

THE STABLE CATEGORY OF GORENSTEIN FLAT SHEAVES ON A NOETHERIAN SCHEME

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ABSTRACT. For a semi-separated noetherian scheme, we show that the category of cotorsion Gorenstein flat quasi-coherent sheaves is Frobenius and a natural non-affine analogue of the category of Gorenstein projective modules over a noetherian ring. We show that this coheres perfectly with the work of Murfet and Salarian that identifies the pure derived category of \mathbf{F} -totally acyclic complexes of flat quasi-coherent sheaves as the natural non-affine analogue of the homotopy category of totally acyclic complexes of projective modules.

INTRODUCTION

A classic result due to Buchweitz [2] says that the singularity category of a Gorenstein local ring A is equivalent to the homotopy category $\mathbf{K}_{\text{tac}}(\text{prj}(A))$ of totally acyclic complexes of finitely generated projective A -modules. The latter category is also equivalent to the stable category of finitely generated maximal Cohen-Macaulay A -modules or, in a different terminology, to the stable category $\text{StGprj}(A)$ of finitely generated Gorenstein projective A -modules. This second equivalence extends beyond the realm of Gorenstein local rings and finitely generated modules: For every ring A , the category $\mathbf{K}_{\text{tac}}(\text{Prj}(A))$ of totally acyclic complexes of projective A -modules is equivalent to the stable category $\text{StGPrj}(A)$ of Gorenstein projective A -modules. We obtain this folklore result as a special case of [5, Corollary 3.9]. What is the analogue in the non-affine setting?

Murfet and Salarian [23] offer a non-affine analogue of the category $\mathbf{K}_{\text{tac}}(\text{Prj}(A))$ over a semi-separated noetherian scheme X in the form of the Verdier quotient,

$$\mathbf{D}_{\mathbf{F}\text{-tac}}(\text{Flat}(X)) = \frac{\mathbf{K}_{\mathbf{F}\text{-tac}}(\text{Flat}(X))}{\mathbf{K}_{\text{pac}}(\text{Flat}(X))},$$

of the homotopy category of \mathbf{F} -totally acyclic complexes of flat quasi-coherent sheaves on X by its subcategory of pure-acyclic complexes. Indeed, for a commutative noetherian ring A of finite Krull dimension and $X = \text{Spec}(A)$, the categories $\mathbf{K}_{\text{tac}}(\text{Prj}(A))$ and $\mathbf{D}_{\mathbf{F}\text{-tac}}(\text{Flat}(X))$ are equivalent by [23, Lemma 4.22]. What remains is to identify an analogue of the category $\text{StGPrj}(A)$ in the non-affine setting, and that is the goal of this paper.

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The stable category of Gorenstein projective modules is a standard construction that applies to any Frobenius category. The category of Gorenstein flat modules is rarely Frobenius; it is essentially only Frobenius if it coincides with the category of Gorenstein projective modules, see [5, Theorem 4.5]. The cotorsion Gorenstein flat modules, however, do form a Frobenius category, and a special case of [5, Corollary 5.9] says that for a commutative noetherian ring A of finite Krull dimension, the category $\text{StGPrj}(A)$ is equivalent to the stable category $\text{StGFC}(A)$ of cotorsion Gorenstein flat modules. This identifies a candidate category and, indeed, the goal stated above is obtained (in 4.6) with

Theorem A. *Let X be a semi-separated noetherian scheme. The stable category $\text{StGFC}(X)$ of cotorsion Gorenstein flat sheaves is equivalent to $\mathbf{D}_{\mathbf{F}\text{-tac}}(\text{Flat}(X))$.*

In the statement of this theorem, and everywhere else in this paper, a sheaf means a quasi-coherent sheaf. A sheaf on X is called cotorsion if it is right Ext-orthogonal to flat sheaves on X .

A crucial step towards the equivalence in Theorem A is to prove (in 4.2 and 4.3) that the sheaves that are both cotorsion and Gorenstein flat are precisely the sheaves that arise as cycles in \mathbf{F} -totally acyclic complexes of flat-cotorsion sheaves. This is exactly what happens in the affine case, and it transpires that the main take-away from [5] also applies in the non-affine setting: One should work with sheaves that are both cotorsion and Gorenstein flat rather than all Gorenstein flat sheaves! One manifestation is a result (4.7) that sharpens [23, Theorem 4.27]:

Theorem B. *A semi-separated noetherian scheme X is Gorenstein if and only if every acyclic complex of flat-cotorsion sheaves on X is \mathbf{F} -totally acyclic.*

A second manifestation—actually the result behind Theorem A—is that the category $\mathbf{D}_{\mathbf{F}\text{-tac}}(\text{Flat}(X))$ considered by Murfet and Salarian is equivalent to the homotopy category $\mathbf{K}_{\mathbf{F}\text{-tac}}(\text{Flat}(X) \cap \text{Cot}(X))$ of \mathbf{F} -totally acyclic complexes of flat cotorsion sheaves (see 4.5). That is, passing from the affine to the non-affine setting, one can replace the homotopy category $\mathbf{K}_{\text{tac}}(\text{Prj}(A))$ by another homotopy category.

Up to equivalence, the category $\mathbf{K}_{\mathbf{F}\text{-tac}}(\text{Flat}(X) \cap \text{Cot}(X))$ arises in a related, yet different, context. The category of Gorenstein flat sheaves on a semi-separated noetherian scheme X is part of a complete hereditary cotorsion pair (see 2.2), and one that is comparable to the cotorsion pair of flat sheaves and cotorsion sheaves on X . Through work of Hovey [21] and Gillespie [16], these cotorsion pairs induce a model structure on the category of sheaves on X . We prove (see 4.4) that the associated homotopy category is equivalent to $\mathbf{K}_{\mathbf{F}\text{-tac}}(\text{Flat}(X) \cap \text{Cot}(X))$.

1. GORENSTEIN FLAT SHEAVES

In this paper, the symbol X denotes a scheme with structure sheaf \mathcal{O}_X . By a sheaf on X we shall always mean a quasi-coherent sheaf, and $\text{Qcoh}(X)$ denotes the category of (quasi-coherent) sheaves on X . We frequently add the assumption that X is *semi-separated*, by which we mean that X has an open affine covering \mathcal{U} such that $U \cap V$ is affine for all $U, V \in \mathcal{U}$; such a covering is referred to as *semi-separating*. We use standard cohomological notation for cochain complexes.

In this first section we show that over a semi-separated noetherian scheme, one can equivalently define Gorenstein flatness of sheaves globally, locally, or stalkwise. Let A be a commutative ring. An acyclic complex F of flat A -modules is called

F-totally acyclic if the complex $I \otimes_A F$ is acyclic for every injective A -module I . An A -module M is *Gorenstein flat* if there exists an **F-totally acyclic** complex F with $M = Z^0(F)$. Denote by $\mathbf{GFlat}(A)$ the category of Gorenstein flat A -modules.

Remark 1.1. For an acyclic complex \mathcal{F} of flat sheaves on X there is a global, a local, and a stalkwise notion of **F-total acyclicity**:

- For every injective sheaf \mathcal{I} on X the complex $\mathcal{I} \otimes \mathcal{F}$ is acyclic.
- For every open affine $U \subseteq X$ the $\mathcal{O}_X(U)$ -complex $\mathcal{F}(U)$ is **F-totally acyclic**.
- For every $x \in X$ the $\mathcal{O}_{X,x}$ -complex \mathcal{F}_x is **F-totally acyclic**.

