

# A REFINEMENT OF GORENSTEIN FLAT DIMENSION VIA THE FLAT-COTORSION THEORY

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ABSTRACT. We introduce a refinement of the Gorenstein flat dimension for complexes over an associative ring—the *Gorenstein flat-cotorsion dimension*—and prove that it, unlike the Gorenstein flat dimension, behaves as one expects of a homological dimension without extra assumptions on the ring. Crucially, we show that it coincides with the Gorenstein flat dimension for complexes where the latter is finite, and for complexes over right coherent rings—the setting where the Gorenstein flat dimension is known to behave as expected.

## INTRODUCTION

The introduction of the G-dimension by Auslander and Bridger [1], and the subsequent broader notion of Gorenstein projective dimension by Enochs and Jenda [8], provided for an elegant characterization of Gorenstein rings in terms of finiteness of homological invariants. It is modeled on the characterization by Auslander, Buchsbaum, and Serre of commutative regular local rings as rings of finite global dimension. Further pursuit of this analogy led to the introduction of the Gorenstein injective and Gorenstein flat dimensions with the aim of building a theory of Gorenstein homological dimensions modeled on the classic projective, injective, and flat dimensions.

This program has largely been successful, but from a homological algebra point of view not entirely so: While the Gorenstein projective and injective dimensions behave as one expects of homological dimensions—in particular, they can be computed in terms of vanishing of cohomology—the Gorenstein flat dimension only exhibits such behavior under coherence assumptions on the ring; see Holm [13]. Our goal is to give a new perspective on the Gorenstein flat dimension: One that puts it on the same footing as the Gorenstein projective and Gorenstein injective dimensions and ensures that it behaves nicely without assumptions on the ring.

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In recent years, it has become apparent that one ought to pay special attention to the narrower class of Gorenstein flat modules that are also cotorsion. For example, work of Gillespie [12] shows that under coherence assumptions on the ring, the category of modules that are Gorenstein flat and cotorsion is Frobenius, while the category of Gorenstein flat modules rarely is; in fact, it only happens when every Gorenstein flat module is cotorsion, see [4, Thm. 4.5]. Motivated in part by this, *Gorenstein flat-cotorsion* modules were introduced in [4], and it was shown that over a right coherent ring they are precisely the modules that are Gorenstein flat and cotorsion.

To push the Gorenstein flat dimension beyond the setting of coherent rings, we introduce the *Gorenstein flat-cotorsion dimension*, not so much to introduce a new homological dimension but rather to refine the already established dimension. The Gorenstein flat-cotorsion dimension is defined in terms of the Hom functor—rather than the tensor product functor which is used for the Gorenstein flat dimension—and in this way it behaves more like the Gorenstein projective and Gorenstein injective dimensions.

Let  $R$  be an associative ring. The flat, Gorenstein flat, and Gorenstein flat-cotorsion dimensions of an  $R$ -complex  $M$  are written  $\text{fd}_R M$ ,  $\text{Gfd}_R M$  and  $\text{Gfcd}_R M$ , respectively. The next two statements, which are extracted from Theorems 4.5, 5.7, and 5.12, capture the essence of the new dimension.

**Theorem A.** *Let  $M$  be an  $R$ -complex and  $n$  an integer. If  $\text{Gfcd}_R M$  is finite and  $n \geq \sup M$ , then the following conditions are equivalent.*

- (i)  $\text{Gfcd}_R M \leq n$ .
- (ii)  $\text{Ext}_R^i(M, C) = 0$  for all  $i > n$  and every  $R$ -module  $C$  that is flat and cotorsion.
- (iii)  $\text{Ext}_R^i(M, C) = 0$  for all  $i > n$  and every cotorsion  $R$ -module  $C$  of finite flat dimension.
- (iv)  $\text{Ext}_R^{n+1}(M, C) = 0$  for every cotorsion  $R$ -module  $C$  of finite flat dimension.

**Theorem B.** *Let  $M$  be an  $R$ -complex. There are inequalities,*

$$\text{Gfcd}_R M \leq \text{Gfd}_R M \leq \text{fd}_R M,$$

*and if any of these quantities is finite, then it equals those to the left of it.*

While the proof of Theorem A is standard fare homological algebra, the connection to the Gorenstein flat dimension captured by Theorem B relies crucially on recent work of Šaroch and Štovíček [20]. It is their work that allows us to say that the Gorenstein flat-cotorsion dimension is not so much a new homological dimension as it is a new perspective on an old one. To further illustrate the utility of this perspective, we briefly introduce a version of Tate cohomology associated to the theory of Gorenstein flat-cotorsion modules. It is shown that this yields a generalization of a result of Hu and Ding [14] that characterizes complexes of finite flat dimension among those of finite Gorenstein flat dimension; see Theorem 6.7.

