HOMOTOPY CATEGORIES OF TOTALLY ACYCLIC COMPLEXES WITH APPLICATIONS TO THE FLAT–COTORSION THEORY

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To S.K. Jain on the occasion of his eightieth birthday

Abstract. We introduce a notion of total acyclicity associated to a subcategory of an abelian category and consider the Gorenstein objects they define. These Gorenstein objects form a Frobenius category, whose induced stable category is equivalent to the homotopy category of totally acyclic complexes. Applied to the flat–cotorsion theory over a coherent ring, this provides a new description of the category of cotorsion Gorenstein flat modules; one that puts it on equal footing with the category of Gorenstein projective modules.

Introduction

Let A be an associative ring. It is classic that the stable category of Gorenstein projective A-modules is triangulated equivalent to the homotopy category of totally acyclic complexes of projective A-modules. Under extra assumptions on A this equivalence can be found already in Buchweitz's 1986 manuscript [6]. In this paper we focus on a corresponding equivalence for Gorenstein flat modules. It could be pieced together from results in the literature, but we develop a framework that provides a direct proof while also exposing how closely the homotopical behavior of cotorsion Gorenstein flat modules parallels that of Gorenstein projective modules.

The category of Gorenstein flat A-modules is rarely Frobenius, indeed we prove in Theorem 4.5 that it only happens when every module is cotorsion. This is evidence that one should restrict attention to the category of cotorsion Gorenstein flat A-modules; in fact, it is already known from work of Gillespie [15] that this category is Frobenius if A is coherent. The associated stable category is equivalent to the homotopy category of **F**-totally acyclic complexes of flat-cotorsion A-modules; this follows from a theorem by Estrada and Gillespie [12] combined with recent work of Bazzoni, Cortés Izurdiaga, and Estrada [3]. The proof in [12] involves model structures on categories of complexes of projective modules, and one goal of this paper—with a view towards extending the result to non-affine schemes [7]—is to give a proof that avoids projective modules; we achieve this with Corollary 5.3.

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The pure derived category of flat A-modules is the Verdier quotient of the homotopy category of complexes of flat A-modules by the subcategory of pure-acyclic complexes; its subcategory of $\mathbf{F}$-totally acyclic complexes was studied by Murfet and Salarian [21]. We show in Theorem 5.6 that this subcategory is equivalent to the homotopy category of $\mathbf{F}$-totally acyclic complexes of flat-cotorsion A-modules, and thus to the stable category of cotorsion Gorenstein flat modules. Combining this with results of Christensen and Kato [8] and Estrada and Gillespie [12], one can derive that under extra assumptions on A, made explicit in Corollary 5.9, the stable category of Gorenstein projective A-modules is equivalent to the stable category of cotorsion Gorenstein flat A-modules.

Underpinning the results we have highlighted above are a framework, developed in Sections 1–3, and two results, Theorems 4.4 and 5.2, that show—as the semantics might suggest—that the cotorsion Gorenstein flat modules are, indeed, the Gorenstein modules naturally attached to the flat–cotorsion theory.

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Let A be an abelian category and U a subcategory of A. In 1.1 we define a *right* U-*totally acyclic complex* to be an acyclic $\hom_{\mathsf{A}}(-, \mathsf{U} \cap \mathsf{U}^{\perp})$-exact complex of objects from U with cycle objects in $\mathsf{U}^{\perp}$. Left U-total acyclicity is defined dually, and in the case of a self-orthogonal subcategory, left and right total acyclicity is the same; see Proposition 1.5. These definitions recover the standard notions of totally acyclic complexes of projective or injective objects; see Example 1.7. In the context of a cotorsion pair (U, V) the natural complexes to consider are right U-totally acyclic complexes, left V-totally acyclic complexes, and $(\mathsf{U} \cap \mathsf{V})$-totally acyclic complexes for the self-orthogonal category $\mathsf{U} \cap \mathsf{V}$.

In Section 2 we define left and right U-Gorenstein objects to be cycles in left and right U-totally acyclic complexes. In the context of a cotorsion pair (U, V), we show that the categories of right U-Gorenstein objects and left V-Gorenstein objects are Frobenius categories whose projective-injective objects are those in $\mathsf{U} \cap \mathsf{V}$; see Theorems 2.11 and 2.12. In Section 3 the stable categories induced by these Frobenius categories are shown to be equivalent to the corresponding homotopy categories of totally acyclic complexes. In particular, Corollary 3.9 recovers the classic results for Gorenstein projective objects and Gorenstein injective objects.

The literature contains a variety of generalized notions of totally acyclic complexes and Gorenstein objects; see for example Sather-Wagstaff, Sharif, and White [23]. We make detailed comparisons in Remark 2.3; at this point it suffices to say that our notion of Gorensteinness differs from the existing generalizations by exhibiting periodicity: For a self-orthogonal category W, the category of (W-Gorenstein)-Gorenstein objects is simply W; see Proposition 2.8.

1. Total acyclicity and other terminology

Throughout this paper, A denotes an abelian category; we write $\hom_{\mathsf{A}}$ for the hom-sets and the induced functor from A to abelian groups. Tacitly, subcategories of A are assumed to be full and closed under isomorphisms. A subcategory of A is called *additively closed* if it is additive and closed under direct summands.

A complex of objects from A is referred to as an A-complex. We use homological notation for complexes, so for a complex T the object in degree i is denoted T_i and $\mathrm{Z}_i(T)$ denotes the cycle subobject in degree i.

Let U and V be subcategories of A. The *right orthogonal* of U is the subcategory

$$\mathsf{U}^{\perp} = \{N \in \mathsf{A} \mid \operatorname{Ext}^1_{\mathsf{A}}(U,N) = 0 \text{ for all } U \in \mathsf{U}\}\ ;$$

the *left orthogonal* of V is the subcategory

$$^{\perp}\mathsf{V} = \{M \in \mathsf{A} \mid \operatorname{Ext}^1_{\mathsf{A}}(M,V) = 0 \text{ for all } V \in \mathsf{V}\}\ .$$

In case $\mathsf{U}^{\perp} = \mathsf{V}$ and $^{\perp}\mathsf{V} = \mathsf{U}$ hold, the pair (U,V) is referred to as a *cotorsion pair*.

In this section and the next, we develop notions of total acyclicity, and corresponding notions of Gorenstein objects, associated to any subcategory of A. Our primary applications are in the context of a cotorsion pair.

1.1 Definition. Let U and V be subcategories of A.

(R) An A-complex T is called *right* U*-totally acyclic* if the following hold:
  (1) T is acyclic.
  (2) For each $i \in \mathbb{Z}$ the object T_i belongs to U.
  (3) For each $i \in \mathbb{Z}$ the object $\mathrm{Z}_i(T)$ belongs to $\mathsf{U}^{\perp}$.
  (4) For each $W \in \mathsf{U} \cap \mathsf{U}^{\perp}$ the complex $\operatorname{hom}_{\mathsf{A}}(T,W)$ is acyclic.

(L) An A-complex T is called *left* V*-totally acyclic* if the following hold:
  (1) T is acyclic.
  (2) For each $i \in \mathbb{Z}$ the object T_i belongs to V.
  (3) For each $i \in \mathbb{Z}$ the object $\mathrm{Z}_i(T)$ belongs to $^{\perp}\mathsf{V}$.
  (4) For each $W \in {}^{\perp}\mathsf{V} \cap \mathsf{V}$ the complex $\operatorname{hom}_{\mathsf{A}}(W,T)$ is acyclic.

1.2 Example. Let U and V be subcategories of A. For every $W \in \mathsf{U} \cap \mathsf{U}^{\perp}$ a complex of the form $0 \longrightarrow W \xrightarrow{=} W \longrightarrow 0$ is right U-totally acyclic; similarly, for every $W \in {}^{\perp}\mathsf{V} \cap \mathsf{V}$ such a complex is left V-totally acyclic.

1.3 Proposition. *Let* U *and* V *be subcategories of* A*.*

(R) *An* A*-complex* T *is right* U*-totally acyclic if and only if the following hold:*
  (1) T *is acyclic.*
  (2) *For each* $i \in \mathbb{Z}$ *the object* T_i *belongs to* $\mathsf{U} \cap \mathsf{U}^{\perp}$*.*
  (3) *For each* $U \in \mathsf{U}$ *the complex* $\operatorname{hom}_{\mathsf{A}}(U,T)$ *is acyclic.*
  (4) *For each* $W \in \mathsf{U} \cap \mathsf{U}^{\perp}$ *the complex* $\operatorname{hom}_{\mathsf{A}}(T,W)$ *is acyclic.*

(L) *An* A*-complex* T *is left* V*-totally acyclic if and only if the following hold:*
  (1) T *is acyclic.*
  (2) *For each* $i \in \mathbb{Z}$ *the object* T_i *belongs to* $^{\perp}\mathsf{V} \cap \mathsf{V}$*.*
  (3) *For each* $V \in \mathsf{V}$ *the complex* $\operatorname{hom}_{\mathsf{A}}(T,V)$ *is acyclic.*
  (4) *For each* $W \in {}^{\perp}\mathsf{V} \cap \mathsf{V}$ *the complex* $\operatorname{hom}_{\mathsf{A}}(W,T)$ *is acyclic.*

Proof. (R): A complex T that satisfies Definition 1.1(R) trivially satisfies conditions (1), (3), and (4), while (2) follows from 1.1(R.2) and 1.1(R.3) as $\mathsf{U}^{\perp}$ is closed under extensions. Conversely, a complex T that satisfies conditions (1)–(4) in the statement trivially satisfies conditions (1), (2), and (4) in Definition 1.1(R). Moreover it follows from (2) and (3) that also condition 1.1(R.3) is satisfied.

The proof of (L) is similar. □

1.4 Example. A right A-totally acyclic complex is a contractible complex of injective objects, and a left A-totally acyclic complex is a contractible complex of projective objects.

In this paper we call a subcategory W of A *self-orthogonal* if $\mathrm{Ext}^1_{\mathsf{A}}(W, W') = 0$ holds for all W and W' in W.

1.5 Proposition. *Let* W *be a subcategory of* A. *The following conditions are equivalent*

(i) W *is self-orthogonal.*

(ii) *Every object in* W *belongs to* $\mathsf{W}^{\perp}$.

