. *There are inequalities*

$$\sup\{\mathrm{Gfcd}_A M \mid M \text{ is an } A\text{-module}\} \ \leqslant \ \mathrm{Ggldim}(A) \ \leqslant \ \mathrm{Gwgldim}(A) + \mathrm{splf}(A) \,.$$

Proof. The second inequality follows immediately from Lemma 2.1. To prove the first inequality, set $n = \mathrm{Ggldim}(A)$; we may assume that it is finite. By [11, Thm. 4.1] one has $\mathrm{spli}(A) = n = \mathrm{silp}(A)$, and a result of Emmanouil and Talelli [12, Prop. 2.1] yields $\mathrm{silf}(A) = n$. We first show that every cotorsion A-module C has $\mathrm{Gfcd}_A C \leqslant n$. To see this, let $C \to I$ be an injective resolution. For every $i \leqslant 0$

there is an exact sequence $0 \to \mathrm{Z}_i(I) \to I_i \to \mathrm{Z}_{i-1}(I) \to 0$ of cotorsion modules. Construct flat resolutions $G \to C = \mathrm{Z}_0(I)$ and $G^{(i)} \to \mathrm{Z}_i(I)$ for $i < 0$ by taking successive flat covers. By [13, Lem. 8.2.1] there are flat resolutions $F^{(i)} \to I_i$ which fit into exact sequences $0 \to G^{(i)} \to F^{(i)} \to G^{(i-1)} \to 0$, such that each module $F_j^{(i)}$ is flat-cotorsion, and each syzygy module $\mathrm{C}_j(F^{(i)})$ is cotorsion. A standard construction, as in [6, Chap. XVII.§1], yields a commutative diagram

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_1 & \longrightarrow & F_1^{(0)} & \longrightarrow & F_1^{(-1)} & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G_0 & \longrightarrow & F_0^{(0)} & \longrightarrow & F_0^{(-1)} & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C & \longrightarrow & I_0 & \longrightarrow & I_{-1} & \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

with exact rows and columns. This induces an exact sequence

$$ 0 \longrightarrow \mathrm{C}_n(G) \longrightarrow \mathrm{C}_n(F^{(0)}) \longrightarrow \mathrm{C}_n(F^{(-1)}) \longrightarrow \cdots . $$

Since $\mathrm{sfli}(A) \leqslant \mathrm{spli}(A) = n$ holds, the modules $\mathrm{C}_n(F^{(i)})$ are flat for all $i \leqslant 0$, and by construction they are cotorsion. The complex $G_{\geqslant n}$ is a resolution of $\mathrm{C}_n(G)$ by flat-cotorsion modules, so $\mathrm{C}_n(G)$ is a syzygy module in an acyclic complex of flat-cotorsion A-modules. As $\mathrm{silf}(A)$ is finite, this complex is totally acyclic—indeed, for an acyclic complex X of flat modules and a cotorsion module Y of finite injective dimension, a standard dimension shifting argument, as in the proof of Theorem 1.1, shows that the complex $\mathrm{Hom}_A(X, Y)$ is acyclic—so the module $\mathrm{C}_n(G)$ is Gorenstein flat-cotorsion. Thus $\mathrm{Gfcd}_A C \leqslant n$.

Finally, let M be an A-module and $F \to M$ a flat resolution built from flat covers. Since $\mathrm{C}_1(F)$ is cotorsion, the module $\mathrm{C}_{n+1}(F)$ is Gorenstein flat-cotorsion. Work of Gillespie [14], see also [7, Fact 2.2], yields an exact sequence,

$$ 0 \longrightarrow F \longrightarrow C \longrightarrow P \longrightarrow 0 , $$

with C degreewise cotorsion and P an acyclic complex of flat modules with flat cycle modules. It follows that C is a semi-flat-cotorsion replacement of M, see [7, Fact 1.4]. For $i \geqslant 1$ the exact sequence $0 \to F_i \to C_i \to P_i \to 0$ shows that P_i is cotorsion. As all the modules $\mathrm{C}_i(P)$ are flat, [7, Lemma 5.6] applied to $P_{\geqslant 1}$ shows that $\mathrm{C}_{n+1}(P)$ is flat-cotorsion. It follows that the exact sequence

$$ 0 \longrightarrow \mathrm{C}_{n+1}(F) \longrightarrow \mathrm{C}_{n+1}(C) \longrightarrow \mathrm{C}_{n+1}(P) \longrightarrow 0 $$

splits, whence $\mathrm{C}_{n+1}(C)$ is Gorenstein flat-cotorsion; in particular, $\mathrm{Gfcd}_A M$ is finite. There is an exact sequence of A-modules, $0 \to M \to C' \to F' \to 0$, with C' cotorsion and F' flat. As we have shown above that $\mathrm{Gfcd}_A C' \leqslant n$ holds, it now follows from [7, Thm. 4.5] that also $\mathrm{Gfcd}_A M \leqslant n$. $\qquad\square$

2.4 Remark. Recall that the invariant

$$\text{FPD}(A) \,=\, \sup\{\text{pd}_A\, M \mid M \text{ has finite projective dimension}\}$$

is known as the *finitistic projective dimension* of A. By a result of Jensen [18, Prop. 6] one has $\text{splf}(A) \leqslant \text{FPD}(A)$. By [15, Thm. 2.28] there is an inequality $\text{FPD}(A) \leqslant \text{Ggldim}(A)$, and equality holds if $\text{Ggldim}(A)$ is finite.