## 1. NOTATION AND TERMINOLOGY

Throughout the paper,  $R$  denotes an associative ring. By an  $R$ -module we mean a left  $R$ -module; right  $R$ -modules are considered modules over the opposite ring  $R^\circ$ . A complex of  $R$ -modules is, for short, called an  $R$ -complex. We use homological notation for complexes, i.e. for  $n \in \mathbb{Z}$  the module in degree  $n$  of an

$R$ -complex  $M$  is denoted  $M_n$ . The submodules of boundaries and cycles are denoted  $B_n(M)$  and  $Z_n(M)$ , respectively. The homology is as always the quotient  $H_n(M) = Z_n(M)/B_n(M)$ . We further use the notation  $C_n(M)$  for the cokernel of the differential  $\partial_{n+1}^M$ , i.e. one has  $C_n(M) = M_n/B_n(M)$ . The invariants

$$\sup M = \sup\{n \in \mathbb{Z} \mid H_n(M) \neq 0\} \quad \text{and} \quad \inf M = \inf\{n \in \mathbb{Z} \mid H_n(M) \neq 0\}$$

capture the homological position of the complex. If  $H_n(M) = 0$  holds for all  $n \in \mathbb{Z}$ , then  $M$  is called *acyclic*. Morphisms of complexes that induce isomorphisms in homology are called *quasi-isomorphisms*; they are characterized by having acyclic mapping cones. For  $s \in \mathbb{Z}$  the  $s$ -fold *shift* of  $M$  is the complex  $\Sigma^s M$  defined by  $(\Sigma^s M)_n = M_{n-s}$  and  $\partial_n^{\Sigma^s M} = (-1)^s \partial_{n-s}^M$ .

For an  $R$ -complex  $M$ , the *hard truncation above* of  $M$  at  $n$  is the complex

$$M_{\leq n} = \cdots \longrightarrow 0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow \cdots$$

with differential induced from  $M$ ; the *hard truncation below* of  $M$  at  $n$ , denoted  $M_{\geq n}$ , is defined similarly. The *soft truncation above* of  $M$  at  $n$  is the complex

$$M_{\leq n} = \cdots \longrightarrow 0 \longrightarrow C_n(M) \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow \cdots$$

with differential induced from  $M$ . The *soft truncation below* of  $M$  at  $n$ , denoted  $M_{\geq n}$ , is defined similarly by replacing  $M_n$  by  $Z_n(M)$ .

An  $R$ -complex  $P$  is called *semi-projective* if it consists of projective modules and the functor  $\text{Hom}_R(P, -)$  preserves acyclicity. Dually, a complex  $I$  is called *semi-injective* if it consists of injective modules and the functor  $\text{Hom}_R(-, I)$  preserves acyclicity. Every  $R$ -complex  $M$  has a semi-projective resolution, i.e. there is a quasi-isomorphism  $\pi: P \xrightarrow{\cong} M$  where  $P$  is semi-projective; one can choose  $\pi$  surjective or one can choose  $P$  with  $P_n = 0$  for  $n < \inf M$ ; see Avramov and Foxby [2, Sect. 1]. Dually, every  $R$ -complex  $M$  has a semi-injective resolution i.e. there is a quasi-isomorphism  $\iota: M \xrightarrow{\cong} I$  where  $I$  is semi-injective; one can choose  $\iota$  injective or one can choose  $I$  with  $I_n = 0$  for  $n > \sup M$ .

An  $R$ -complex  $F$  is called *semi-flat* if it consists of flat modules and the functor  $-\otimes_R F$  preserves acyclicity; see [2, Sect. 1]. Every semi-projective complex is semi-flat and so is every bounded below complex of flat modules.