(iii) *Every object in* W *belongs to* ${}^{\perp}\mathsf{W}$.

(iv) *One has* $\mathsf{W} \cap \mathsf{W}^{\perp} = \mathsf{W} = {}^{\perp}\mathsf{W} \cap \mathsf{W}$.

Moreover, if W *satisfies these conditions, then an* A*-complex is right* W*-totally acyclic if and only if it is left* W*-totally acyclic.*

Proof. Evidently, (i) implies (iv), and (iv) implies both (ii) and (iii). Conditions (ii) and (iii) each precisely mean that $\mathrm{Ext}^1_{\mathsf{A}}(W, W') = 0$ holds for all W and W' in W, so either implies (i).

Now assume that W satisfies (i)–(iv). Parts (1) are the same in Proposition 1.3(R) and 1.3(L), and so are parts (2) per the assumption $\mathsf{W} \cap \mathsf{W}^{\perp} = {}^{\perp}\mathsf{W} \cap \mathsf{W}$. Part (3) in 1.3(R) coincides with part (4) in 1.3(L) by the assumption $\mathsf{W} = {}^{\perp}\mathsf{W} \cap \mathsf{W}$, and 1.3(R.4) coincides with 1.3(L.3) per the assumption $\mathsf{W} \cap \mathsf{W}^{\perp} = \mathsf{W}$. □

1.6 Definition. For a self-orthogonal subcategory W of A, a right, equivalently left, W-totally acyclic complex is simply called a W*-totally acyclic complex.*

1.7 Example. The subcategory $\mathsf{Prj}(\mathsf{A})$ of projective objects in A is self-orthogonal, and a $\mathsf{Prj}(\mathsf{A})$-totally acyclic complex is called a *totally acyclic complex of projective objects*. In the special case where A is the category $\mathsf{Mod}(A)$ of modules over a ring A these were introduced by Auslander and Bridger [1]; see also Enochs and Jenda [10]. The terminology is due to Avramov and Martsinkovsky [2].

Dually, $\mathsf{Inj}(\mathsf{A})$ is the subcategory of injective objects in A, and an $\mathsf{Inj}(\mathsf{A})$-totally acyclic complex is called a *totally acyclic complex of injective objects;* see Krause [19]. The case $\mathsf{A} = \mathsf{Mod}(A)$ was first considered in [10].

1.8 Remark. For a cotorsion pair (U, V) in A, the subcategory $\mathsf{U} \cap \mathsf{V}$ is self-orthogonal. It follows from Proposition 1.3 that every right U-totally acyclic complex and every left V-totally acyclic complex is $(\mathsf{U} \cap \mathsf{V})$-totally acyclic.

2. Gorenstein objects

In line with standard terminology, cycles in totally acyclic complexes are called Gorenstein objects.

2.1 Definition. Let U and V be subcategories of A.

(R) An object M in A is called *right* U*-Gorenstein* if there is a right U-totally acyclic complex T with $\mathrm{Z}_0(T) = M$. Denote by $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ the full subcategory of right U-Gorenstein objects in A.

(L) An object M in A is called *left* V-*Gorenstein* if there is a left V-totally acyclic complex T with $\mathrm{Z}_0(T) = M$. Denote by $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ the full subcategory of left V-Gorenstein objects in A.

For a self-orthogonal subcategory W one has $\mathsf{RGor}_{\mathsf{W}}(\mathsf{A}) = \mathsf{LGor}_{\mathsf{W}}(\mathsf{A})$, see Proposition 1.5; this category is denoted $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$, and its objects are called W-*Gorenstein.*

Notice that if U is an additive subcategory, then so is $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$; similarly for V and $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$.

2.2 Example. Let U and V be subcategories of A. Objects in $\mathsf{U} \cap \mathsf{U}^{\perp}$ are right U-Gorenstein and objects in ${}^{\perp}\mathsf{V} \cap \mathsf{V}$ are left V-Gorenstein; see Example 1.2.

2.3 Remark. We compare our definitions of total acyclicity and Gorenstein objects with others that already appear in the literature.

(1) For an additive category W, Iyengar and Krause [17] define a "totally acyclic complex over W." For additive subcategories U and V of an abelian category, a right U-totally acyclic complex is totally acyclic over $\mathsf{U} \cap \mathsf{U}^{\perp}$ in the sense of [17, def. 5.2], and a left V-totally acyclic complex is totally acyclic over ${}^{\perp}\mathsf{V} \cap \mathsf{V}$. In particular, for a self-orthogonal additive subcategory W of an abelian category, a W-totally acyclic complex is the same as an acyclic complex that is totally acyclic over W in the sense of [17, def. 5.2].

(2) For an additive subcategory W of an abelian category, Sather-Wagstaff, Sharif, and White [23] define a "totally W-acyclic" complex. A right or left W-totally acyclic complex is totally W-acyclic in the sense of [23, def. 4.1]; the converse holds if W is self-orthogonal. For a self-orthogonal additively closed subcategory W of a module category, Geng and Ding [13] study the associated Gorenstein objects.

(3) For subcategories U and V of $\mathsf{Mod}(A)$ with $\mathsf{Prj}(A) \subseteq \mathsf{U}$ and $\mathsf{Inj}(A) \subseteq \mathsf{V}$, Pan and Cai [22] define "(U,V)-Gorenstein projective/injective" modules. In this setting, a right U-Gorenstein module is $(\mathsf{U}, \mathsf{U} \cap \mathsf{U}^{\perp})$-Gorenstein projective in the sense of [22, def. 2.1], and a left V-Gorenstein module is $({}^{\perp}\mathsf{V} \cap \mathsf{V}, \mathsf{V})$-Gorenstein injective in the sense of [22, def. 2.2].

(4) For a complete hereditary cotorsion pair (U,V) in an abelian category, Yang and Chen [26] define a "complete U-resolution." Every right U-totally acyclic complex is a complete U-resolution in the sense of [26, def. 3.1].

(5) For a pair of subcategories (U,V) in an abelian category, Becerril, Mendoza, and Santiago [4] define a "left complete (U,V)-resolution." If (U,V) is a cotorsion pair, then a right U-totally acyclic complex is a left complete $(\mathsf{U}, \mathsf{U} \cap \mathsf{V})$-resolution in the sense of [4, def. 3.2].

The key difference between Definition 1.1 and those cited above is that 1.1—motivated by 4.1—places restrictions on the cycle objects in a totally acyclic complex; the significance of this becomes apparent in Proposition 2.8.

2.4 Remark. Given a cotorsion pair (U,V) in A, it follows from Remark 1.8 that there are containments

$$\mathsf{RGor}_{\mathsf{U}}(\mathsf{A}) \subseteq \mathsf{Gor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \supseteq \mathsf{LGor}_{\mathsf{V}}(\mathsf{A}) .$$

2.5 Example. A right A-Gorenstein object is injective, and a left A-Gorenstein object is projective; see Example 1.4.

The subcategory $\mathsf{Prj(A)}$ is self-orthogonal, and a $\mathsf{Prj(A)}$-Gorenstein object is called *Gorenstein projective*; see [1, 10] for the special case $\mathsf{A} = \mathsf{Mod}(A)$. Similarly, an $\mathsf{Inj(A)}$-Gorenstein object is called *Gorenstein injective;* see [19] and see [10] for the case $\mathsf{A} = \mathsf{Mod}(A)$.

The next three results, especially Proposition 2.8, are motivated in part by [23, Theorem A]. We consider what happens when one iterates the process of constructing Gorenstein objects. Starting from a self-orthogonal additively closed subcategory, our construction iterated twice returns the original subcategory. The construction in [23] is, in contrast, "idempotent."

2.6 Lemma. *Let* U *and* V *be additively closed subcategories of* A*. One has*

$$^{\perp}\mathsf{RGor_U(A)} \cap \mathsf{RGor_U(A)} \;=\; \mathsf{U} \cap \mathsf{U}^{\perp} \;=\; \mathsf{RGor_U(A)} \cap \mathsf{RGor_U(A)}^{\perp} \quad \textit{and}$$

$$^{\perp}\mathsf{LGor_V(A)} \cap \mathsf{LGor_V(A)} \;=\; {}^{\perp}\mathsf{V} \cap \mathsf{V} \;=\; \mathsf{LGor_V(A)} \cap \mathsf{LGor_V(A)}^{\perp}\,.$$

In particular, $\mathsf{RGor_U(A)}$ *is self-orthogonal if and only if* $\mathsf{RGor_U(A)} = \mathsf{U} \cap \mathsf{U}^{\perp}$ *holds, and* $\mathsf{LGor_V(A)}$ *is self-orthogonal if and only if* $\mathsf{LGor_V(A)} = {}^{\perp}\mathsf{V} \cap \mathsf{V}$ *holds.*

For a self-orthogonal category W *one has*

(2.6.1) $$^{\perp}\mathsf{Gor_W(A)} \cap \mathsf{Gor_W(A)} = \mathsf{W} = \mathsf{Gor_W(A)} \cap \mathsf{Gor_W(A)}^{\perp}\,.$$

Proof. Set $\mathsf{W} = \mathsf{U} \cap \mathsf{U}^{\perp}$ and notice that W is self-orthogonal and additively closed. By Example 2.2 objects in W are right U-Gorenstein, and by Proposition 1.3 the subcategory W is contained in both $^{\perp}\mathsf{RGor_U(A)}$ and $\mathsf{RGor_U(A)}^{\perp}$. Let G be a right U-Gorenstein object. By Proposition 1.3 there are exact sequences

$$\eta' \;=\; 0 \to G' \to T' \to G \to 0 \qquad \text{and} \qquad \eta'' \;=\; 0 \to G \to T'' \to G'' \to 0$$

where G' and G'' are right U-Gorenstein, while T' and T'' belong to W. If G belongs to $^{\perp}\mathsf{RGor_U(A)}$, then η' splits, so G is a summand of T' and hence in W. Similarly, if G is in $\mathsf{RGor_U(A)}^{\perp}$, then η'' splits, and it follows that G is in W. This proves the first set of equalities, and the ones pertaining to $\mathsf{LGor_V(A)}$ are proved similarly.