It is proved in [23, Lemmas 4.4 and 4.5] that all three notions agree if the scheme X is semi-separated noetherian. Christensen, Estrada, and Iacob [4, Corollary 2.8] show that the local notion is Zariski-local, and by [4, Proposition 2.10] the local and global notions agree if X is semi-separated and quasi-compact (which is weaker than noetherian).

Definition 1.2. Assume that X is semi-separated noetherian. An acyclic complex \mathcal{F} of flat sheaves on X is called **F-totally acyclic** if it satisfies the equivalent conditions in Remark 1.1. A sheaf \mathcal{M} on X is called *Gorenstein flat* if there exists an **F-totally acyclic** complex \mathcal{F} of flat sheaves on X with $\mathcal{M} = Z^0(\mathcal{F})$. Denote by $\mathbf{GFlat}(X)$ the category of Gorenstein flat sheaves on X .

Over any scheme, Gorenstein flatness can also be defined locally or stalkwise, and we proceed to show that these notions agree with Gorenstein flatness as defined above if the scheme is semi-separated noetherian.

Definition 1.3. A sheaf \mathcal{M} on X is called *locally Gorenstein flat* if for every open affine subset $U \subseteq X$ the $\mathcal{O}_X(U)$ -module $\mathcal{M}(U)$ is Gorenstein flat, and \mathcal{M} is called *stalkwise Gorenstein flat* if \mathcal{M}_x is a Gorenstein flat $\mathcal{O}_{X,x}$ -module for every $x \in X$.

Like local **F-total acyclicity**, local Gorenstein flatness is a Zariski-local property, at least under mild assumptions on the scheme. As shown in [4], this follows from the next proposition.

Proposition 1.4. *Let $\varphi: A \rightarrow B$ be a flat homomorphism of commutative rings.*

- (a) *If M is a Gorenstein flat A -module, then $B \otimes_A M$ is a Gorenstein flat B -module.*
- (b) *Assume that A is coherent and φ is faithfully flat. An A -module M is Gorenstein flat if the B -module $B \otimes_A M$ is Gorenstein flat.*

Proof. (a) Let F be an **F-totally acyclic** complex of flat A -modules with $M = Z^0(F)$. By [4, Proposition 2.7(1)] the B -complex $B \otimes_A F$ is an **F-totally acyclic** complex of flat B -modules, so $B \otimes_A M = Z^0(B \otimes_A F)$ is a Gorenstein flat B -module.

(b) It follows from work of Šaroch and Šťovíček [28, Corollary 4.12] that the category $\mathbf{GFlat}(A)$ is closed under extensions, so the assertion is immediate from a result of Christensen, Köksal, and Liang [6, Theorem 1.1]. \square

Corollary 1.5. *Assume that X is locally coherent. A sheaf \mathcal{M} on X is locally Gorenstein flat if there exists an open affine covering \mathcal{U} of X such that the $\mathcal{O}_X(U)$ -module $\mathcal{M}(U)$ is Gorenstein flat for every $U \in \mathcal{U}$.*

Proof. Proposition 1.4 shows that Gorenstein flatness is an ascent–descent property for modules over commutative coherent rings. Now invoke [4, Lemma 2.4]. \square

Theorem 1.6. *Assume that X is semi-separated noetherian. For a sheaf \mathcal{M} on X the following conditions are equivalent.*

- (i) \mathcal{M} is Gorenstein flat.
- (ii) \mathcal{M} is locally Gorenstein flat.
- (iii) \mathcal{M} is stalkwise Gorenstein flat.

Proof. The implication (i) \Rightarrow (ii) is trivial by the definition of Gorenstein flatness; see Remark 1.1.

(ii) \Rightarrow (iii): Let $x \in X$ and choose an open affine subset $U \subseteq X$ with $x \in U$. Localization is exact and commutes with tensor products, so it preserves Gorenstein flatness of modules, whence the module $\mathcal{M}_x \cong \mathcal{M}(U)_x$ is Gorenstein flat over the local ring $\mathcal{O}_{X,x} \cong \mathcal{O}_X(U)_x$.

(iii) \Rightarrow (i): This argument is inspired by Yang and Liu [29, Lemmas 3.8 and 3.9]. Let $\mathcal{U} = \{U_0, \dots, U_n\}$ be a semi-separating open affine covering of X . For every $x \in X$ there is a short exact sequence of $\mathcal{O}_{X,x}$ -modules,

$$(1) \quad 0 \longrightarrow \mathcal{M}_x \longrightarrow F_x \longrightarrow T_x \longrightarrow 0,$$

where F_x is flat and T_x is Gorenstein flat. For $x \in X$ and $U \in \mathcal{U}$ consider the canonical maps

$$i_x: \text{Spec}(\mathcal{O}_{X,x}) \longrightarrow X \quad \text{and} \quad i_U: \text{Spec}(\mathcal{O}_X(U)) \longrightarrow X;$$

for $x \in U$ the map i_x factors through i_U . The map $\mathcal{M} \rightarrow \prod_{x \in X} (i_x)_*(\widetilde{\mathcal{M}}_x)$ is a monomorphism locally at every $y \in X$, as one has

$$\prod_{x \in X} (i_x)_*(\widetilde{\mathcal{M}}_x) \cong (i_y)_*(\widetilde{\mathcal{M}}_y) \oplus \prod_{x \in X \setminus \{y\}} (i_x)_*(\widetilde{\mathcal{M}}_x),$$

so it is a monomorphism in $\text{Qcoh}(X)$. Now (1) yields a monomorphism

$$\mathcal{M} \longrightarrow \prod_{x \in X} (i_x)_*(\widetilde{F}_x),$$

so with $\mathcal{F}^0 = \prod_{x \in X} (i_x)_*(\widetilde{F}_x)$ there is an exact sequence in $\text{Qcoh}(X)$

$$(2) \quad 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{K}^1 \longrightarrow 0.$$

The first goal is to show that \mathcal{F}^0 is flat. For every $x \in X$ choose only one element U_k in \mathcal{U} with $x \in U_k$, and for $k = 0, \dots, n$ let $I_k \subseteq U_k$ denote the corresponding subset such that X is the disjoint union $\bigcup_{k=0}^n I_k$. Now one has

$$\begin{aligned} \mathcal{F}^0 &= \prod_{x \in X} (i_x)_*(\widetilde{F}_x) = \bigoplus_{k=0}^n \prod_{x \in I_k} (i_{U_k})_*(\widetilde{F}_x) \cong \\ &\stackrel{(\dagger)}{\cong} \bigoplus_{k=0}^n (i_{U_k})_* \left(\prod_{x \in I_k} \widetilde{F}_x \right) \cong \bigoplus_{k=0}^n (i_{U_k})_* \left(\widetilde{\prod_{x \in I_k} F_x} \right), \end{aligned}$$

where the isomorphism (\dagger) holds as $(i_{U_k})_*$, being a right adjoint functor, preserves direct products. Since F_x is a flat $\mathcal{O}_X(U_k)$ -module, and $\mathcal{O}_X(U_k)$ is noetherian, it follows that $\prod_{x \in I_k} F_x$ is a flat $\mathcal{O}_X(U_k)$ -module. Hence \mathcal{F}^0 is a flat sheaf.

The second goal is to show that \mathcal{K}^1 is Gorenstein flat locally at every point $y \in X$. Consider the commutative diagram of $\mathcal{O}_{X,y}$ -modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_y & \longrightarrow & \mathcal{F}_y^0 & \longrightarrow & \mathcal{K}_y^1 & \longrightarrow & 0 \\ & & \parallel & & \downarrow \pi & & \downarrow \varpi & & \\ 0 & \longrightarrow & \mathcal{M}_y & \longrightarrow & \mathcal{F}_y & \longrightarrow & \mathcal{T}_y & \longrightarrow & 0 \end{array}$$

where π is the canonical projection with kernel L ; this is a flat module as it is the kernel of an epimorphism between flat $\mathcal{O}_{X,x}$ -modules. By the Snake Lemma ϖ is surjective with kernel L , so \mathcal{K}_y^1 is Gorenstein flat; see e.g. [28, Corollary 4.12].