An  $R$ -complex  $C$  is called *semi-cotorsion* if it consists of cotorsion modules and  $\text{Hom}_R(F, C)$  is acyclic for every acyclic semi-flat  $R$ -complex  $F$ . Every semi-injective complex is semi-cotorsion and so is every bounded above complex of cotorsion  $R$ -modules; this is a standard consequence of [5, Lem. 2.5]. In fact, one can do better:

**1.1 Fact.** An acyclic complex of cotorsion  $R$ -modules has cotorsion cycle modules, and it follows that every complex of cotorsion  $R$ -modules is semi-cotorsion. This is proved by Bazzoni, Cortés Izurdiaga, and Estrada [3, Thm. 1.3].

**1.2 Lemma.** For every  $R^\circ$ -complex  $M$  the  $R$ -complex  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is semi-cotorsion.

**Proof.** Let  $F$  be an acyclic semi-flat  $R$ -complex; it is a direct limit of contractible complexes of finitely generated free  $R$ -modules, see [6, Thm. 7.3], so  $M \otimes_R F$  is acyclic. It follows by Hom–tensor adjunction that  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is a complex of cotorsion  $R$ -modules and, further, that there is an isomorphism

$$\text{Hom}_R(F, \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \text{Hom}_{\mathbb{Z}}(M \otimes_R F, \mathbb{Q}/\mathbb{Z}).$$

The right-hand complex is acyclic, so  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is semi-cotorsion.  $\square$

**1.3 Lemma.** *Let  $0 \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0$  be an exact sequence of  $R$ -complexes. If  $C'$  is semi-cotorsion, then  $C$  is semi-cotorsion if and only if  $C''$  is semi-cotorsion.*

**Proof.** This is an immediate consequence of Fact 1.1.  $\square$

The next fact is standard and proved in much the same way as Lemma 1.3.

**1.4 Fact.** Let  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  be an exact sequence of  $R$ -complexes. If  $F''$  is semi-flat, then  $F$  is semi-flat if and only if  $F'$  is semi-flat.

From [18, A.1] we recall:

**1.5 Fact.** For a semi-flat  $R$ -complex  $F$  and a semi-cotorsion  $R$ -complex  $C$  there is an isomorphism  $\mathbf{R}\mathrm{Hom}_R(F, C) \simeq \mathrm{Hom}_R(F, C)$  in the derived category.

**1.6 Proposition.** *A complex  $F$  of flat  $R$ -modules is semi-flat if and only if the complex  $\mathrm{Hom}_R(F, C)$  is acyclic for every acyclic semi-cotorsion complex  $C$ .*

**Proof.** The “only if” follows from Fact 1.5. For the converse, let  $M$  be an acyclic  $R^\circ$ -complex. The  $R$ -complex  $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is semi-cotorsion by Lemma 1.2 and acyclic. Thus it follows from the adjunction isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M \otimes_R F, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_R(F, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))$$

and faithful injectivity of  $\mathbb{Q}/\mathbb{Z}$  that  $M \otimes_R F$  is acyclic.  $\square$

**1.7 Proposition.** *Let  $C$  be a semi-cotorsion  $R$ -complex and  $\beta: F \xrightarrow{\simeq} F'$  a quasi-isomorphism of semi-flat  $R$ -complexes. For every morphism  $\alpha: F \rightarrow C$  there is a morphism  $\gamma: F' \rightarrow C$  with  $\gamma\beta \sim \alpha$ , and  $\gamma$  is unique up to homotopy.*

**Proof.** The mapping cone of  $\beta$  is acyclic and semi-flat, so the induced morphism  $\mathrm{Hom}_R(\beta, C)$  is a quasi-isomorphism. It follows that there exists a morphism  $\gamma$  in  $Z_0(\mathrm{Hom}_R(F', C))$  such that

$$[\alpha] = H_0(\mathrm{Hom}_R(\beta, C))([\gamma]) = [\gamma\beta];$$

that is,  $\alpha - \gamma\beta$  is in  $B_0(\mathrm{Hom}_R(F, C))$ . Given another morphism  $\gamma'$  such that  $\gamma'\beta \sim \alpha$ , one has  $[\alpha] = [\gamma'\beta]$  and, therefore  $0 = [(\gamma - \gamma')\beta] = H_0(\mathrm{Hom}_R(\beta, C))([\gamma - \gamma'])$ . It follows that the homology class  $[\gamma - \gamma']$  is 0 as  $H_0(\mathrm{Hom}_R(\beta, C))$  is an isomorphism, so  $\gamma - \gamma'$  is in  $B_0(\mathrm{Hom}_R(F', C))$ . That is,  $\gamma$  and  $\gamma'$  are homotopic.  $\square$

## 2. SEMI-FLAT-COTORSION COMPLEXES

We recall from [18] that an  $R$ -complex  $W$  is referred to as *semi-flat-cotorsion* if it is semi-flat and semi-cotorsion.