The remaining assertions are immediate in view of Proposition 1.5. □

2.7 Remark. Let U be an additively closed subcategory of A. It follows from Example 2.2 and Lemma 2.6 that objects in $\mathsf{U} \cap \mathsf{U}^{\perp}$ are both right U-Gorenstein and right $\mathsf{RGor_U(A)}$-Gorenstein. On the other hand, a right $\mathsf{RGor_U(A)}$-Gorenstein object belongs by Definition 1.1(R.3) to $\mathsf{RGor_U(A)}^{\perp}$, so any object that is both right U- and right $\mathsf{RGor_U(A)}$-Gorenstein belongs to $\mathsf{U} \cap \mathsf{U}^{\perp}$. In symbols,

$$\mathsf{RGor_U(A)} \cap \mathsf{RGor}_{\mathsf{RGor_U(A)}}\mathsf{(A)} \;=\; \mathsf{U} \cap \mathsf{U}^{\perp}\,.$$

For an additively closed subcategory V, a similar argument yields

$$\mathsf{LGor_V(A)} \cap \mathsf{LGor}_{\mathsf{LGor_V(A)}}\mathsf{(A)} \;=\; {}^{\perp}\mathsf{V} \cap \mathsf{V}\,.$$

2.8 Proposition. *Let* W *be a self-orthogonal additively closed subcategory of* A*. A right or left* $\mathsf{Gor_W(A)}$*-totally acyclic complex is a contractible complex of objects from* W*. In particular, one has*

(2.8.1) $$\mathsf{LGor}_{\mathsf{Gor_W(A)}}\mathsf{(A)} = \mathsf{W} = \mathsf{RGor}_{\mathsf{Gor_W(A)}}\mathsf{(A)}\,.$$

Moreover, the following hold

- *If* $(\mathsf{Gor_W(A)}, \mathsf{Gor_W(A)}^{\perp})$ *is a cotorsion pair, then one has*

(2.8.2) $$\mathsf{LGor}_{\mathsf{Gor_W(A)}^{\perp}}\mathsf{(A)} = \mathsf{Gor_W(A)}\,.$$

- *If* $({}^{\perp}\mathsf{Gor}_{\mathsf{W}}(\mathsf{A}), \mathsf{Gor}_{\mathsf{W}}(\mathsf{A}))$ *is a cotorsion pair, then one has*

$$\mathsf{RGor}_{{}^{\perp}\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})}(\mathsf{A}) = \mathsf{Gor}_{\mathsf{W}}(\mathsf{A})\,. \tag{2.8.3}$$

Proof. A right $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$-totally acyclic complex T is by Proposition 1.3 and (2.6.1) an acyclic complex of objects from W, and by Definition 1.1 the cycles $\mathrm{Z}_i(T)$ belong to $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})^{\perp}$. As W is contained in $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$, it follows from Proposition 1.3(R.3) that the cycles $\mathrm{Z}_i(T)$ are contained in $\mathsf{W}^{\perp}$. It now follows from Definition 1.1 that T is W-totally acyclic, whence the cycles $\mathrm{Z}_i(T)$ belong to $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$ and hence to W, see (2.6.1). Thus T is an acyclic complex of objects from W with cycles in $\mathsf{W} \subset \mathsf{W}^{\perp}$ and, therefore, contractible. A parallel argument shows that a left $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$-totally acyclic complex is contractible.

Assume that $(\mathsf{Gor}_{\mathsf{W}}(\mathsf{A}), \mathsf{Gor}_{\mathsf{W}}(\mathsf{A})^{\perp})$ is a cotorsion pair; by (2.6.1) and Remark 2.4 one has $\mathsf{LGor}_{\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})^{\perp}}(\mathsf{A}) \subseteq \mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$. To prove the opposite containment, let T be a W-totally acyclic complex. By Definition 1.1 it is an acyclic complex of objects from $\mathsf{W} \subseteq \mathsf{Gor}_{\mathsf{W}}(\mathsf{A})^{\perp}$, see (2.6.1), and $\mathrm{hom}_{\mathsf{A}}(W,T)$ is acyclic for every object W in $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A}) \cap \mathsf{Gor}_{\mathsf{W}}(\mathsf{A})^{\perp}$. Moreover, the cycle objects $\mathrm{Z}_i(T)$ belong to $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})$ by Definition 2.1, so T is per Definition 1.1 a left $\mathsf{Gor}_{\mathsf{W}}(\mathsf{A})^{\perp}$-totally acyclic complex.

A parallel argument proves the last assertion. □

2.9 Example. Let A be a ring. Šaroch and Šťovíček [24, thm. 4.6] show that the subcategory $\mathsf{Gor}_{\mathsf{Inj}}(A)$ of Gorenstein injective A-modules is the right half of a cotorsion pair, so by Proposition 2.8 one has $\mathsf{LGor}_{\mathsf{Gor}_{\mathsf{Inj}}(A)}(A) = \mathsf{Inj}(A) = \mathsf{RGor}_{\mathsf{Gor}_{\mathsf{Inj}}(A)}(A)$ and $\mathsf{RGor}_{{}^{\perp}\mathsf{Gor}_{\mathsf{Inj}}(A)}(A) = \mathsf{Gor}_{\mathsf{Inj}}(A)$.

2.10 Lemma. *Let* U *and* V *be additive subcategories of* A.

(R) *The subcategory* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ *is closed under extensions.*

(L) *The subcategory* $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ *is closed under extensions.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence where M' and M'' are right U-Gorenstein objects. Let T' and T'' be right U-totally acyclic complexes with $\mathrm{Z}_0(T') = M'$ and $\mathrm{Z}_0(T'') = M''$. Per Remark 2.3(1) it follows from [23, prop. 4.4] that there exists an A-complex T that satisfies conditions (1), (2), and (4) in Definition 1.1(R), has $\mathrm{Z}_0(T) = M$, and fits in an exact sequence

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0\,.$$

The functor $\mathrm{Z}(-)$ is left exact, and since T' is acyclic a standard application of the Snake Lemma yields an exact sequence

$$0 \longrightarrow \mathrm{Z}_i(T') \longrightarrow \mathrm{Z}_i(T) \longrightarrow \mathrm{Z}_i(T'') \longrightarrow 0$$

for every $i \in \mathbb{Z}$. As $\mathsf{U}^{\perp}$ is closed under extensions, it follows that $\mathrm{Z}_i(T)$ belongs to $\mathsf{U}^{\perp}$ for each i and thus T is right U-totally acyclic by Definition 1.1. This proves (R) and a similar argument proves (L). □

2.11 Theorem. *Let* U *be an additively closed subcategory of* A. *The category* $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ *is Frobenius and* $\mathsf{U} \cap \mathsf{U}^{\perp}$ *is the subcategory of projective-injective objects.*

Proof. Set $\mathsf{W} = \mathsf{U} \cap \mathsf{U}^{\perp}$ and notice that W is additively closed. It follows from Lemma 2.10 that $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is an exact category. It is immediate from Example 2.2 and Proposition 1.3 that objects in W are both projective and injective in $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$. It is now immediate from Definition 2.1 that $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ has enough projectives and injectives. It remains to show that every projective and every injective object in $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ belongs to W.

Let P be a projective object in $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$. By Definition 2.1 and Proposition 1.3 there is an exact sequence $0 \to P' \to W \to P \to 0$ in A with $P' \in \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ and $W \in \mathsf{W}$. As all three objects belong to $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ it follows by projectivity of P that the sequence splits, so P is a summand of W, and thus in W. A dual argument shows that every injective object in $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ belongs to W. Thus $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is a Frobenius category and W is the subcategory of projective-injective objects. □

2.12 Theorem. *Let V be an additively closed subcategory of A. The category $\mathsf{LGor}_{\mathsf{V}}(\mathsf{A})$ is Frobenius and ${}^{\perp}\mathsf{V} \cap \mathsf{V}$ is the subcategory of projective-injective objects.*

Proof. Parallel to the proof of Theorem 2.11. □

3. An equivalence of triangulated categories

Generalizing the classic result, we prove here that the stable category of right/left Gorenstein objects is equivalent to the homotopy category of right/left totally acyclic complexes.

3.1 Lemma. *Let U be a subcategory of A; let T and T' be right U-totally acyclic complexes. Every morphism $\varphi\colon \mathrm{Z}_0(T) \to \mathrm{Z}_0(T')$ in A lifts to a morphism $\phi\colon T \to T'$ of A-complexes.*

Proof. Let a morphism $\varphi\colon \mathrm{Z}_0(T) \to \mathrm{Z}_0(T')$ be given; to see that it lifts to a morphism $\phi\colon T \to T'$ of complexes it is sufficient to show that φ lifts to morphisms $\phi_1\colon T_1 \to T'_1$ and $\phi_0\colon T_0 \to T'_0$. As T_1 is in U and T' is right U-totally acyclic, one obtains per Proposition 1.3(R.3) an exact sequence

$$0 \longrightarrow \hom_{\mathsf{A}}(T_1, \mathrm{Z}_1(T')) \longrightarrow \hom_{\mathsf{A}}(T_1, T'_1) \longrightarrow \hom_{\mathsf{A}}(T_1, \mathrm{Z}_0(T')) \longrightarrow 0\,.$$

In particular, there is a $\phi_1 \in \hom_{\mathsf{A}}(T_1, T'_1)$ with $\partial_1^{T'}\phi_1 = \varphi\partial_1^{T}$. As T'_0 is in $\mathsf{U} \cap \mathsf{U}^{\perp}$ and T is right U-totally acyclic, it follows that the sequence

$$0 \longrightarrow \hom_{\mathsf{A}}(\mathrm{Z}_{-1}(T), T'_0) \longrightarrow \hom_{\mathsf{A}}(T_0, T'_0) \longrightarrow \hom_{\mathsf{A}}(\mathrm{Z}_0(T), T'_0) \longrightarrow 0$$

is exact, whence there exists a $\phi_0 \in \hom_{\mathsf{A}}(T_0, T'_0)$ that lifts φ. □