Let \mathcal{I} be an injective sheaf on X ; we argue that (2) remains exact after tensoring with \mathcal{I} by showing that $\mathrm{Tor}_1^{\mathrm{Qcoh}(X)}(\mathcal{I}, \mathcal{K}^1) = 0$ holds. For $x \in X$ let $J(x)$ be the sheaf on $\mathrm{Spec}(\mathcal{O}_{X,x})$ associated to the injective hull of the residue field of the local ring $\mathcal{O}_{X,x}$. One has

$$\mathcal{I} \cong \bigoplus_{x \in X} (i_x)_* J(x)^{(\Lambda_x)}$$

for some index sets Λ_x ; see Hartshorne [19, Proposition II.7.17]. Therefore, it suffices to verify that $\mathrm{Tor}_1^{\mathrm{Qcoh}(X)}((i_x)_* J(x), \mathcal{K}^1) = 0$ holds for every $x \in X$. This can be verified locally, and every localization $\mathrm{Tor}_1^{\mathrm{Qcoh}(X)}((i_x)_* J(x), \mathcal{K}^1)_{x'}$ is 0 or isomorphic to $\mathrm{Tor}_1^{\mathcal{O}_{X,x}}(J(x), \mathcal{K}_x^1)$, and the latter is also 0 as \mathcal{K}_x^1 is a Gorenstein flat $\mathcal{O}_{X,x}$ -module.

Repeating this process, one gets an exact sequence of sheaves

$$(3) \quad 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \mathcal{F}^2 \longrightarrow \dots$$

which remains exact after tensoring with any injective sheaf on X . Since X , in particular, is semi-separated quasi-compact, every sheaf is a homomorphic image of a flat sheaf; see for example Efimov and Positselski [7, Lemma A.1]. Therefore, there is an exact sequence

$$(4) \quad 0 \longrightarrow \mathcal{K}^{-1} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{M} \longrightarrow 0$$

with \mathcal{F}^{-1} a flat sheaf. The class of Gorenstein flat modules is closed under kernels of epimorphisms, see e.g. [28, Corollary 4.12], so \mathcal{K}_x^{-1} is a Gorenstein flat $\mathcal{O}_{X,x}$ -module for every $x \in X$. By the same argument as above the sequence (4) remains exact after tensoring with any injective sheaf on X . Repeating this process, one obtains an exact sequence

$$(5) \quad \dots \longrightarrow \mathcal{F}^{-3} \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{M} \longrightarrow 0$$

that remains exact after tensoring with any injective sheaf on X . Splicing together (3) and (5) one gets per Definition 1.2 an \mathbf{F} -totally acyclic complex of flat sheaves, $\mathcal{F} = \dots \rightarrow \mathcal{F}^{-1} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots$. Thus, $\mathcal{M} = \mathbf{Z}^0(\mathcal{F})$ is Gorenstein flat. \square

Henceforth we work mainly over semi-separated noetherian schemes. In that setting we consistently refer to the sheaves described in Theorem 1.6 by their shortest name: Gorenstein flat; some proofs, though, rely crucially on their local properties.

2. THE GORENSTEIN FLAT MODEL STRUCTURE ON $\mathbf{Qcoh}(X)$

Let \mathcal{G} be a Grothendieck category, that is, an abelian category that has colimits, exact direct limits (filtered colimits), and a generator. A class \mathcal{C} of objects in \mathcal{G} is called *resolving* if it contains all projective objects and is closed under extensions and kernels of epimorphisms. To a class \mathcal{C} of objects in \mathcal{G} one associates the orthogonal classes

$$\begin{aligned}\mathcal{C}^\perp &= \{G \in \mathcal{G} \mid \mathrm{Ext}_{\mathcal{G}}^1(C, G) = 0 \text{ for all } C \in \mathcal{C}\} \text{ and} \\ {}^\perp\mathcal{C} &= \{G \in \mathcal{G} \mid \mathrm{Ext}_{\mathcal{G}}^1(G, C) = 0 \text{ for all } C \in \mathcal{C}\}.\end{aligned}$$

Let $\mathcal{S} \subseteq \mathcal{C}$ be a set. The pair $(\mathcal{C}, \mathcal{C}^\perp)$ is said to be *generated by the set \mathcal{S}* if an object G belongs to \mathcal{C}^\perp if and only if $\mathrm{Ext}_{\mathcal{G}}^1(C, G) = 0$ holds for all $C \in \mathcal{S}$. A pair $(\mathcal{F}, \mathcal{C})$ of classes in \mathcal{G} with $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$ is called a *cotorsion pair*. The intersection $\mathcal{F} \cap \mathcal{C}$ is called the *core* of the cotorsion pair.

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in \mathcal{G} is called *hereditary* if for all $F \in \mathcal{F}$ and $C \in \mathcal{C}$ one has $\mathrm{Ext}_{\mathcal{G}}^i(F, C) = 0$ for all $i \geq 1$. Notice that the class \mathcal{F} in this case is resolving.

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ in \mathcal{G} is called *complete* provided that for every $G \in \mathcal{G}$ there are short exact sequences $0 \rightarrow C \rightarrow F \rightarrow G \rightarrow 0$ and $0 \rightarrow G \rightarrow C' \rightarrow F' \rightarrow 0$ with $F, F' \in \mathcal{F}$ and $C, C' \in \mathcal{C}$.

Abelian model category structures from cotorsion pairs. Gillespie [16] shows how to construct a hereditary abelian model structure on \mathcal{G} from two comparable cotorsion pairs. Namely, if $(\mathcal{Q}, \tilde{\mathcal{R}})$ and $(\tilde{\mathcal{Q}}, \mathcal{R})$ are complete hereditary cotorsion pairs in \mathcal{G} with $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, $\tilde{\mathcal{Q}} \subseteq \mathcal{Q}$, and $\mathcal{Q} \cap \tilde{\mathcal{R}} = \tilde{\mathcal{Q}} \cap \mathcal{R}$, then there exists a unique thick (i.e. full, closed under direct summands, and having the two-out-of-three property) subcategory \mathcal{W} of \mathcal{G} such that $\tilde{\mathcal{Q}} = \mathcal{Q} \cap \mathcal{W}$ and $\tilde{\mathcal{R}} = \mathcal{R} \cap \mathcal{W}$. In other words $(\mathcal{Q}, \mathcal{W}, \mathcal{R})$ is a so-called *Hovey triple*, and from work of Hovey [21] it is known that there is a unique abelian model structure on \mathcal{G} in which \mathcal{Q} , \mathcal{R} , and \mathcal{W} are the classes of cofibrant, fibrant, and trivial objects, respectively; refer to [21] for this standard terminology. We are now going to apply this machine to cotorsion pairs with $\tilde{\mathcal{Q}}$ and \mathcal{Q} the categories of flat and Gorenstein flat sheaves on X .

Remark 2.1. If X is semi-separated quasi-compact, then $\mathbf{Qcoh}(X)$ is a locally finitely presentable Grothendieck category. This was proved already in EGA [18, I.6.9.12], though not using that terminology. Being a Grothendieck category, $\mathbf{Qcoh}(X)$ has a generator and hence, by [7, Lemma A.1], a flat generator. Slávik and Šťovíček [26] have recently proved that if X is quasi-separated and quasi-compact, then $\mathbf{Qcoh}(X)$ has a flat generator if and only if X is semi-separated.