**2.1 Proposition.** *Let  $W$  be a semi-flat-cotorsion  $R$ -complex and  $F$  be a semi-flat  $R$ -complex. If  $\beta: W \rightarrow F$  is a quasi-isomorphism, then there is a quasi-isomorphism  $\gamma: F \rightarrow W$  such that  $\gamma\beta \sim 1^W$ . In particular, a quasi-isomorphism of semi-flat-cotorsion  $R$ -complexes is a homotopy equivalence.*

**Proof.** For the first assertion apply Proposition 1.7 with  $\alpha = 1^W$  to get a morphism  $\gamma: F \rightarrow W$  with  $\gamma\beta \sim 1^W$ . As  $\beta$  and  $1^W$  are quasi-isomorphisms, so is  $\gamma$ . Next notice that if  $F$  is semi-flat-cotorsion then the same argument applies to yield a morphism  $\beta': W \rightarrow F$  with  $\beta'\gamma \sim 1^F$ . It follows that  $\gamma$  and hence  $\beta$  is a homotopy equivalence.  $\square$

Gillespie [11] studies how a cotorsion pair in the category of modules induces cotorsion pairs in the category of complexes. The short exact sequences below are often referred to as *approximations*; they exist<sup>1</sup> by [11, Cor. 4.10].

**2.2 Fact.** For every  $R$ -complex  $M$  there are exact sequences of  $R$ -complexes

$$0 \longrightarrow M \longrightarrow C \longrightarrow F \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow C' \longrightarrow F' \longrightarrow M \longrightarrow 0$$

where  $C$  and  $C'$  are semi-cotorsion,  $F$  and  $F'$  are semi-flat, and  $F$  and  $C'$  are acyclic.

**2.3 Definition.** Let  $M$  be an  $R$ -complex. A semi-flat-cotorsion complex that is isomorphic to  $M$  in the derived category is called a *semi-flat-cotorsion replacement* of  $M$ .

The next construction recapitulates the proof of [18, Thm. A.6]; it is one of many ways to construct a semi-flat-cotorsion replacement, Remark 6.6 presents another.

**2.4 Construction.** Let  $M$  be an  $R$ -complex and consider a semi-projective resolution  $\pi^M: P^M \xrightarrow{\simeq} M$ . By Fact 2.2 there is an exact sequence

$$0 \longrightarrow P^M \xrightarrow{\varepsilon^M} C^M \longrightarrow A^M \longrightarrow 0$$

where  $C^M$  is semi-cotorsion and  $A^M$  is acyclic and semi-flat. It follows from Fact 1.4 that  $C^M$  is semi-flat and isomorphic to  $P^M$  and hence to  $M$  in the derived category, so  $C^M$  is a semi-flat-cotorsion replacement of  $M$ .

The Gorenstein flat-cotorsion dimension is defined based on semi-flat-cotorsion replacements. Below we collect some technical results for later use; the first one is about comparison of semi-flat-cotorsion replacements.

**2.5 Proposition.** *Let  $M$  be an  $R$ -complex. If  $W$  and  $W'$  are semi-flat-cotorsion replacements of  $M$ , then there is a homotopy equivalence  $W \rightarrow W'$ .*

**Proof.** Let  $P \xrightarrow{\simeq} M$  be a semi-projective resolution; there are quasi-isomorphisms  $W \xleftarrow{\simeq} P \xrightarrow{\simeq} W'$ , see [2, 1.4.P]. By Proposition 1.7 there is a quasi-isomorphism  $W \xrightarrow{\simeq} W'$ , and by Proposition 2.1 it is a homotopy equivalence.  $\square$

**2.6 Remark.** While, say, a semi-projective resolution is a quasi-isomorphism between complexes, there may not be a quasi-isomorphism between a complex and its semi-flat-cotorsion replacement. In certain cases, though, such maps do exist.

Let  $M$  be a semi-cotorsion complex. By Fact 2.2 there is an exact sequence of complexes  $0 \longrightarrow C \longrightarrow F \xrightarrow{\varpi} M \longrightarrow 0$  with  $F$  semi-flat and  $C$  semi-cotorsion and acyclic. It follows from Lemma 1.3 that  $F$  is semi-flat-cotorsion and, since  $C$  is acyclic the morphism  $\varpi$  is a quasi-isomorphism.