3.2 Lemma. *Let U be a subcategory of A and $\phi\colon T \to T'$ be a morphism of right U-totally acyclic complexes. If the cycle subobject $\mathrm{Z}_0(T)$ has a decomposition $\mathrm{Z}_0(T) = Z \oplus \widetilde{Z}$ with $Z \subseteq \ker\phi_0$ and $\widetilde{Z} \in \mathsf{U}$, then ϕ is null-homotopic.*

Proof. The goal is to construct a family of morphisms $\sigma_i\colon T_i \to T'_{i+1}$ such that $\phi_i = \partial_{i+1}^{T'}\sigma_i + \sigma_{i-1}\partial_i^{T}$ holds for all $i \in \mathbb{Z}$. Set $\widetilde{\varphi} = \phi_0|_{\widetilde{Z}}$. By Definition 1.1 each object $\mathrm{Z}_i(T')$ is in $\mathsf{U}^{\perp}$. It follows that there is an exact sequence,

$$0 \longrightarrow \hom_{\mathsf{A}}(\widetilde{Z}, \mathrm{Z}_1(T')) \longrightarrow \hom_{\mathsf{A}}(\widetilde{Z}, T'_1) \longrightarrow \hom_{\mathsf{A}}(\widetilde{Z}, \mathrm{Z}_0(T')) \longrightarrow 0\,.$$

In particular, there is a $\widetilde{\sigma} \in \hom_{\mathsf{A}}(\widetilde{Z}, T'_1)$ with $\partial_1^{T'}\widetilde{\sigma} = \widetilde{\varphi}$. Set $\widetilde{\sigma}_0 = 0 \oplus \widetilde{\sigma}$; by exactness of the sequence

$$0 \longrightarrow \hom_{\mathsf{A}}(\mathrm{Z}_{-1}(T), T'_1) \longrightarrow \hom_{\mathsf{A}}(T_0, T'_1) \longrightarrow \hom_{\mathsf{A}}(\mathrm{Z}_0(T), T'_1) \longrightarrow 0\,,$$

$\widetilde{\sigma}_0$ lifts to a morphism $\sigma_0\colon T_0 \to T'_1$.

We proceed by induction to construct the morphisms σ_i for $i \geq 1$. The image of the morphism $\phi_1 - \sigma_0\partial_1^T$ is in $\mathrm{Z}_1(T')$ as one has

$$\begin{aligned}\partial_1^{T'}(\phi_1 - \sigma_0\partial_1^T) &= \phi_0\partial_1^T - \partial_1^{T'}\sigma_0\partial_1^T \\ &= (0 \oplus \widetilde{\varphi})\partial_1^T - \partial_1^{T'}(0 \oplus \widetilde{\sigma})\partial_1^T \\ &= (0 \oplus (\widetilde{\varphi} - \partial_1^{T'}\widetilde{\sigma}))\partial_1^T \\ &= 0\,.\end{aligned}$$

As T_1 is in U and $\mathrm{Z}_2(T')$ is in $\mathsf{U}^\perp$ per Definition 1.1, there is an exact sequence

$$0 \longrightarrow \mathrm{hom}_{\mathsf{A}}(T_1, \mathrm{Z}_2(T')) \longrightarrow \mathrm{hom}_{\mathsf{A}}(T_1, T_2') \longrightarrow \mathrm{hom}_{\mathsf{A}}(T_1, \mathrm{Z}_1(T')) \longrightarrow 0\,.$$

In particular, there is a $\sigma_1 \in \mathrm{hom}_{\mathsf{A}}(T_1, T_2')$ with $\partial_2^{T'}\sigma_1 = \phi_1 - \sigma_0\partial_1^T$. Now let $i \geq 1$ and assume that σ_j has been constructed for $0 \leq j \leq i$. The standard computation

$$\begin{aligned}\partial_{i+1}^{T'}(\phi_{i+1} - \sigma_i\partial_{i+1}^T) &= (\phi_i - \partial_{i+1}^{T'}\sigma_i)\partial_{i+1}^T \\ &= (\sigma_{i-1}\partial_i^T)\partial_{i+1}^T \\ &= 0\end{aligned}$$

shows that the image of $\phi_{i+1} - \sigma_i\partial_{i+1}^T$ is in $\mathrm{Z}_{i+1}(T')$. As T_{i+1} is in U and $\mathrm{Z}_{i+2}(T')$ is in $\mathsf{U}^\perp$, the existence of the desired σ_{i+1} follows as for $i = 0$.

Finally, we prove the existence of the morphisms σ_i for $i \leq -1$ by descending induction. The morphism $\phi_0 - \partial_1^{T'}\sigma_0 \colon T_0 \to T_0'$ restricts to 0 on $\mathrm{Z}_0(T)$; indeed one has

$$(\phi_0 - \partial_1^{T'}\sigma_0)|_{\mathrm{Z}_0(T)} = 0 \oplus \widetilde{\varphi} - \partial_1^{T'}(0 \oplus \widetilde{\sigma}) = 0 \oplus (\widetilde{\varphi} - \partial_1^{T'}\widetilde{\sigma}) = 0\,.$$

Thus it induces a morphism ζ_{-1} from $T_0/\mathrm{Z}_0(T) \cong \mathrm{Z}_{-1}(T)$ to T_0' with $\zeta_{-1}\partial_0^T = \phi_0 - \partial_1^{T'}\sigma_0$. As T_0' is in $\mathsf{U} \cap \mathsf{U}^\perp$ it follows that the sequence

$$0 \longrightarrow \mathrm{hom}_{\mathsf{A}}(\mathrm{Z}_{-2}(T), T_0') \longrightarrow \mathrm{hom}_{\mathsf{A}}(T_{-1}, T_0') \longrightarrow \mathrm{hom}_{\mathsf{A}}(\mathrm{Z}_{-1}(T), T_0') \longrightarrow 0$$

is exact. In particular, there is a $\sigma_{-1} \in \mathrm{hom}_{\mathsf{A}}(T_{-1}, T_0')$ with $\sigma_{-1}|_{\mathrm{Z}_{-1}(T)} = \zeta_{-1}$ and, therefore, $\sigma_{-1}\partial_0^T = \phi_0 - \partial_1^{T'}\sigma_0$. Now let $i \leq -1$ and assume that σ_j has been constructed for $0 \geq j \geq i$. The standard computation

$$(\phi_i - \partial_{i+1}^{T'}\sigma_i)\partial_{i+1}^T = \partial_{i+1}^{T'}(\phi_{i+1} - \sigma_i\partial_{i+1}^T) = \partial_{i+1}^{T'}(\partial_{i+2}^{T'}\sigma_{i+1}) = 0$$

shows that the morphism $\phi_i - \partial_{i+1}^{T'}\sigma_i$ restricts to 0 on $\mathrm{Z}_i(T)$. It follows that it induces a morphism ζ_{i-1} on $T_i/\mathrm{Z}_i(T) \cong \mathrm{Z}_{i-1}(T)$ with $\zeta_{i-1}\partial_i^T = \phi_i - \partial_{i+1}^{T'}\sigma_i$. Since T_i' is in $\mathsf{U} \cap \mathsf{U}^\perp$, it follows as for $i = 0$ that the desired σ_{i-1} exists. □

3.3 Proposition. *Let U be a subcategory of A. Let T and T' be right U-totally acyclic complexes and $\varphi\colon \mathrm{Z}_0(T) \to \mathrm{Z}_0(T')$ be a morphism in A.*

(a) *If $\phi\colon T \to T'$ and $\psi\colon T \to T'$ are morphisms that lift φ, then $\phi - \psi$ is null-homotopic.*

(b) *If φ is an isomorphism and $\phi\colon T \to T'$ is a morphism that lifts φ, then ϕ is a homotopy equivalence.*

Proof. (a): Immediate from Lemma 3.2 as $(\phi - \psi)|_{\mathrm{Z}_0(T)} = \varphi - \varphi = 0$.

(b): Let $\phi'\colon T' \to T$ be a lift of φ^{-1}; see Lemma 3.1. The restriction of $1^T - \phi'\phi$ to $\mathrm{Z}_0(T)$ is 0, so it follows from part (a) that $1^T - \phi'\phi$ is null-homotopic. Similarly, $1^{T'} - \phi\phi'$ is null-homotopic; that is, ϕ is a homotopy equivalence. □

3.4 Definition. Let U and V be subcategories of A. Denote by $\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})$ and $\mathsf{K}^{\mathrm{L}}_{\mathsf{V}\text{-tac}}({}^{\perp}\mathsf{V}\cap\mathsf{V})$ the homotopy categories of right U-totally acyclic complexes and left V-totally acyclic complexes.

The subcategory $\mathsf{U}\cap\mathsf{U}^{\perp}$ is self-orthogonal, so the categories $\mathsf{K}^{\mathrm{R}}_{(\mathsf{U}\cap\mathsf{U}^{\perp})\text{-tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})$ and $\mathsf{K}^{\mathrm{L}}_{(\mathsf{U}\cap\mathsf{U}^{\perp})\text{-tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})$ coincide, see Proposition 1.5, and are denoted $\mathsf{K}_{\mathrm{tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})$. The self-orthogonal subcategory ${}^{\perp}\mathsf{V}\cap\mathsf{V}$ similarly gives a category $\mathsf{K}_{\mathrm{tac}}({}^{\perp}\mathsf{V}\cap\mathsf{V})$. For a cotorsion pair (U,V) all of these homotopy categories are $\mathsf{K}_{\mathrm{tac}}(\mathsf{U}\cap\mathsf{V})$.

If U is an additive subcategory, then the homotopy category $\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})$ is triangulated; similarly for V and $\mathsf{K}^{\mathrm{L}}_{\mathsf{V}\text{-tac}}({}^{\perp}\mathsf{V}\cap\mathsf{V})$.

3.5 Lemma. *Let U be an additively closed subcategory of A. Let T be a right U-totally acyclic complex; if $\mathrm{Z}_i(T)$ belongs to U for some $i\in\mathbb{Z}$, then T is contractible.*

Proof. Set $\mathsf{W}=\mathsf{U}\cap\mathsf{U}^{\perp}$ and notice that W is additively closed. To prove that T is contractible it is enough to show that $Z_i:=\mathrm{Z}_i(T)$ belongs to W for every $i\in\mathbb{Z}$. There are exact sequences

$$(*)\qquad 0\longrightarrow Z_{j+1}\longrightarrow T_{j+1}\longrightarrow Z_j\longrightarrow 0$$

with T_{j+1} in W and $Z_{j+1},Z_j\in\mathsf{U}^{\perp}$; see Definition 1.1 and Proposition 1.3. Without loss of generality, assume that Z_0 is in U and hence in W.