Theorem 2.2. *Assume that X is semi-separated noetherian. The pair*

$$(\mathbf{GFlat}(X), \mathbf{GFlat}(X)^\perp)$$

is a complete hereditary cotorsion pair.

Proof. For an open affine subset $U \subseteq X$ we write $\mathbf{GFlat}(U)$ for the category $\mathbf{GFlat}(\mathcal{O}_X(U))$ of Gorenstein flat $\mathcal{O}_X(U)$ -modules. For every open affine subset $U \subseteq X$ the pair $(\mathbf{GFlat}(U), \mathbf{GFlat}(U)^\perp)$ is a complete hereditary cotorsion pair; see Enochs, Jenda, and López-Ramos [10, Theorems 2.11 and 2.12]. The proof of [10, Theorem 2.11] shows that the pair is generated by a set \mathcal{S}_U ; see also the more precise statement in [28, Corollary 4.12].

A result of Estrada, Guil Asensio, Prest, and Trlifaj [11, Corollary 3.15] now shows that $(\mathbf{GFlat}(X), \mathbf{GFlat}(X)^\perp)$ is a complete cotorsion pair. Indeed, the flat generator of $\mathbf{Qcoh}(X)$ belongs to $\mathbf{GFlat}(X)$. As the quiver in [11, Notation 3.12] one takes the quiver with vertices all open affine subsets of X , and the class \mathcal{L} in [11, Corollary 3.15] is in this case

$$\mathcal{L} = \{\mathcal{L} \in \mathbf{Qcoh}(X) \mid \mathcal{L}(U) \in \mathcal{S}_U \text{ for every open affine subset } U \subseteq X\}.$$

Moreover since $(\mathbf{GFlat}(U), \mathbf{GFlat}(U)^\perp)$ is hereditary, the class $\mathbf{GFlat}(U)$ is resolving for every open affine subset $U \subseteq X$. It follows that $\mathbf{GFlat}(X)$ is also resolving, whence $(\mathbf{GFlat}(X), \mathbf{GFlat}(X)^\perp)$ is hereditary as $\mathbf{GFlat}(X)$ contains a generator; see Saorín and Šťovíček [25, Lemma 4.25]. \square

Let X be a semi-separated noetherian scheme. By $\mathbf{Flat}(X)$ we denote the category of flat sheaves on X . The proof of the next result is modeled on an argument due to Estrada, Iacob, and Pérez [12, Proposition 4.1].

Lemma 2.3. *Assume that X is semi-separated noetherian. In $\mathbf{Qcoh}(X)$ one has*

$$\mathbf{GFlat}(X) \cap \mathbf{GFlat}(X)^\perp = \mathbf{Flat}(X) \cap \mathbf{Flat}(X)^\perp.$$

Proof. “ \subseteq ”: Let $\mathcal{M} \in \mathbf{GFlat}(X) \cap \mathbf{GFlat}(X)^\perp$. The inclusion $\mathbf{Flat}(X) \subseteq \mathbf{GFlat}(X)$ yields $\mathbf{GFlat}(X)^\perp \subseteq \mathbf{Flat}(X)^\perp$, so it remains to show that \mathcal{M} is flat. Since \mathcal{M} is in $\mathbf{GFlat}(X)$ there is an exact sequence in $\mathbf{Qcoh}(X)$,

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow 0,$$

with \mathcal{F} a flat sheaf and \mathcal{N} a Gorenstein flat sheaf on X . Since \mathcal{M} belongs to $\mathbf{GFlat}(X)^\perp$ the sequence splits, whence \mathcal{M} is flat.

“ \supseteq ”: Let $\mathcal{M} \in \mathbf{Flat}(X) \cap \mathbf{Flat}(X)^\perp$. As the inclusion $\mathbf{Flat}(X) \subseteq \mathbf{GFlat}(X)$ holds, it remains to show that \mathcal{M} is in $\mathbf{GFlat}(X)^\perp$. Since $(\mathbf{GFlat}(X), \mathbf{GFlat}(X)^\perp)$ is a complete cotorsion pair in $\mathbf{Qcoh}(X)$, see Theorem 2.2, there is an exact sequence in $\mathbf{Qcoh}(X)$,

$$(6) \quad 0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{G} \longrightarrow \mathcal{N} \longrightarrow 0,$$

with $\mathcal{G} \in \mathbf{GFlat}(X)^\perp$ and $\mathcal{N} \in \mathbf{GFlat}(X)$. Moreover, since $\mathbf{GFlat}(X)$ is closed under extensions by Theorem 2.2, also \mathcal{G} belongs to $\mathbf{GFlat}(X)$. Thus the sheaf \mathcal{G} is in $\mathbf{GFlat}(X) \cap \mathbf{GFlat}(X)^\perp$, so by the containment already proved \mathcal{G} is flat. Since \mathcal{M} is also flat, it follows that

$$\text{flat dim}_{\mathcal{O}_X(U)} \mathcal{N}(U) \leq 1$$

holds for every open affine subset $U \subseteq X$. Thus $\mathcal{N}(U)$ is a Gorenstein flat $\mathcal{O}_X(U)$ -module of finite flat dimension and, therefore, flat; see [9, Corollary 10.3.4]. It follows that \mathcal{N} is a flat sheaf. Since $\mathcal{M} \in \mathbf{Flat}(X)^\perp$ by assumption, the sequence (6) splits. Therefore, \mathcal{M} is a direct summand of \mathcal{G} and thus in $\mathbf{GFlat}(X)^\perp$. \square

We call sheaves in the subcategory $\mathbf{Cot}(X) = \mathbf{Flat}(X)^\perp$ of $\mathbf{Qcoh}(X)$ *cotorsion*. Sheaves in the intersection $\mathbf{Flat}(X) \cap \mathbf{Cot}(X)$ are called *flat-cotorsion*.

Remark 2.4. Assume that X is semi-separated quasi-compact. In this case the category $\mathbf{Flat}(X)$ contains a generator for $\mathbf{Qcoh}(X)$, so it follows from work of Enochs and Estrada [8, Corollary 4.2] that $(\mathbf{Flat}(X), \mathbf{Cot}(X))$ is a complete cotorsion pair, and since $\mathbf{Flat}(X)$ is resolving it follows from [25, Lemma 4.25] that the pair $(\mathbf{Flat}(X), \mathbf{Cot}(X))$ is hereditary. This fact can also be deduced from work of Gillespie [14, Proposition 6.4] and Hovey [21, Corollary 6.6].

The next theorem establishes what we call the *Gorenstein flat model structure* on $\mathbf{Qcoh}(X)$; it may be regarded as a non-affine version of [17, Theorem 3.3].