For a cotorsion module  $M$ , a semi-flat-cotorsion resolution  $F \xrightarrow{\simeq} M$  can be constructed by taking successive flat covers, which have cotorsion kernels; in particular, one has  $F_i = 0$  for  $i < 0$ .

The next result is a Schanuel's lemma for semi-flat-cotorsion replacements.

<sup>1</sup>Although [11, cor. 4.10] is stated for commutative rings, it is standard that the result remains valid without this assumption; see for example the discussion before [10, Thm. 4.2].

**2.7 Lemma.** *Let  $M$  be an  $R$ -complex and  $W$  and  $W'$  be semi-flat-cotorsion replacements of  $M$ . For every  $n \in \mathbb{Z}$  there exist flat-cotorsion  $R$ -modules  $V$  and  $V'$  with  $C_n(W) \oplus V \cong C_n(W') \oplus V'$ .*

**Proof.** By Proposition 2.5 there is a homotopy equivalence  $\alpha: W \rightarrow W'$ . The complex  $\text{Cone } \alpha$  is contractible and semi-flat-cotorsion by Lemma 1.3 and Fact 1.4. It follows that every cycle module  $Z_n(\text{Cone } \alpha)$  is flat-cotorsion. The soft truncated morphism  $\alpha_{\leq n}: W_{\leq n} \rightarrow W'_{\leq n}$  is also a homotopy equivalence, so  $\text{Cone}(\alpha_{\leq n})$ , i.e. the complex

$$0 \longrightarrow C_n(W) \longrightarrow C_n(W') \oplus W_{n-1} \longrightarrow W'_{n-1} \oplus W_{n-2} \xrightarrow{\partial_{n-1}^{\text{Cone } \alpha}} W'_{n-2} \oplus W_{n-3} \longrightarrow \cdots$$

is contractible. Hence one has  $C_n(W) \oplus Z_{n-1}(\text{Cone } \alpha) \cong C_n(W') \oplus W_{n-1}$ .  $\square$

**2.8 Lemma.** *Let  $W$  be a semi-flat-cotorsion complex. For every  $n \in \mathbb{Z}$  the truncated complex  $W_{\geq n}$  is semi-flat-cotorsion.*

**Proof.** As  $W_{\geq n}$  is a bounded below complex of flat modules it is semi-flat, and it is semi-cotorsion by Fact 1.1. Thus  $W_{\geq n}$  is semi-flat-cotorsion.  $\square$

### 3. GORENSTEIN FLAT-COTORSION MODULES

Recall from [4, Def. 4.3 and Prop. 1.3] that a *totally acyclic complex of flat-cotorsion modules* is an acyclic complex  $T$  of flat-cotorsion modules such that the complexes  $\text{Hom}_R(T, W)$  and  $\text{Hom}_R(W, T)$  are acyclic for every flat-cotorsion  $R$ -module  $W$ . An  $R$ -module  $G$  is called *Gorenstein flat-cotorsion* if there exists a totally acyclic complex  $T$  of flat-cotorsion  $R$ -modules with  $Z_0(T) = G$ .

**3.1 Remark.** To show that an acyclic complex  $T$  of flat-cotorsion  $R$ -modules is totally acyclic it suffices to verify that  $\text{Hom}_R(T, W)$  is acyclic for every flat-cotorsion  $R$ -module  $W$ ; the acyclicity of  $\text{Hom}_R(W, T)$  is automatic as the cycles in  $T$  are cotorsion by Fact 1.1.

**3.2 Lemma.** *An  $R$ -module  $G$  is Gorenstein flat-cotorsion if and only if the following conditions are satisfied.*

- (1)  $G$  is cotorsion.
- (2)  $\text{Ext}_R^{i \geq 1}(G, W) = 0$  holds for every flat-cotorsion  $R$ -module  $W$ .
- (3) There exists a complex  $T = T_0 \rightarrow T_{-1} \rightarrow \cdots$  of flat-cotorsion  $R$ -modules and an injective quasi-isomorphism  $\varepsilon: G \rightarrow T$  such that  $\text{Hom}_R(\varepsilon, W)$  is a quasi-isomorphism for every flat-cotorsion  $R$ -module  $W$ .