Let $j\geq 0$ and assume that Z_j is W. The sequence $(*)$ splits as Z_j is in U and Z_{j+1} is in $\mathsf{U}^{\perp}$. It follows that Z_{j+1} is in W, so by induction Z_i is in W for all $i\geq 0$.

Now let $j<0$ and assume that Z_{j+1} is in W. The sequence $(*)$ splits as $\mathrm{hom}_{\mathsf{A}}(T,Z_{j+1})$ is acyclic by Definition 1.1. It follows that Z_j belongs to W, so by descending induction Z_i is in W for all $i\leq 0$. □

3.6 Proposition. *Let U be an additively closed subcategory of A.*

- *For every right U-Gorenstein object M fix a right U-totally acyclic complex T with $\mathrm{Z}_0(T)=M$ and set $\dot{\mathrm{T}}_{\mathrm{R}}(M)=T$.*
- *For every morphism $\varphi\colon M\to M'$ of right U-Gorenstein objects fix by 3.1 a lift $\phi\colon\ \dot{\mathrm{T}}_{\mathrm{R}}(M)\to\dot{\mathrm{T}}_{\mathrm{R}}(M')$ of φ and set $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi)=[\phi]$.*

This defines a functor

$$\dot{\mathrm{T}}_{\mathrm{R}}\colon \mathsf{RGor}_{\mathsf{U}}(\mathsf{A})\longrightarrow \mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})\;.$$

For every morphism φ in $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ that factors through an object in $\mathsf{U}\cap\mathsf{U}^{\perp}$ one has $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi)=[0]$. In particular, $\dot{\mathrm{T}}_{\mathrm{R}}(M)$ is contractible for every M in $\mathsf{U}\cap\mathsf{U}^{\perp}$.

Proof. Let M be a right U-Gorenstein object. Denote by $\iota^M\colon\ \dot{\mathrm{T}}_{\mathrm{R}}(M)\to\dot{\mathrm{T}}_{\mathrm{R}}(M)$ the fixed lift of 1^M; that is, $[\iota^M]=\dot{\mathrm{T}}_{\mathrm{R}}(1^M)$. As the morphisms $1^{\dot{\mathrm{T}}_{\mathrm{R}}(M)}$ and ι^M agree on $\mathrm{Z}_0(\dot{\mathrm{T}}_{\mathrm{R}}(M))=M$, it follows from Lemma 3.2 that the difference $1^{\dot{\mathrm{T}}_{\mathrm{R}}(M)}-\iota^M$ is null-homotopic. That is, one has $\dot{\mathrm{T}}_{\mathrm{R}}(1^M)=[1^{\dot{\mathrm{T}}_{\mathrm{R}}(M)}]$, which is the identity on $\dot{\mathrm{T}}_{\mathrm{R}}(M)$ in $\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U}\cap\mathsf{U}^{\perp})$.

Let $M'\xrightarrow{\varphi'}M\xrightarrow{\varphi}M''$ be morphisms of right U-Gorenstein objects. The restrictions of $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi\varphi')$ and $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi)\,\dot{\mathrm{T}}_{\mathrm{R}}(\varphi')$ to $\mathrm{Z}_0(\dot{\mathrm{T}}_{\mathrm{R}}(M'))$ are both $\varphi\varphi'$. It now follows from Lemma 3.2 that the homotopy classes $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi\varphi')$ and $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi)\,\dot{\mathrm{T}}_{\mathrm{R}}(\varphi')$ are equal. Thus $\dot{\mathrm{T}}_{\mathrm{R}}$ is a functor.

For an object M in the additively closed subcategory $\mathsf{U}\cap\mathsf{U}^{\perp}$ it follows from Lemma 3.5 that $\dot{\mathrm{T}}_{\mathrm{R}}(M)$ is contractible. Finally, if a morphism $\varphi\colon M'\to M''$ in

$\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ factors as

$$M' \xrightarrow{\psi'} M \xrightarrow{\psi} M''$$

where M is in $\mathsf{U} \cap \mathsf{U}^{\perp}$, then $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi) = \dot{\mathrm{T}}_{\mathrm{R}}(\psi\psi') = \dot{\mathrm{T}}_{\mathrm{R}}(\psi)\,\dot{\mathrm{T}}_{\mathrm{R}}(\psi')$ factors through the contractible complex $\dot{\mathrm{T}}_{\mathrm{R}}(M)$, so one has $\dot{\mathrm{T}}_{\mathrm{R}}(\varphi) = [0][0] = [0]$. □

3.7 Remark. Let U be an additively closed subcategory of A.

Let M be a right U-Gorenstein object in A and T a right U-totally acyclic complex with $\mathrm{Z}_0(T) \cong M$. It follows from Proposition 3.3 that T and $\dot{\mathrm{T}}_{\mathrm{R}}(M)$ are isomorphic in $\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U} \cap \mathsf{U}^{\perp})$.

Let $\varphi\colon M \to M'$ be a morphism of right U-Gorenstein objects in A. For every morphism $\phi\colon\ \dot{\mathrm{T}}_{\mathrm{R}}(M) \to \dot{\mathrm{T}}_{\mathrm{R}}(M')$ that lifts φ, Proposition 3.3 yields $[\phi] = \dot{\mathrm{T}}_{\mathrm{R}}(\varphi)$.

Let U be an additively closed subcategory of A. Recall from Theorem 2.11 that $\mathsf{RGor}_{\mathsf{U}}(\mathsf{A})$ is a Frobenius category with $\mathsf{U}\cap\mathsf{U}^{\perp}$ the subcategory of projective-injective objects. Denote by $\mathsf{StRGor}_{\mathsf{U}}(\mathsf{A})$ the associated stable category. It is a triangulated category, see for example Krause [20, 7.4], and it is immediate from Proposition 3.6 that $\dot{\mathrm{T}}_{\mathrm{R}}$ induces a triangulated functor $\mathrm{T}_{\mathrm{R}}\colon \mathsf{StRGor}_{\mathsf{U}}(\mathsf{A}) \longrightarrow \mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U} \cap \mathsf{U}^{\perp})$.

3.8 Theorem. *Let* U *be an additively closed subcategory of* A*. There is a biadjoint triangulated equivalence*

$$\mathsf{StRGor}_{\mathsf{U}}(\mathsf{A}) \underset{\mathrm{Z}_0}{\overset{\mathrm{T}_{\mathrm{R}}}{\rightleftarrows}} \mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{U} \cap \mathsf{U}^{\perp})\,.$$

Proof. Set $\mathsf{W} = \mathsf{U} \cap \mathsf{U}^{\perp}$ and notice that W is additively closed. The functors T_{R} and Z_0 are triangulated. We prove that $(\mathrm{T}_{\mathrm{R}}, \mathrm{Z}_0)$ is an adjoint pair; a parallel argument shows that $(\mathrm{Z}_0, \mathrm{T}_{\mathrm{R}})$ is an adjoint pair. Let M be a right U-Gorenstein object and T be a right U-totally acyclic complex. The assignment $[\phi] \longmapsto [\mathrm{Z}_0(\phi)]$ defines a map

$$\Phi^{M,T}\colon\ \mathrm{hom}_{\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{W})}(\mathrm{T}_{\mathrm{R}}(M), T) \ \longrightarrow\ \mathrm{hom}_{\mathsf{StRGor}_{\mathsf{U}}(\mathsf{A})}(M, \mathrm{Z}_0(T))\,.$$

By Lemma 3.1 there is a morphism of A-complexes $\varepsilon^T\colon\ \mathrm{T}_{\mathrm{R}}(\mathrm{Z}_0(T)) \to T$. The assignment $[\varphi] \longmapsto [\varepsilon^T]\,\mathrm{T}_{\mathrm{R}}(\varphi)$ defines a map $\Psi^{M,T}$ in the opposite direction.

Let $[\phi] \in \mathrm{hom}_{\mathsf{K}^{\mathrm{R}}_{\mathsf{U}\text{-tac}}(\mathsf{W})}(\mathrm{T}_{\mathrm{R}}(M), T)$ be given. Let $\phi_{M,T}\colon\ \mathrm{T}_{\mathrm{R}}(M) \to \mathrm{T}_{\mathrm{R}}(\mathrm{Z}_0(T))$ be a representative of the homotopy class $\mathrm{T}_{\mathrm{R}}(\mathrm{Z}_0(\phi))$, cf. Remark 3.7. The composite $\varepsilon^T\phi_{M,T}$ agrees with ϕ on $M = \mathrm{Z}_0(\mathrm{T}_{\mathrm{R}}(M))$, so Lemma 3.2 yields

$$[\phi] = [\varepsilon^T\phi_{M,T}] = [\varepsilon^T]\,\mathrm{T}_{\mathrm{R}}(\mathrm{Z}_0(\phi)) = \Psi^{M,T}\Phi^{M,T}([\phi])\,.$$

Now let $[\varphi] \in \mathrm{hom}_{\mathsf{StRGor}_{\mathsf{U}}(\mathsf{A})}(M, \mathrm{Z}_0(T))$ be given. Let $\varphi_{M,T}\colon\ \mathrm{T}_{\mathrm{R}}(M) \to \mathrm{T}_{\mathrm{R}}(\mathrm{Z}_0(T))$ be a lift of φ; that is, a representative of the homotopy class $\mathrm{T}_{\mathrm{R}}(\varphi)$. One now has

$$\begin{aligned}
\Phi^{M,T}\Psi^{M,T}([\varphi]) &= \Phi^{M,T}([\varepsilon^T]\,\mathrm{T}_{\mathrm{R}}(\varphi)) \\
&= \Phi^{M,T}([\varepsilon^T\varphi_{M,T}]) \\
&= [\mathrm{Z}_0(\varepsilon^T\varphi_{M,T})] \\
&= [\mathrm{Z}_0(\varepsilon^T)\,\mathrm{Z}_0(\varphi_{M,T})] \\
&= [1^{\mathrm{Z}_0(T)}\varphi] \\
&= [\varphi]\,.
\end{aligned}$$

Thus $\Phi^{M,T}$ is an isomorphism.