Theorem 2.5. *Assume that X is semi-separated noetherian. There exists a unique abelian model structure on $\mathbf{Qcoh}(X)$ with $\mathbf{GFlat}(X)$ the class of cofibrant objects and $\mathbf{Cot}(X)$ the class of fibrant objects. In this structure $\mathbf{Flat}(X)$ is the class of trivially cofibrant objects and $\mathbf{GFlat}(X)^\perp$ is the class of trivially fibrant objects.*

Proof. It follows from Theorem 2.2 and Remark 2.4, that $(\mathbf{GFlat}(X), \mathbf{GFlat}(X)^\perp)$ and $(\mathbf{Flat}(X), \mathbf{Cot}(X))$ are complete hereditary cotorsion pairs. Every flat sheaf is Gorenstein flat, and by Lemma 2.3 the two pairs have the same core, so they satisfy the conditions in [16, Theorem 1.2]. Thus the pairs determine a Hovey triple, and by [21, Theorem 2.2] a unique abelian model category structure on $\mathbf{Qcoh}(X)$ with fibrant and cofibrant objects as asserted. \square

Corollary 2.6. *Assume that the scheme X is semi-separated noetherian. The category $\mathbf{Cot}(X) \cap \mathbf{GFlat}(X)$ is Frobenius and the projective–injective objects are the flat-cotorsion sheaves. Its associated stable category is equivalent to the homotopy category of the Gorenstein flat model structure.*

Proof. Applied to the Gorenstein flat model structure from the theorem, [15, Proposition 5.2(4)] shows that $\mathbf{Cot}(X) \cap \mathbf{GFlat}(X)$ is a Frobenius category with the stated projective–injective objects. The last assertion follows from [15, Corollary 5.4]. \square

3. ACYCLIC COMPLEXES OF COTORSION SHEAVES

We assume throughout this section that X is semi-separated quasi-compact. The category of cochain complexes of sheaves on X is denoted $\mathbf{C}(\mathbf{Qcoh}(X))$. The goal is to establish a result, Theorem 3.3 below, which in the affine case is proved by Bazzoni, Cortés Izurdiaga, and Estrada [1, Theorem 1.3]. It says, in part, that every acyclic complex of cotorsion sheaves has cotorsion cycles. Our proof is inspired by arguments of Hosseini [20] and Šťovíček [27].

Let $\mathbf{C}_{\text{ac}}^{\mathbb{Z}}(\mathbf{Flat}(X))$ denote the full subcategory of $\mathbf{C}(\mathbf{Qcoh}(X))$ whose objects are the acyclic complexes \mathcal{F} of flat sheaves with $Z^n(\mathcal{F}) \in \mathbf{Flat}(X)$ for every $n \in \mathbb{Z}$; similarly, let $\mathbf{C}_{\text{ac}}^{\mathbb{Z}}(\mathbf{Cot}(X))$ denote the full subcategory whose objects are the acyclic complexes \mathcal{C} of cotorsion sheaves with $Z^n(\mathcal{C}) \in \mathbf{Cot}(X)$ for every $n \in \mathbb{Z}$. Further, $\mathbf{C}_{\text{semi}}(\mathbf{Cot}(X))$ denotes the category of complexes \mathcal{C} of cotorsion sheaves with the property that the total Hom complex $\mathbf{Hom}(\mathcal{F}, \mathcal{C})$ of abelian groups is acyclic for every complex $\mathcal{F} \in \mathbf{C}_{\text{ac}}^{\mathbb{Z}}(\mathbf{Flat}(X))$. In the literature such complexes are referred to as dg- or semi-cotorsion complexes; it is part of Theorem 3.3 that every complex of cotorsion sheaves on X has this property.

Remark 3.1. The pair $(\mathbf{C}_{\text{ac}}^{\mathbb{Z}}(\mathbf{Flat}(X)), \mathbf{C}_{\text{semi}}(\mathbf{Cot}(X)))$ is by [14, Theorem 6.7] and [21, Theorem 2.2] a complete cotorsion pair in $\mathbf{C}(\mathbf{Qcoh}(X))$.

For complexes \mathcal{A} and \mathcal{B} of sheaves on X , let $\mathbf{Hom}(\mathcal{A}, \mathcal{B})$ denote the standard total Hom complex of abelian groups. There is an isomorphism

$$(7) \quad \text{Ext}_{\mathbf{C}(\mathbf{Qcoh}(X)), \text{dw}}^1(\mathcal{A}, \Sigma^{n-1}\mathcal{B}) \cong \mathbf{H}^n \mathbf{Hom}(\mathcal{A}, \mathcal{B}),$$

where $\text{Ext}_{\mathbf{C}(\mathbf{Qcoh}(X)), \text{dw}}^1(\mathcal{A}, \Sigma^{n-1}\mathcal{B})$ is the subgroup of $\text{Ext}_{\mathbf{C}(\mathbf{Qcoh}(X))}^1(\mathcal{A}, \Sigma^{n-1}\mathcal{B})$ consisting of degreewise split short exact sequences; see e.g. [13, Lemma 2.1]. For

a complex \mathcal{F} of flat sheaves and a complex \mathcal{C} of cotorsion sheaves, every extension $0 \rightarrow \mathcal{C} \rightarrow \mathcal{X} \rightarrow \mathcal{F} \rightarrow 0$ is degreewise split, so (7) reads

$$(8) \quad \mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\mathcal{F}, \Sigma^{n-1}\mathcal{C}) \cong \mathrm{H}^n \mathrm{Hom}(\mathcal{F}, \mathcal{C}).$$

Lemma 3.2. *Let $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be a direct system of complexes in $\mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X))$. If each complex \mathcal{F}_λ is contractible, then*

$$\mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda, \mathcal{C}) \cong \mathrm{H}^1 \mathrm{Hom}(\varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda, \mathcal{C}) = 0$$

holds for every complex \mathcal{C} of cotorsion sheaves on X .

Proof. The category $\mathrm{Qcoh}(X)$ is locally finitely presentable and, therefore, finitely accessible; see Remark 2.1. It follows that the results, and arguments, in [27] apply. The argument in the proof of [27, Proposition 5.3] yields an exact sequence,

$$(9) \quad 0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda \longrightarrow \varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda \longrightarrow 0,$$

where \mathcal{K} is filtered by finite direct sums of complexes \mathcal{F}_λ . That is, there is an ordinal number β and a filtration $(\mathcal{K}_\alpha \mid \alpha < \beta)$, where $\mathcal{K}_0 = 0$, $\mathcal{K}_\beta = \mathcal{K}$, and $\mathcal{K}_{\alpha+1}/\mathcal{K}_\alpha \cong \bigoplus_{\lambda \in J_\alpha} \mathcal{F}_\lambda$ for $\alpha < \beta$ and J_α a finite set.

Let \mathcal{C} be a complex of cotorsion sheaves on X . As $\mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X))$ is closed under direct limits, one has $\varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda \in \mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X))$. Thus, $\mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1((\varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda)^i, \mathcal{C}^j) = 0$ holds for all $i, j \in \mathbb{Z}$, whence there is an exact sequence of complexes of abelian groups:

$$(10) \quad 0 \longrightarrow \mathrm{Hom}(\varinjlim_{\lambda \in \Lambda} \mathcal{F}_\lambda, \mathcal{C}) \longrightarrow \mathrm{Hom}(\bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda, \mathcal{C}) \longrightarrow \mathrm{Hom}(\mathcal{K}, \mathcal{C}) \longrightarrow 0.$$

By (8) it now suffices to show that the left-hand complex in this sequence is acyclic. The middle complex is acyclic because each complex \mathcal{F}_λ and, therefore, the direct sum $\bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda$ is contractible. Thus it is enough to prove that $\mathrm{Hom}(\mathcal{K}, \mathcal{C})$ is acyclic. Since $\mathrm{Flat}(X)$ is resolving, it follows from (9) that \mathcal{K} is a complex of flat sheaves. As \mathcal{C} is a complex of cotorsion sheaves, (8) yields

$$\mathrm{H}^n \mathrm{Hom}(\mathcal{K}, \mathcal{C}) \cong \mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\mathcal{K}, \Sigma^{n-1}\mathcal{C}).$$