**Proof.** The “only if” is in view of Remark 3.1 clear from the definition of Gorenstein flat-cotorsion modules. For the “if” recall from Remark 2.6 that since  $G$  is cotorsion, there is a semi-flat-cotorsion complex  $T'$  and a surjective quasi-isomorphism  $\pi: T' \xrightarrow{\cong} G$ . Splicing  $\Sigma T'$  with the complex  $T$  yields an acyclic complex  $T''$  of flat-cotorsion modules with  $Z_0(T'') = G$ . For every flat-cotorsion module  $W$  one has, by way of Fact 1.5, that

$$H_i(\text{Hom}_R(T'', W)) = H_{i+1}(\text{Hom}_R(T', W)) = \text{Ext}_R^{-i-1}(G, W) = 0$$

for  $i \leq -2$ , and for  $i \geq 0$  one has  $H_i(\text{Hom}_R(T'', W)) = H_i(\text{Cone } \text{Hom}_R(\varepsilon, W)) = 0$ . Finally  $H_{-1}(\text{Hom}_R(T'', W)) = 0$  holds as one has

$$\text{Im}(\text{Hom}_R(\varepsilon_0, W)) = \text{Hom}_R(G, W) = \text{Ker}(\text{Hom}_R(\partial_1^{T'}, W)),$$

where the first equality holds as  $\text{Hom}_R(\varepsilon, W)$  is a quasi-isomorphism and the second holds by left exactness of  $\text{Hom}_R(-, W)$ .  $\square$

**3.3 Proposition.** *The class of Gorenstein flat-cotorsion  $R$ -modules is closed under finite direct sums and direct summands.*

**Proof.** To see that the class is closed under finite direct sums, let  $T$  and  $T'$  be totally acyclic complexes of flat-cotorsion  $R$ -modules. The direct sum  $T \oplus T'$  is a totally acyclic complex of flat-cotorsion modules with  $Z_0(T \oplus T') = Z_0(T) \oplus Z_0(T')$ .

Assume that  $G$  and  $G'$  are  $R$ -modules such that  $G \oplus G'$  is Gorenstein flat-cotorsion. By additivity of  $\text{Ext}$ , it follows from Lemma 3.2 that  $G$  and  $G'$  are cotorsion with  $\text{Ext}_R^{i \geq 1}(G, W) = 0 = \text{Ext}_R^{i \geq 1}(G', W)$  for every flat-cotorsion  $R$ -module  $W$ . It remains to verify part (3) of Lemma 3.2 for, say, the module  $G$ .

By definition there is an exact sequence of  $R$ -modules,

$$0 \longrightarrow G \oplus G' \longrightarrow T_0 \longrightarrow G'' \longrightarrow 0,$$

with  $T_0$  flat-cotorsion and  $G''$  Gorenstein flat-cotorsion, in particular cotorsion. Consider the push-out diagram with exact rows and columns:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G & \xlongequal{\quad} & G & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G \oplus G' & \longrightarrow & T_0 & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & G' & \longrightarrow & X_{-1} & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Let  $W$  be a flat-cotorsion  $R$ -module. As  $\text{Ext}_R^{i \geq 1}(G', W) = 0 = \text{Ext}_R^{i \geq 1}(G'', W)$ , again by Lemma 3.2, one has  $\text{Ext}_R^{i \geq 1}(X_{-1}, W) = 0$ . Thus the exact sequence

$$(1) \quad 0 \longrightarrow G \longrightarrow T_0 \longrightarrow X_{-1} \longrightarrow 0$$

is  $\text{Hom}_R(-, W)$ -exact for every flat-cotorsion  $R$ -module  $W$ .

Next interchange the roles of  $G'$  and  $G$  and consider the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & G' & \xlongequal{\quad} & G' & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G \oplus G' & \longrightarrow & T_0 & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & G & \longrightarrow & X_{-1} & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns. As above,  $\text{Ext}_R^{i \geq 1}(X'_{-1}, W) = 0$  holds for every flat-cotorsion  $R$ -module  $W$ . In particular, the exact sequence

$$(2) \quad 0 \longrightarrow G' \longrightarrow T_0 \longrightarrow X'_{-1} \longrightarrow 0$$

is  $\text{Hom}_R(-, W)$ -exact for every flat-cotorsion  $R$ -module  $W$ . The direct sum of the sequences (1) and (2) makes up the upper row in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & G \oplus G' & \longrightarrow & T_0 \oplus T_0 & \longrightarrow & X_{-1} \oplus X'_{-1} \longrightarrow 0 \\ & & \parallel & & \alpha \downarrow & & \downarrow \\ 0 & \longrightarrow & G \oplus G' & \longrightarrow & T_0 & \longrightarrow & G'' \longrightarrow 0 \end{array}$$