The unit of the adjunction is the identity as one has $\mathrm{Z}_0(\mathrm{T}_\mathrm{R}(-)) = 1^{\mathsf{StRGor}_\mathsf{U}(\mathsf{A})}$, and it is straightforward to check that ε^T defined above determines the counit $\varepsilon\colon \mathrm{T}_\mathrm{R}(\mathrm{Z}_0(-)) \to 1^{\mathsf{K}^\mathrm{R}_{\mathsf{U}\text{-tac}}(\mathsf{W})}$. To show that ε is an isomorphism, let $T \in \mathsf{K}^\mathrm{R}_{\mathsf{U}\text{-tac}}(\mathsf{W})$ be given and consider a lift of the identity $\mathrm{Z}_0(T) \to \mathrm{Z}_0(\mathrm{T}_\mathrm{R}(\mathrm{Z}_0(T)))$ to a morphism $\iota^T : T \to \mathrm{T}_\mathrm{R}(\mathrm{Z}_0(T))$; see Lemma 3.1. The composite $\varepsilon^T\iota^T$ agrees with 1^T on $\mathrm{Z}_0(T)$, so $\varepsilon^T\iota^T$ is a homotopy equivalence by Lemma 3.2. Similarly, $\iota^T\varepsilon^T$ is a homotopy equivalence. It follows that ε^T is a homotopy equivalence, i.e. $[\varepsilon^T]$ is an isomorphism in $\mathsf{K}^\mathrm{R}_{\mathsf{U}\text{-tac}}(\mathsf{W})$. □

3.9 Corollary. *Let* (U,V) *be a cotorsion pair in* A. *There is a biadjoint triangulated equivalence*

$$\mathsf{StGor}_{\mathsf{U}\cap\mathsf{V}}(\mathsf{A}) \underset{\mathrm{Z}_0}{\overset{\mathrm{T}_\mathrm{R}}{\rightleftarrows}} \mathsf{K}_{\mathrm{tac}}(\mathsf{U}\cap\mathsf{V})\ .$$

Proof. This is Theorem 3.8 applied to the self-orthogonal additively closed subcategory $\mathsf{U}\cap\mathsf{V}$ and written in the notation from Definitions 2.1 and 3.4. □

3.10 Example. Applied to the cotorsion pair $(\mathsf{A},\mathsf{Inj}(\mathsf{A}))$, Corollary 3.9 recovers the well-known equivalence of the stable category of Gorenstein injective objects and the homotopy category of totally acyclic complexes of injective objects; see [19, prop. 7.2]. Applied to the cotorsion pair $(\mathsf{Prj}(\mathsf{A}),\mathsf{A})$, the corollary yields the corresponding equivalence $\mathsf{StGor}_{\mathsf{Prj}}(\mathsf{A}) \simeq \mathsf{K}_{\mathrm{tac}}(\mathsf{Prj}(\mathsf{A}))$.

3.11 Remark. Let U and V be additively closed subcategories of A. In 3.1–3.8 we have focused on right U-totally acyclic complexes and right U-Gorenstein objects. There are, of course, parallel results about left V-totally acyclic complexes and left V-Gorenstein objects. In particular, there is a biadjoint triangulated equivalence

$$\mathsf{StLGor}_{\mathsf{V}}(\mathsf{A}) \underset{\mathrm{Z}_0}{\overset{\mathrm{T}_\mathrm{L}}{\rightleftarrows}} \mathsf{K}^\mathrm{L}_{\mathsf{V}\text{-tac}}({}^\perp\mathsf{V}\cap\mathsf{V})\ .$$

Notice that applied to a cotorsion pair (U,V) this also yields Corollary 3.9.

4. Gorenstein flat-cotorsion modules

In this section and the next, A is an associative ring. We adopt the convention that an A-module is a left A-module; right A-modules are considered to be modules over the opposite ring A°. The category of A-modules is denoted $\mathsf{Mod}(A)$.

Given a cotorsion pair (U,V) in $\mathsf{Mod}(A)$ the natural categories of Gorenstein objects to consider are $\mathsf{RGor}_\mathsf{U}(A)$, $\mathsf{LGor}_\mathsf{V}(A)$, and $\mathsf{Gor}_{(\mathsf{U}\cap\mathsf{V})}(A)$; see Remark 1.8. For each of the absolute cotorsion pairs $(\mathsf{Prj}(A),\mathsf{Mod}(A))$ and $(\mathsf{Mod}(A),\mathsf{Inj}(A))$, two of these categories of Gorenstein objects coincide and the third is trivial. We start this section by recording the non-trivial fact that the cotorsion pair $(\mathsf{Flat}(A),\mathsf{Cot}(A))$ exhibits the same behavior. For brevity we denote the self-orthogonal subcategory $\mathsf{Flat}(A)\cap\mathsf{Cot}(A)$ of flat-cotorsion modules by $\mathsf{FlatCot}(A)$.

Bazzoni, Cortés Izurdiaga, and Estrada [3, thm. 1.3] prove:

4.1 Fact. An acyclic complex of cotorsion A-modules has cotorsion cycle modules.

4.2 Proposition. *A* $\mathsf{FlatCot}(A)$*-totally acyclic complex is right* $\mathsf{Flat}(A)$*-totally acyclic, and a left* $\mathsf{Cot}(A)$*-totally acyclic complex is contractible. In particular, one has*

$$\mathsf{RGor}_{\mathsf{Flat}}(A) = \mathsf{Gor}_{\mathsf{FlatCot}}(A) \qquad \text{and} \qquad \mathsf{LGor}_{\mathsf{Cot}}(A) = \mathsf{FlatCot}(A)\ .$$

Proof. In a $\mathsf{FlatCot}(A)$-totally acyclic complex, the cycle modules are cotorsion by 4.1, whence the complex is right $\mathsf{Flat}(A)$-totally acyclic by Definition 1.1. By Remark 1.8 every right $\mathsf{Flat}(A)$-totally acyclic complex is $\mathsf{FlatCot}(A)$-totally acyclic, so the first equality of categories follows from Definition 2.1. In a left $\mathsf{Cot}(A)$-totally acyclic complex, the cycle modules are flat-cotorsion, again by Definition 1.1 and 4.1, so such a complex is contractible, and the second equality follows. □

We introduce a less symbol-heavy terminology.

4.3 Definition. A $\mathsf{FlatCot}(A)$-totally acyclic complex is called a *totally acyclic complex of flat-cotorsion modules.* A cycle module in such a complex, that is, a $\mathsf{FlatCot}(A)$-Gorenstein module, is called a *Gorenstein flat-cotorsion module.*

Recall that a complex T of flat A-modules is called $\mathbf{F}$-*totally acyclic* if it is acyclic and the complex $I \otimes_A T$ is acyclic for every injective A°-module I.

4.4 Theorem. *Let A be right coherent. For an A-complex T the following conditions are equivalent*

(i) *T is a totally acyclic complex of flat-cotorsion modules.*

(ii) *T is a complex of flat-cotorsion modules and $\mathbf{F}$-totally acyclic.*

(iii) *T is right $\mathsf{Flat}(A)$-totally acyclic.*

Proof. Per Remark 1.8 condition (iii) implies (i).

(i)$\Longrightarrow$(ii): If T is a totally acyclic complex of flat-cotorsion modules, then by Proposition 1.3 it is an acyclic complex of flat-cotorsion modules. For every injective A°-module I the A-module $\operatorname{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$ is flat-cotorsion, as A is right coherent. Now it follows by the isomorphism

$$(*) \qquad \operatorname{Hom}_A(T, \operatorname{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(I \otimes_A T, \mathbb{Q}/\mathbb{Z})$$

and faithful injectivity of $\mathbb{Q}/\mathbb{Z}$ that $I \otimes_A T$ is acyclic.

(ii)$\Longrightarrow$(iii): If T is a complex of flat-cotorsion A-modules and $\mathbf{F}$-totally acyclic, then T satisfies conditions (R.1) and (R.2) in Definition 1.1. By 4.1 the cycles modules of T are cotorsion, so T also satisfies condition (R.3). Further, as A is right coherent, every flat-cotorsion A-module is a direct summand of a module of the form $\operatorname{Hom}_{\mathbb{Z}}(I, \mathbb{Q}/\mathbb{Z})$, for some injective A°-module I; see e.g. Xu [25, lem. 3.2.3]. Now it follows from the isomorphism ($*$) that $\operatorname{Hom}_A(T, W)$ is acyclic for every $W \in \mathsf{FlatCot}(A)$. That is, T also satisfies condition 1.1(R.4). □

Recall that an A-module M is called *Gorenstein flat* if there exists an $\mathbf{F}$-totally acyclic complex F of flat A-modules with $\mathrm{Z}_0(F) = M$. The full subcategory of $\mathsf{Mod}(A)$ whose objects are the Gorenstein flat modules is denoted $\mathsf{GFlat}(A)$.

Gillespie [15, cor. 3.4] proved that the category $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$ is Frobenius if A is right coherent. That it remains true without the coherence assumption is an immediate consequence of [24, cor. 3.12] discussed *ibid.*; for convenience we include the statement as part of the next result.

4.5 Theorem. *The category $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$ is Frobenius and $\mathsf{FlatCot}(A)$ is the subcategory of projective-injective objects. Moreover, the following conditions are equivalent.*

(i) *A is left perfect.*

(ii) *The category $\mathsf{GFlat}(A)$ is Frobenius.*

(*iii*) *One has* $\mathsf{GFlat}(A) = \mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$.