Hence, it suffices to show that $\mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\mathcal{K}, \Sigma^{n-1}\mathcal{C}) = 0$ holds for all $n \in \mathbb{Z}$. Let $(\mathcal{K}_\alpha \mid \alpha \leq \lambda)$ be the filtration of \mathcal{K} described above. For every $n \in \mathbb{Z}$ one has

$$\mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\bigoplus_{\lambda \in J_\alpha} \mathcal{F}_\lambda, \Sigma^{n-1}\mathcal{C}) = 0,$$

so Eklof's lemma [27, Proposition 2.10] yields $\mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\mathcal{K}, \Sigma^{n-1}\mathcal{C}) = 0$. \square

Theorem 3.3. *Assume that X is semi-separated quasi-compact. Every complex of cotorsion sheaves on X belongs to $\mathcal{C}_{\mathrm{semi}}(\mathrm{Cot}(X))$, and every acyclic complex of cotorsion sheaves belongs to $\mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Cot}(X))$.*

Proof. As $(\mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X)), \mathcal{C}_{\mathrm{semi}}(\mathrm{Cot}(X)))$ is a cotorsion pair, see Remark 3.1, the first assertion is that for every complex \mathcal{M} of cotorsion sheaves and every \mathcal{F} in $\mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X))$ one has $\mathrm{Ext}_{\mathcal{C}(\mathrm{Qcoh}(X))}^1(\mathcal{F}, \mathcal{M}) = 0$. Fix $\mathcal{F} \in \mathcal{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X))$ and a semi-separating open affine covering $\mathcal{U} = \{U_0, \dots, U_d\}$ of X . Consider the double complex of sheaves obtained by taking the Čech resolutions of each term in \mathcal{F} ;

see Murfet [22, Section 3.1]. The rows of the double complex form a sequence in $\mathbf{C}(\mathrm{Qcoh}(X))$:

$$(11) \quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \longrightarrow \mathcal{C}^d(\mathcal{U}, \mathcal{F}) \longrightarrow 0$$

with

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{j_0 < \cdots < j_p} i_* (\mathcal{F}(\widetilde{U_{j_0, \dots, j_p}})),$$

where j_0, \dots, j_p belong to the set $\{0, \dots, d\}$ and $i: U_{j_0, \dots, j_p} \rightarrow X$ is the inclusion of the open affine subset $U_{j_0, \dots, j_p} = U_{j_0} \cap \cdots \cap U_{j_p}$ of X . For a tuple of indices $j_0 < \cdots < j_p$, the complex $\mathcal{F}(U_{j_0, \dots, j_p})$ is an acyclic complex of flat $\mathcal{O}_X(U_{j_0, \dots, j_p})$ -modules whose cycle modules are also flat. It follows that the complex $\mathcal{F}(U_{j_0, \dots, j_p})$ is a direct limit

$$\mathcal{F}(U_{j_0, \dots, j_p}) = \varinjlim_{\lambda \in \Lambda} P_\lambda^{U_{j_0, \dots, j_p}}$$

of contractible complexes of projective, hence flat, $\mathcal{O}_X(U_{j_0, \dots, j_p})$ -modules; see for example Neeman [24, Theorem 8.6]. The functor i_* preserves split exact sequences,

so $i_*(P_\lambda^{U_{j_0, \dots, j_p}})$ is for every $\lambda \in \Lambda$ a contractible complex of flat sheaves. The functor also preserves direct limits, so $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is a finite direct sum of direct limits of contractible complexes in $\mathbf{C}_{\mathrm{ac}}^Z(\mathrm{Flat}(X))$, hence $\mathcal{C}^p(\mathcal{U}, \mathcal{F})$ is itself a direct limit of contractible complexes in $\mathbf{C}_{\mathrm{ac}}^Z(\mathrm{Flat}(X))$. For every complex \mathcal{M} of cotorsion sheaves and every $n \in \mathbb{Z}$, Lemma 3.2 now yields

$$\mathrm{H}^n \mathrm{Hom}(\mathcal{C}^p(\mathcal{U}, \mathcal{F}), \mathcal{M}) \cong \mathrm{H}^1 \mathrm{Hom}(\mathcal{C}^p(\mathcal{U}, \mathcal{F}), \Sigma^{n-1} \mathcal{M}) = 0 \quad \text{for } 0 \leq p \leq d.$$

That is, the complex $\mathrm{Hom}(\mathcal{C}^p(\mathcal{U}, \mathcal{F}), \mathcal{M})$ is acyclic for every $0 \leq p \leq d$ and every complex \mathcal{M} of cotorsion sheaves. Applying $\mathrm{Hom}(-, \mathcal{M})$ to the exact sequence

$$0 \longrightarrow \mathcal{L}_{d-1} \longrightarrow \mathcal{C}^{d-1}(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^d(\mathcal{U}, \mathcal{F}) \longrightarrow 0,$$

one gets an exact sequence of complexes of abelian groups

$$0 \longrightarrow \mathrm{Hom}(\mathcal{C}^d(\mathcal{U}, \mathcal{F}), \mathcal{M}) \longrightarrow \mathrm{Hom}(\mathcal{C}^{d-1}(\mathcal{U}, \mathcal{F}), \mathcal{M}) \longrightarrow \mathrm{Hom}(\mathcal{L}_{d-1}, \mathcal{M}) \longrightarrow 0.$$

The first two terms are acyclic, and hence so is $\mathrm{Hom}(\mathcal{L}_{d-1}, \mathcal{M})$. Repeating this argument $d-1$ more times, one concludes that $\mathrm{Hom}(\mathcal{F}, \mathcal{M})$ is acyclic, whence one has $\mathrm{Ext}_{\mathbf{C}(\mathrm{Qcoh}(X))}^1(\mathcal{F}, \mathcal{M}) = 0$ per (8).

The second assertion now follows from [14, Corollary 3.9] which applies as $(\mathrm{Flat}(X), \mathrm{Cot}(X))$ is a complete hereditary cotorsion pair and $\mathrm{Flat}(X)$ contains a generator for $\mathrm{Qcoh}(X)$; see Remark 2.1. \square

4. THE STABLE CATEGORY OF GORENSTEIN FLAT-COTORSION SHEAVES

In this last section, we give a description of the stable category associated to the cotorsion pair of Gorenstein flat sheaves described in Theorem 2.2. In particular, we prove Theorems A and B from the introduction.

Here we use the symbol hom to denote the morphism sets in $\mathrm{Qcoh}(X)$ as well as the induced functor to abelian groups. Further, the tensor product on $\mathrm{Qcoh}(X)$ has a right adjoint functor denoted $\mathcal{H}om_{\mathrm{qc}}$; see for example [23, 2.1].

We recall from [5, Definition 1.1, Proposition 1.3, and Definition 2.1]:

Definition 4.1. An acyclic complex \mathcal{F} of flat-cotorsion sheaves on X is called *totally acyclic* if the complexes $\mathrm{hom}(\mathcal{C}, \mathcal{F})$ and $\mathrm{hom}(\mathcal{F}, \mathcal{C})$ are acyclic for every flat-cotorsion sheaf \mathcal{C} on X .

A sheaf \mathcal{M} on X is called *Gorenstein flat-cotorsion* if there exists a totally acyclic complex \mathcal{F} of flat-cotorsion sheaves on X with $\mathcal{M} = Z^0(\mathcal{F})$. Denote by $\mathrm{GFC}(X)$ the category of Gorenstein flat-cotorsion sheaves on X .¹

We proceed to show that the sheaves defined in 4.1 are precisely the cotorsion Gorenstein flat sheaves, i.e. the sheaves that are both cotorsion and Gorenstein flat.

The next result is analogous to [5, Theorem 4.4].