2.5 Corollary. *Assume that A is right coherent. There are inequalities*

$$\text{Gwgldim}(A) \,\leqslant\, \text{Ggldim}(A) \,\leqslant\, \text{Gwgldim}(A) + \text{splf}(A)\,.$$

Moreover, the following conditions are equivalent.

   (i) $\text{Gwgldim}(A)$ *and* $\text{splf}(A)$ *are finite.*
  (ii) $\text{Ggldim}(A)$ *is finite.*

Proof. Since A is right coherent, the equality $\text{Gfcd}_A\, M = \text{Gfd}_A\, M$ holds for every A-module M by [7, Cor. 5.8]. Hence, one has

$$\sup\{\text{Gfcd}_A\, M \mid M \text{ is an } R\text{-module}\} \,=\, \text{Gwgldim}(A)\,,$$

and the asserted inequalities follow from Theorem 2.3.

It is immediate from the second inequality that (i) implies (ii). For the converse, assume that $\text{Ggldim}(A)$ is finite; it follows from the first inequality that $\text{Gwgldim}(A)$ is finite, and $\text{splf}(A)$ is finite by Remark 2.4. □

The next corollary applies, in particular, to commutative rings.

2.6 Corollary. *If A and A° are isomorphic, then the next conditions are equivalent.*

   (i) $\text{Gwgldim}(A)$ *and* $\text{splf}(A)$ *are finite.*
  (ii) $\text{Ggldim}(A)$ *is finite.*

Proof. Immediate from Theorems 1.4 and 2.3, along with Remark 2.4. □

2.7 Corollary. *If $\text{Gwgldim}(A)$ is finite, then $\text{Ggldim}(A) = \text{FPD}(A)$ holds, and the invariants $\text{splf}(A)$ and $\text{FPD}(A)$ are simultaneously finite.*

Proof. By Remark 2.4 the equality holds if $\text{Ggldim}(A)$ is finite. Assume that $\text{Gwgldim}(A)$ is finite. If $\text{FPD}(A)$ is finite, then $\text{splf}(A)$ is finite by Remark 2.4, so $\text{Ggldim}(A)$ is finite by Theorem 2.3. This proves the equality, and the last assertion follows as finiteness of $\text{splf}(A)$ by 2.3 implies finiteness of $\text{Ggldim}(A)$. □

Simson [19] shows that a ring A of cardinality $\aleph_n$ has $\text{splf}(A) \leqslant n + 1$; for commutative rings this was shown earlier by Jensen [17, Thm. 5.8]. Thus, the next corollary yields, in particular, that the Gorenstein global dimension is symmetric for countable coherent rings.

2.8 Corollary. *If A is coherent, then the following conditions are equivalent.*

   (i) $\text{Ggldim}(A)$ *and* $\text{splf}(A^{\circ})$ *are finite.*
  (ii) $\text{Ggldim}(A^{\circ})$ *and* $\text{splf}(A)$ *are finite.*

Proof. The assertion follows from Corollary 2.5 combined with Corollary 1.5. □

2.9 Remark. For noetherian rings Beligiannis [2, Cor. 6.11] proved that the Goren-
stein global dimension is symmetric. Corollary 2.8 provides us with non-trivial new
examples of rings that exhibit this kind of Gorenstein symmetry; by non-trivial
we here mean rings of infinite global dimension. For instance, let R be a non-
commutative artinian ring and S a commutative coherent, but not noetherian, ring
of cardinality $\leqslant \aleph_n$. By [19] the direct product ring $R \times S$ satisfies the assumptions
in Corollary 2.8. Coherent quotients of the product ring $M_n(\mathbb{Q}) \times \mathbb{Q}[x_0, x_1, \ldots]$ are
simple examples of such rings.

One can replace the coherent assumption in Corollary 2.8 with assumptions of
Gorenstein flatness of Gorenstein projective modules.

2.10 Corollary. *The next conditions are equivalent.*
- (i) $\mathrm{Ggldim}(A)$ *and* $\mathrm{splf}(A^\circ)$ *are finite and every Gorenstein projective A-module
  is Gorenstein flat.*
- (ii) $\mathrm{Ggldim}(A^\circ)$ *and* $\mathrm{splf}(A)$ *are finite and every Gorenstein projective A°-module
  is Gorenstein flat.*

Proof. If $\mathrm{Ggldim}(A)$ is finite and every Gorenstein projective A-module is Goren-
stein flat, then it follows that $\mathrm{Gwgldim}(A)$ is finite. By Corollary 1.5 this implies
that $\mathrm{Gwgldim}(A^\circ)$ is finite. It follows that every cycle in an acyclic complex of flat
A°-modules is Gorenstein flat; in particular, every Gorenstein projective A°-module
is Gorenstein flat. Finally, Theorem 2.3 yields $\mathrm{Ggldim}(A^\circ) < \infty$ and $\mathrm{splf}(A)$ is finite
by Remark 2.4, □

2.11 Remark. The result of Beligiannis, [2, Cor. 6.11], mentioned in Remark 2.9
shows that a noetherian ring is Iwanaga–Gorenstein—that is, of finite self-injective
dimension on both sides—if it has finite Gorenstein global dimension on one side.
It follows from Theorem 1.4 and [7, Cor. 5.10] that a noetherian ring is Iwanaga–
Gorenstein if it has finite Gorenstein weak global dimension on one side. This
improves [13, Thm. 12.3.1].

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