Furthermore, if A *is right coherent then these conditions are equivalent to*

(*iv*) *An* A*-module is Gorenstein flat if and only if it is Gorenstein projective.*

Proof. By [24, cor. 3.12] the category $\mathsf{GFlat}(A)$ is closed under extensions, and $\mathsf{GFlat}(A) \cap \mathsf{GFlat}(A)^{\perp}$ is the subcategory $\mathsf{FlatCot}(A)$ of flat-cotorsion modules. It follows that $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$ is closed under extensions, and that modules in $\mathsf{FlatCot}(A)$ are both projective and injective in $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$. Let P be a projective object in $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$; it fits in an exact sequence

$$(*) \qquad 0 \longrightarrow C \longrightarrow F \longrightarrow P \longrightarrow 0$$

where F is flat and C is cotorsion; see Bican, El Bashir, and Enochs [5]. As P is cotorsion it follows that F is flat-cotorsion. By [24, cor. 3.12] the category $\mathsf{GFlat}(A)$ is resolving, so C is Gorenstein flat. Thus, $(*)$ is an exact sequence in $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$, whence it splits by the assumption on P. In particular, P is flat-cotorsion. Now let I be an injective object in $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$. It fits by [24, cor. 3.12] in an exact sequence

$$(\dagger) \qquad 0 \longrightarrow I \longrightarrow F \longrightarrow G \longrightarrow 0$$

where F belongs to $\mathsf{GFlat}(A)^{\perp}$ and G is Gorenstein flat. It follows that F is Gorenstein flat and hence flat-cotorsion, still by [24, cor. 3.12]. Finally, G is cotorsion as both I and F are cotorsion. Thus, $(\dagger)$ is an exact sequence in $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$, whence it splits by the assumption on I. In particular, I is flat-cotorsion.

(*i*)$\Longrightarrow$(*iii*): Assuming that A is left perfect, every flat A-module module is projective, whence every A-module is cotorsion.

(*iii*)$\Longrightarrow$(*ii*): Evident as $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$ is Frobenius as shown above.

(*ii*)$\Longrightarrow$(*i*): Assume that $\mathsf{GFlat}(A)$ is Frobenius and denote by W its subcategory of projective-injective objects. To prove that A is left perfect it suffices by a result of Guil Asensio and Herzog [16, cor. 20] to show that the free module $A^{(\mathbb{N})}$ is cotorsion. As $A^{(\mathbb{N})}$ is flat, in particular Gorenstein flat, and as $\mathsf{GFlat}(A)$ by assumption has enough projectives, there is an exact sequence $0 \to K \to W \to A^{(\mathbb{N})} \to 0$ with $W \in \mathsf{W}$. The sequence splits because $A^{(\mathbb{N})}$ is projective, so it suffices to show that modules in W are cotorsion. Fix $W \in \mathsf{W}$, let F be a flat A-module, and consider an extension

$$(\ddagger) \qquad 0 \longrightarrow W \longrightarrow E \longrightarrow F \longrightarrow 0\,.$$

As $\mathsf{GFlat}(A)$ by [24, cor. 3.12] is closed under extensions, the module E is Gorenstein flat. As W is injective in $\mathsf{GFlat}(A)$ it follows that the sequence $(\ddagger)$ splits, i.e. one has $\operatorname{Ext}^1_A(F,W) = 0$. That it, W is cotorsion.

(*iv*)$\Longrightarrow$(*ii*): By Theorem 2.11 the category of Gorenstein projective A-modules is Frobenius.

(*i*)$\Longrightarrow$(*iv*): If A is perfect and right coherent, then it follows from Theorem 4.4 that an A-module is Gorenstein flat if and only if it is Gorenstein projective. ☐

By Theorem 4.5 the category $\mathsf{GFlat}(A)$ is only Frobenius when every A-module is cotorsion, and the take-away is that the appropriate Frobenius category to focus on is $\mathsf{Cot}(A) \cap \mathsf{GFlat}(A)$. If A is right coherent ring, then this category contains $\mathsf{Gor}_{\mathsf{FlatCot}}(A)$, by Theorem 4.4 and 4.1, and one goal of the next section is to prove the reverse inclusion; that is Theorem 5.2.

5. The stable category of Gorenstein flat-cotorsion modules

Recall that an A-complex P is called *pure-acyclic* if the complex $N \otimes_A P$ is acyclic for every A°-module N. In particular, an acyclic complex P of flat A-modules is pure-acyclic if and only if all cycle modules $\mathrm{Z}_i(P)$ are flat.

5.1 Fact. Let M be an A-complex. It follows[1] from Gillespie [14, cor. 4.10] that there exists an exact sequence of A-complexes

$$0 \longrightarrow M \longrightarrow C \longrightarrow P \longrightarrow 0$$

where C is a complex of cotorsion modules and P is a pure-acyclic complex of flat modules.

The first theorem of this section shows that if A is right coherent, then the cotorsion modules in $\mathsf{GFlat}(A)$ are precisely the non-trivial Gorenstein modules associated to the cotorsion pair $(\mathsf{Flat}(A), \mathsf{Cot}(A))$; namely the Gorenstein flat-cotorsion modules or, equivalently, the right $\mathsf{Flat}(A)$-Gorenstein modules.

5.2 Theorem. *Let A be right coherent. There are equalities*

$$\mathsf{Cot}(A) \cap \mathsf{GFlat}(A) = \mathsf{Gor}_{\mathsf{FlatCot}}(A) = \mathsf{RGor}_{\mathsf{Flat}}(A)\,.$$

Proof. The second equality is by Proposition 4.2, and the containment

$$\mathsf{Cot}(A) \cap \mathsf{GFlat}(A) \supseteq \mathsf{Gor}_{\mathsf{FlatCot}}(A)$$

follows from 4.1 and Theorem 4.4. It remains to show the reverse containment.

Let M be a Gorenstein flat A-module that is also cotorsion. By definition, there is an $\mathbf{F}$-totally acyclic complex F of flat A-modules with $\mathrm{Z}_0(F) = M$. Further, 5.1 yields an exact sequence of A-complexes

$$0 \longrightarrow F \xrightarrow{\iota'} T \xrightarrow{\pi'} P \longrightarrow 0 \tag{1}$$

where T is a complex of cotorsion modules and P is a pure-acyclic complex of flat modules. It follows that T is a complex of flat modules; moreover, since P is trivially $\mathbf{F}$-totally acyclic, so is T. As A is right coherent, it now follows from Theorem 4.4 that T is a totally acyclic complex of flat-cotorsion modules.

The functor $\mathrm{Z}(-)$ is left exact, and since F is acyclic a standard application of the Snake Lemma yields the exact sequence

$$0 \longrightarrow M \xrightarrow{\iota} \mathrm{Z}_0(T) \xrightarrow{\pi} \mathrm{Z}_0(P) \longrightarrow 0 \tag{2}$$

where ι and π are the restrictions of the morphisms from (1). As M is cotorsion and $\mathrm{Z}_0(P)$ is flat, (2) splits. Set $Z = \mathrm{Z}_0(P)$ and denote by ϱ the section with $\pi\varrho = 1^Z$. By 4.1 the module $\mathrm{Z}_0(T)$ is cotorsion, so it follows that Z is a flat-cotorsion module. Now, as $\mathrm{Z}_{-1}(P)$ is flat, the exact sequence

$$0 \longrightarrow Z \xrightarrow{\varepsilon_0^P} P_0 \longrightarrow \mathrm{Z}_{-1}(P) \longrightarrow 0$$

[1] Although [14, cor. 4.10] is stated for commutative rings, it is standard that the result remains valid without this assumption; see for example the discussion before [12, thm. 4.2].

splits; denote by σ the section with $\sigma\varepsilon_0^P = 1^Z$. By commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Z}_0(T) & \xrightarrow{\pi} & Z \\ \big\downarrow{\scriptstyle \varepsilon_0^T} & & \big\downarrow{\scriptstyle \varepsilon_0^P} \\ T_0 & \xrightarrow{\pi_0'} & P_0 \end{array}$$

one has $\sigma\pi_0'\varepsilon_0^T\varrho = \sigma\varepsilon_0^P\pi\varrho = 1^Z$. It follows that $\sigma\pi_0'\colon T_0 \to Z$ is a split surjection with section $\varepsilon_0^T\varrho$.

As Z is flat and $\mathrm{Z}_1(T)$ is cotorsion, there is an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(Z, \mathrm{Z}_1(T)) \longrightarrow \mathrm{Hom}_A(Z, T_1) \longrightarrow \mathrm{Hom}_A(Z, \mathrm{Z}_0(T)) \longrightarrow 0\,.$$

It follows that there is a homomorphism $\zeta\colon Z \to T_1$ with $\partial_1^T\zeta = \varrho$ and, therefore, $\partial_1^T\zeta = \varepsilon_0^T\varrho$ as homomorphisms from Z to T_0. As $\partial_0^T\varepsilon_0^T\varrho = 0$ trivially holds, the homomorphisms ζ and $\varepsilon_0^T\varrho$ yield a morphism of complexes:

$$\begin{array}{ccccccccccccc} D & = & \cdots & \longrightarrow & 0 & \longrightarrow & Z & \xrightarrow{=} & Z & \longrightarrow & 0 & \longrightarrow & \cdots \\ \big\downarrow{\scriptstyle \rho} & & & & \big\downarrow & & \big\downarrow{\scriptstyle \zeta} & & \big\downarrow{\scriptstyle \varepsilon_0^T\varrho} & & \big\downarrow & & \\ T & = & \cdots & \longrightarrow & T_2 & \xrightarrow{\partial_2^T} & T_1 & \xrightarrow{\partial_1^T} & T_0 & \xrightarrow{\partial_0^T} & T_{-1} & \longrightarrow & \cdots \end{array}$$

This is evidently a split embedding in the category of complexes whose section given by the homomorphisms $\sigma\pi_0'\partial_1^T\colon T_1 \to Z$ and $\sigma\pi_0'\colon T_0 \to Z$. The restriction of the split exact sequence of complexes

$$0 \longrightarrow D \longrightarrow T \longrightarrow T' \longrightarrow 0 \tag{3}$$

to cycles is isomorphic to the split exact sequence $0 \longrightarrow Z \xrightarrow{\varrho} \mathrm{Z}_0(T) \longrightarrow M \longrightarrow 0$, see (2), so it follows that the complex T' has $\mathrm{Z}_0(T') \cong M$.