Proposition 4.2. *Assume that X is semi-separated noetherian. An acyclic complex of flat-cotorsion sheaves on X is totally acyclic if and only if it is \mathbf{F} -totally acyclic.*

Proof. Let \mathcal{F} be a totally acyclic complex of flat-cotorsion sheaves. Let \mathcal{I} be an injective sheaf and \mathcal{E} an injective cogenerator in $\mathrm{Qcoh}(X)$. By [23, Lemma 3.2], the sheaf $\mathcal{H}om_{\mathrm{qc}}(\mathcal{I}, \mathcal{E})$ is flat-cotorsion. The adjunction isomorphism

$$(12) \quad \mathrm{hom}(\mathcal{I} \otimes \mathcal{F}, \mathcal{E}) \cong \mathrm{hom}(\mathcal{F}, \mathcal{H}om_{\mathrm{qc}}(\mathcal{I}, \mathcal{E}))$$

along with faithful injectivity of \mathcal{E} implies that $\mathcal{I} \otimes \mathcal{F}$ is acyclic, hence \mathcal{F} is \mathbf{F} -totally acyclic.

For the converse, let \mathcal{F} be an \mathbf{F} -totally acyclic complex of flat-cotorsion sheaves and \mathcal{C} be a flat-cotorsion sheaf. Recall from [23, Proposition 3.3] that \mathcal{C} is a direct summand of $\mathcal{H}om_{\mathrm{qc}}(\mathcal{I}, \mathcal{E})$ for some injective sheaf \mathcal{I} and injective cogenerator \mathcal{E} . Thus (12) shows that $\mathrm{hom}(\mathcal{F}, \mathcal{C})$ is acyclic. Moreover, it follows from Theorem 3.3 that $Z^n(\mathcal{F})$ is cotorsion for every $n \in \mathbb{Z}$, so $\mathrm{hom}(\mathcal{C}, \mathcal{F})$ is acyclic. \square

Theorem 4.3. *Assume that X is semi-separated noetherian. A sheaf on X is Gorenstein flat-cotorsion if and only if it is cotorsion and Gorenstein flat; that is,*

$$\mathrm{GFC}(X) = \mathrm{Cot}(X) \cap \mathrm{GFlat}(X).$$

Proof. The containment “ \subseteq ” is immediate by Theorem 3.3 and Proposition 4.2. For the reverse containment, let \mathcal{M} be a cotorsion Gorenstein flat sheaf on X . There exists an \mathbf{F} -totally acyclic complex \mathcal{F} of flat sheaves with $\mathcal{M} = Z^0(\mathcal{F})$. As $(\mathrm{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X)), \mathrm{C}_{\mathrm{semi}}(\mathrm{Cot}(X)))$ is a complete cotorsion pair, see Remark 3.1, there is an exact sequence in $\mathrm{C}(\mathrm{Qcoh}(X))$

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow \mathcal{P} \longrightarrow 0$$

with $\mathcal{T} \in \mathrm{C}_{\mathrm{semi}}(\mathrm{Cot}(X))$ and $\mathcal{P} \in \mathrm{C}_{\mathrm{ac}}^{\mathbb{Z}}(\mathrm{Flat}(X))$. As \mathcal{F} and \mathcal{P} are \mathbf{F} -totally acyclic so is \mathcal{T} ; in particular, $Z^n(\mathcal{T})$ is cotorsion for every $n \in \mathbb{Z}$; see Theorem 3.3. The argument in [5, Theorem 5.2] now applies verbatim to finish the proof. \square

Remark 4.4. One upshot of Theorem 4.3 is that the Frobenius category described in Corollary 2.6 coincides with the one associated to $\mathrm{GFC}(X)$ per [5, Theorem 2.11]. In particular, the associated stable categories are equal. One of these is equivalent to the homotopy category of the Gorenstein flat model structure and the other is by [5, Corollary 3.9] and Proposition 4.2 equivalent to the homotopy category

$$\mathrm{K}_{\mathbf{F}\text{-tac}}(\mathrm{Flat}(X) \cap \mathrm{Cot}(X))$$

of \mathbf{F} -totally acyclic complexes of flat-cotorsion sheaves on X .

¹In [5] this category is denoted $\mathrm{Gor}_{\mathrm{FlatCot}}(A)$ in the case of an affine scheme $X = \mathrm{Spec} A$.

In [23, 2.5] the *pure derived category of flat sheaves* on X is the Verdier quotient

$$\mathbf{D}(\mathbf{Flat}(X)) = \frac{\mathbf{K}(\mathbf{Flat}(X))}{\mathbf{K}_{\text{pac}}(\mathbf{Flat}(X))},$$

where $\mathbf{K}_{\text{pac}}(\mathbf{Flat}(X))$ is the full subcategory of $\mathbf{K}(\mathbf{Flat}(X))$ of pure acyclic complexes; that is, the objects in $\mathbf{K}_{\text{pac}}(\mathbf{Flat}(X))$ are precisely the objects in $\mathbf{C}_{\text{ac}}^{\mathbf{Z}}(\mathbf{Flat}(X))$. Still following [23] we denote by $\mathbf{D}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X))$ the full subcategory of $\mathbf{D}(\mathbf{Flat}(X))$ whose objects are \mathbf{F} -totally acyclic. As the category $\mathbf{K}_{\text{pac}}(\mathbf{Flat}(X))$ is contained in $\mathbf{K}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X))$, it can be expressed as the Verdier quotient

$$\mathbf{D}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X)) = \frac{\mathbf{K}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X))}{\mathbf{K}_{\text{pac}}(\mathbf{Flat}(X))}.$$

Theorem 4.5. *Assume that X is semi-separated noetherian. The composite of canonical functors*

$$\mathbf{K}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X) \cap \mathbf{Cot}(X)) \longrightarrow \mathbf{K}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X)) \longrightarrow \mathbf{D}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X))$$

is a triangulated equivalence of categories.

Proof. In view of Theorem 3.3 and the fact that $(\mathbf{C}_{\text{ac}}^{\mathbf{Z}}(\mathbf{Flat}(X)), \mathbf{C}_{\text{semi}}(\mathbf{Cot}(X)))$ is a complete cotorsion pair, see Remark 3.1, the proof of [5, Theorem 5.6] applies *mutatis mutandis*. \square

We denote by $\mathbf{StGFC}(X)$ the stable category of Gorenstein flat-cotorsion sheaves; cf. Remark 4.4. Let A be a commutative noetherian ring of finite Krull dimension. For the affine scheme $X = \text{Spec } A$ this category is by [5, Corollary 5.9] equivalent to the stable category $\mathbf{StGPrj}(A)$ of Gorenstein projective A -modules. This, together with the next result, suggests that $\mathbf{StGFC}(X)$ is a natural non-affine analogue of $\mathbf{StGPrj}(A)$. Indeed, the category $\mathbf{D}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X))$ is Murfet and Salarian's non-affine analogue of the homotopy category of totally acyclic complexes of projective modules; see [23, Lemma 4.22].

Corollary 4.6. *There is a triangulated equivalence of categories*

$$\mathbf{StGFC}(X) \simeq \mathbf{D}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X)).$$

Proof. Combine the equivalence $\mathbf{StGFC}(X) \simeq \mathbf{K}_{\mathbf{F}\text{-tac}}(\mathbf{Flat}(X) \cap \mathbf{Cot}(X))$ from Remark 4.4 with Theorem 4.5. \square

We emphasize that Proposition 4.2 and Theorem 4.5 offer another equivalent of the category $\mathbf{StGFC}(X)$, namely the homotopy category of totally acyclic complexes of flat-cotorsion sheaves.

A noetherian scheme X is called Gorenstein if the local ring $\mathcal{O}_{X,x}$ is Gorenstein for every $x \in X$. We close this paper with a characterization of Gorenstein schemes in terms of flat-cotorsion sheaves, it sharpens [23, Theorem 4.27]. In a paper in progress [3] we show that Gorensteinness of a scheme X can be characterized by the equivalence of the category $\mathbf{StGFC}(X)$ to a naturally defined singularity category.

Theorem 4.7. *Assume that X is semi-separated noetherian. The following conditions are equivalent.*

- (i) X is Gorenstein.
- (ii) Every acyclic complex of flat sheaves on X is \mathbf{F} -totally acyclic.
- (iii) Every acyclic complex of flat-cotorsion sheaves on X is \mathbf{F} -totally acyclic.
- (iv) Every acyclic complex of flat-cotorsion sheaves on X is totally acyclic.

Proof. The equivalence of conditions (i) and (ii) is [23, Theorem 4.27], and conditions (iii) and (iv) are equivalent by Proposition 4.2. As (ii) evidently implies (iii), it suffices to argue the converse.

Assume that every acyclic complex of flat-cotorsion sheaves is \mathbf{F} -totally acyclic. Let \mathcal{F} be an acyclic complex of flat sheaves. As $(\mathbf{C}_{\text{ac}}^{\mathbf{Z}}(\text{Flat}(X)), \mathbf{C}_{\text{semi}}(\text{Cot}(X)))$ is a complete cotorsion pair, see Remark 3.1, there is an exact sequence in $\mathbf{C}(\text{Qcoh}(X))$,

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C} \longrightarrow \mathcal{P} \longrightarrow 0,$$

with $\mathcal{C} \in \mathbf{C}_{\text{semi}}(\text{Cot}(X))$ and $\mathcal{P} \in \mathbf{C}_{\text{ac}}^{\mathbf{Z}}(\text{Flat}(X))$. Since \mathcal{F} and \mathcal{P} are acyclic, the complex \mathcal{C} is also acyclic. Moreover, \mathcal{C} is a complex of flat-cotorsion sheaves. By assumption \mathcal{C} is \mathbf{F} -totally acyclic, and so is \mathcal{P} , whence it follows that \mathcal{F} is \mathbf{F} -totally acyclic. \square

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REFERENCES

- [1] Silvana Bazzoni, Manuel Cortés Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes*, Algebr. Represent. Theory, online 2019.
- [2] Ragnar-Olaf Buchweitz, *Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings*, University of Hannover, 1986, available at <http://hdl.handle.net/1807/16682>.
- [3] Lars Winther Christensen, Nanqing Ding, Sergio Estrada, Jiangsheng Hu, Huanhuan Li, and Peder Thompson, *Singularity categories of exact categories with applications to the flat-cotorsion theory*, in preparation.
- [4] Lars Winther Christensen, Sergio Estrada, and Alina Iacob, *A Zariski-local notion of \mathbf{F} -total acyclicity for complexes of sheaves*, Quaest. Math. **40** (2017), no. 2, 197–214. MR3630500
- [5] Lars Winther Christensen, Sergio Estrada, and Peder Thompson, *Homotopy categories of totally acyclic complexes with applications to the flat-cotorsion theory*, Categorical, Homological and Combinatorial Methods in Algebra, Contemp. Math., vol. 751, Amer. Math. Soc., Providence, RI, 2020, pp. 99–118.
- [6] Lars Winther Christensen, Fatih Köksal, and Li Liang, *Gorenstein dimensions of unbounded complexes and change of base (with an appendix by Driss Bennis)*, Sci. China Math. **60** (2017), no. 3, 401–420. MR3600932
- [7] Alexander I. Efimov and Leonid Positselski, *Coherent analogues of matrix factorizations and relative singularity categories*, Algebra Number Theory **9** (2015), no. 5, 1159–1292. MR3366002
- [8] Edgar Enochs and Sergio Estrada, *Relative homological algebra in the category of quasi-coherent sheaves*, Adv. Math. **194** (2005), no. 2, 284–295. MR2139915
- [9] Edgar E. Enochs and Overtoun M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR1753146
- [10] Edgar E. Enochs, Overtoun M. G. Jenda, and Juan A. López-Ramos, *The existence of Gorenstein flat covers*, Math. Scand. **94** (2004), no. 1, 46–62. MR2032335
- [11] Sergio Estrada, Pedro A. Guil Asensio, Mike Prest, and Jan Trlifaj, *Model category structures arising from Drinfeld vector bundles*, Adv. Math. **231** (2012), no. 3–4, 1417–1438. MR2964610
- [12] Sergio Estrada, Alina Iacob, and Marco Perez, *Model structures and relative Gorenstein flat modules*, Categorical, Homological and Combinatorial Methods in Algebra, Contemp. Math., vol. 751, Amer. Math. Soc., Providence, RI, 2020, pp. 135–176.
- [13] James Gillespie, *The flat model structure on $\text{Ch}(R)$* , Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390. MR2052954
- [14] James Gillespie, *Kaplansky classes and derived categories*, Math. Z. **257** (2007), no. 4, 811–843. MR2342555

- [15] James Gillespie, *Model structures on exact categories*, J. Pure Appl. Algebra **215** (2011), no. 12, 2892–2902. MR2811572
- [16] James Gillespie, *How to construct a Hovey triple from two cotorsion pairs*, Fund. Math. **230** (2015), no. 3, 281–289. MR3351474
- [17] James Gillespie, *The flat stable module category of a coherent ring*, J. Pure Appl. Algebra **221** (2017), no. 8, 2025–2031. MR3623182
- [18] A. Grothendieck and J. A. Dieudonné, *Eléments de géométrie algébrique. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 166, Springer-Verlag, Berlin, 1971. MR3075000
- [19] Robin Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, vol. 20, Springer-Verlag, Berlin, 1966. MR0222093
- [20] Esmail Hosseini, *The pure derived category of quasi-coherent sheaves*, Comm. Algebra **47** (2019), no. 9, 3781–3788. MR3969478
- [21] Mark Hovey, *Cotorsion pairs, model category structures, and representation theory*, Math. Z. **241** (2002), no. 3, 553–592. MR1938704
- [22] Daniel Murfet, *The mock homotopy category of projectives and Grothendieck duality*, Ph.D. thesis, Australian National University, 2007, x+145 pp., Available from www.therisingsea.org.
- [23] Daniel Murfet and Shokrollah Salarian, *Totally acyclic complexes over Noetherian schemes*, Adv. Math. **226** (2011), no. 2, 1096–1133. MR2737778
- [24] Amnon Neeman, *The homotopy category of flat modules, and Grothendieck duality*, Invent. Math. **174** (2008), no. 2, 255–308. MR2439608
- [25] Manuel Saorín and Jan Šťovíček, *On exact categories and applications to triangulated adjoints and model structures*, Adv. Math. **228** (2011), no. 2, 968–1007. MR2822215
- [26] Alexander Slávik and Jan Šťovíček, *On the existence of flat generators of quasicoherent sheaves*, preprint [arXiv:1902.05740](https://arxiv.org/abs/1902.05740) [math.AG]; 8 pp.
- [27] Jan Šťovíček, *On purity and applications to coderived and singularity categories*, preprint, [arXiv:1412.1615](https://arxiv.org/abs/1412.1615) [math.CT]; 45 pp.
- [28] Jan Šaroch and Jan Šťovíček, *Singular compactness and definability for Σ -cotorsion and Gorenstein modules*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 23, 40. MR4076700
- [29] Gang Yang and Zhongkui Liu, *Stability of Gorenstein flat categories*, Glasg. Math. J. **54** (2012), no. 1, 177–191. MR2862396

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