In (3) both D and T are complexes of flat-cotorsion modules and **F**-totally acyclic, so also T' is a complex of flat-cotorsion modules and **F**-totally acyclic. Now it follows from Theorem 4.4 that T' is a totally acyclic complex of flat-cotorsion modules, whence the module $M \cong \mathrm{Z}_0(T')$ is Gorenstein flat-cotorsion. □

5.3 Corollary. *Let A be right coherent. There is a triangulated equivalence*

$$\mathsf{StGor}_{\mathsf{FlatCot}}(A) \simeq \mathsf{K}_{\text{F-tac}}(\mathsf{FlatCot}(A))\,.$$

Proof. Immediate from Theorems 3.8, 4.4, and 5.2; see also the diagram in 5.7. □

5.4 Corollary. *Let A be right coherent. The category $\mathsf{Gor}_{\mathsf{FlatCot}}(A)$ is closed under direct summands.*

Proof. Immediate from the theorem as both $\mathsf{Cot}(A)$ and $\mathsf{GFlat}(A)$ are closed under direct summands; for the latter see [24, cor. 3.12]. □

Gorenstein flat A-modules are, within the framework of Sections 1–2, not born out of a cotorsion pair, not even out of a self-orthogonal subcategory of $\mathsf{Mod}(A)$. However, they form the left half of a cotorsion pair, and also out of that pair comes the Gorenstein flat-cotorsion modules.

5.5 Remark. Let A be right coherent. Enochs, Jenda, and Lopez-Ramos [11, thm. 2.11] show that $\mathsf{GFlat}(A)$ is the left half of a cotorsion pair, and Gillespie [15, prop. 3.2] shows that $\mathsf{GFlat}(A) \cap \mathsf{GFlat}(A)^{\perp}$ is $\mathsf{FlatCot}(A)$.[2]

A right $\mathsf{GFlat}(A)$-totally acyclic complex as well as a left $\mathsf{GFlat}(A)^{\perp}$-totally acyclic complex is by Remark 1.8 and Definition 4.3 a totally acyclic complex of flat-cotorsion modules. For a right $\mathsf{GFlat}(A)$-totally acyclic complex T, it follows from Definition 1.1 that $\operatorname{Hom}_A(G,T)$ is acyclic for every Gorenstein flat A-module G, in particular for every Gorenstein flat-cotorsion module. That is, such a complex is contractible and, therefore, a right $\mathsf{GFlat}(A)$-Gorenstein module is flat-cotorsion. On the other hand, the cycles in a left $\mathsf{GFlat}(A)^{\perp}$-totally acyclic complex are by Definition 1.1 Gorenstein flat and by 4.1 cotorsion, so a left $\mathsf{GFlat}(A)^{\perp}$-Gorenstein module is by Theorem 5.2 Gorenstein flat-cotorsion.

Let $\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))$ denote the full subcategory of $\mathsf{K}(\mathsf{Flat}(A))$ whose objects are pure-acyclic; notice that it is contained in $\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))$. Via 4.1 and the dual of 5.1 one could obtain the next theorem as a consequence of a standard result [18, prop. 10.2.7]; we opt for a direct argument.

5.6 Theorem. *The composite*

$$\mathrm{I}\colon \mathsf{K}_{\text{F-tac}}(\mathsf{FlatCot}(A)) \longrightarrow \mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A)) \longrightarrow \frac{\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))}{\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))}$$

of canonical functors is a triangulated equivalence of categories.

Proof. Let I be the composite of the inclusion followed by Verdier localization; notice that I is the identity on objects. We argue that the functor I is essentially surjective, full, and faithful.

Let F be an **F**-totally acyclic complex of flat modules. By 5.1 there is an exact sequence

$$(*) \qquad 0 \longrightarrow F \longrightarrow C^F \longrightarrow P^F \longrightarrow 0$$

where C^F is a complex of cotorsion modules and P^F is in $\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))$. As F and P^F are **F**-totally acyclic complexes of flat A-modules so is C^F; that is, C^F belongs to $\mathsf{K}_{\text{F-tac}}(\mathsf{FlatCot}(A))$. It follows from $(*)$ that F and C^F are isomorphic in the Verdier quotient $\frac{\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))}{\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))}$. Thus I is essentially surjective.

Let F and F' be **F**-totally acyclic complexes of flat-cotorsion modules. A morphism $F \to F'$ in $\frac{\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))}{\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))}$ is a diagram in $\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))$

$$(*) \qquad F \xrightarrow{[\alpha]} X \underset{\simeq}{\xleftarrow{[\varphi]}} F'$$

such that the complex $\operatorname{Cone}\varphi$ belongs to $\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))$. Let ι be the embedding $X \to C^X$ from 5.1. It is elementary to verify that the composite $\iota\varphi\colon F' \to C^X$ has a pure-acyclic mapping cone; see [9, lem. 2.7]. Since F' and C^X are complexes of flat-cotorsion modules, so is $\operatorname{Cone}\iota\varphi$. It now follows by way of 4.1 that $\operatorname{Cone}\iota\varphi$ is contractible; that is, $\iota\varphi$ is a homotopy equivalence. Thus $[\iota\varphi]$ has an

[2] Šaroch and Šťovíček [24, cor. 3.12] show that all of this is true without assumptions on A, and we used that crucially in the proof of Theorem 4.5. The results from [11] and [15] suffice to prove 4.5 for a right coherent ring.

inverse in $\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))$, i.e. $[\iota\varphi]^{-1} = [\psi]$ for some morphism $\psi\colon C^X \to F'$. The commutative diagram

$$
\begin{array}{ccccc}
 & & X & & \\
 & \overset{[\alpha]}{\nearrow} & \simeq\ \downarrow [\iota] & \underset{\simeq}{\overset{[\varphi]}{\nwarrow}} & \\
F & \xrightarrow{[\iota\alpha]} & C^X & \xleftarrow[\simeq]{[\iota\varphi]} & F' \\
 & \underset{[\psi\iota\alpha]}{\overset{\simeq}{\searrow}} & \simeq\ \uparrow [\iota\varphi] & \underset{[1^{F'}]}{\overset{\simeq}{\swarrow}} & \\
 & & F' & &
\end{array}
$$

now shows that the morphism $(*)$ is equivalent to $F \xrightarrow{[\psi\iota\alpha]} F' \xleftarrow{[1^{F'}]} F'$, which is $\mathrm{I}(\psi\iota\alpha)$. This shows that I is full.

Finally, let $\alpha\colon F \to F'$ be a morphism of $\mathbf{F}$-totally acyclic complexes of flat-cotorsion modules, and assume that $\mathrm{I}([\alpha])$ is zero. It follows that there is a commutative diagram in $\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))$,

$$
\begin{array}{ccccc}
 & & F' & & \\
 & \overset{[\alpha]}{\nearrow} & \simeq\ \downarrow [\varphi] & \underset{\simeq}{\overset{[1^{F'}]}{\nwarrow}} & \\
F & \xrightarrow{[\varphi\alpha]} & X & \xleftarrow[\simeq]{[\varphi]} & F' \\
 & \underset{[0]}{\overset{\simeq}{\searrow}} & \simeq\ \uparrow [\varphi] & \underset{[1^{F'}]}{\overset{\simeq}{\swarrow}} & \\
 & & F' & &
\end{array}
$$

where the mapping cone of φ is in $\mathsf{K}_{\text{pac}}(\mathsf{Flat}(A))$. The diagram yields $[\varphi\alpha] = [0]$ and, therefore, $[\iota\varphi][\alpha] = [\iota\varphi\alpha] = [0]$ where ι is the embedding $X \to C^X$ from 5.1. As above, $[\iota\varphi]$ is invertible in $\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))$, so one has $[\alpha] = [0]$ in $\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))$. That is, α is null-homotopic, and hence $[\alpha] = 0$ in $\mathsf{K}_{\text{F-tac}}(\mathsf{FlatCot}(A))$. □

5.7 Summary. Let A be right coherent. By Theorems 3.8 and 5.6 there are triangulated equivalences

$$
\begin{array}{ccccc}
 & & \mathsf{K}_{\text{F-tac}}(\mathsf{FlatCot}(A)) & \xrightarrow[\simeq]{\ \mathrm{I}\ } & \dfrac{\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))}{\mathsf{K}_{\text{pac}}(\mathsf{Flat}(A))} \\
 & & \| & & \\
\mathsf{StRGor}_{\mathsf{Flat}}(A) & \xrightarrow[\simeq]{\mathrm{T_R}} & \mathsf{K}^{\mathrm{R}}_{\mathsf{Flat}\text{-tac}}(\mathsf{FlatCot}(A)) & & \\
\| & & \| & & \\
\mathsf{StGor}_{\mathsf{FlatCot}}(A) & & \mathsf{K}_{\text{tac}}(\mathsf{FlatCot}(A)) & &
\end{array}
$$

where the equalities come from Proposition 4.2 and Theorems 4.4 and 5.2.

5.8 Corollary. *Let A be right coherent. There is a triangulated equivalence*

$$\mathsf{StGor}_{\mathsf{FlatCot}}(A) \simeq \frac{\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))}{\mathsf{K}_{\text{pac}}(\mathsf{Flat}(A))}\,.$$

Proof. See the diagram in 5.7. □

In the special case where A is commutative noetherian of finite Krull dimension, the next result is immediate from [21, lem. 4.22] and Corollary 5.8.

5.9 Corollary. *Let A be right coherent ring such that all flat A-modules have finite projective dimension. There is a triangulated equivalence of categories*

$$\mathsf{StGor}_{\mathsf{Prj}}(A) \simeq \mathsf{StGor}_{\mathsf{FlatCot}}(A)\;.$$

Proof. Under the assumptions on A, a complex of projective A-modules is totally acyclic if and only if it **F**-totally acyclic; see [8, claims 2.4 and 2.5]. By [12, thm. 5.1] there is now a triangulated equivalence of categories

$$\mathsf{K}_{\mathrm{tac}}(\mathsf{Prj}(A)) \simeq \frac{\mathsf{K}_{\text{F-tac}}(\mathsf{Flat}(A))}{\mathsf{K}_{\mathrm{pac}}(\mathsf{Flat}(A))}\;.$$

Now apply the equivalences from Example 3.10 and Corollary 5.8. □

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