where  $\alpha: T_0 \oplus T_0 \rightarrow T_0$  is the epimorphism given by  $\alpha(x, y) = x + y$ . By the Snake Lemma one gets the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & T_0 & \xlongequal{\quad} & T_0 \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G \oplus G' & \longrightarrow & T_0 \oplus T_0 & \longrightarrow & X_{-1} \oplus X'_{-1} \longrightarrow 0 \\ & & \parallel & & \alpha \downarrow & & \downarrow \\ 0 & \longrightarrow & G \oplus G' & \longrightarrow & T_0 & \longrightarrow & G'' \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array}$$

As  $T_0$  is flat-cotorsion, the right-hand column splits, so there is an isomorphism  $X_{-1} \oplus X'_{-1} \cong T_0 \oplus G''$ . As the class of Gorenstein flat-cotorsion modules is closed under finite direct sums, it follows that  $X_{-1} \oplus X'_{-1}$  is Gorenstein flat-cotorsion.

Applying the same process to  $X_{-1} \oplus X'_{-1}$  one gets exact sequences

$$\begin{array}{l} 0 \longrightarrow X_{-1} \longrightarrow T_{-1} \longrightarrow X_{-2} \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow X'_{-1} \longrightarrow T_{-1} \longrightarrow X'_{-2} \longrightarrow 0 \end{array}$$

of  $R$ -modules where  $T_{-1}$  is flat-cotorsion and  $X_{-2} \oplus X'_{-2}$  is Gorenstein flat-cotorsion. Moreover, both sequences are  $\text{Hom}_R(-, W)$ -exact for every flat-cotorsion  $R$ -module  $W$ . Continuing this process, one gets an exact sequence

$$0 \longrightarrow G \longrightarrow T_0 \longrightarrow T_{-1} \longrightarrow T_{-2} \longrightarrow \cdots$$

with each  $T_i$  flat-cotorsion, and the sequence is  $\text{Hom}_R(-, W)$ -exact for every flat-cotorsion  $R$ -module  $W$ .  $\square$

**3.4 Proposition.** *Let  $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$  be an exact sequence of cotorsion  $R$ -modules.*

- (a) *If  $G''$  is Gorenstein flat-cotorsion, then  $G$  is Gorenstein flat-cotorsion if and only if  $G'$  is Gorenstein flat-cotorsion.*
- (b) *If  $G'$  and  $G$  are Gorenstein flat-cotorsion, then  $G''$  is Gorenstein flat-cotorsion if and only if  $\text{Ext}_R^1(G'', W) = 0$  holds for every flat-cotorsion  $R$ -module  $W$ .*



**Proof.** (a): Let  $0 \rightarrow G' \xrightarrow{\alpha'} G \xrightarrow{\alpha} G'' \rightarrow 0$  be an exact sequence of cotorsion  $R$ -modules. If  $G'$  and  $G''$  are Gorenstein flat-cotorsion, then it follows from [4, Prop. 4.2 and Lem. 2.10] that  $G$  is Gorenstein flat-cotorsion.

Assume now that  $G$  and  $G''$  are Gorenstein flat-cotorsion. By definition there exist exact sequences of  $R$ -modules

$$0 \rightarrow G \xrightarrow{\tilde{\varepsilon}} \tilde{T} \rightarrow C \rightarrow 0 \quad \text{and} \quad 0 \rightarrow G'' \xrightarrow{\varepsilon''} T'' \rightarrow \text{Coker } \varepsilon'' \rightarrow 0$$

with  $\tilde{T}$  and  $T''$  flat-cotorsion and  $C$  and  $\text{Coker } \varepsilon''$  Gorenstein flat-cotorsion. One has  $\text{Ext}_R^1(C, T'') = 0$  by Lemma 3.2, and so there is a homomorphism  $\tilde{\tau}: \tilde{T} \rightarrow T''$  such that  $\tilde{\tau}\tilde{\varepsilon} = \varepsilon''\alpha$  holds. The module  $T = \tilde{T} \oplus T''$  is flat-cotorsion, and the surjective homomorphism  $\tau = (\tilde{\tau}, 1_{T''}): T \rightarrow T''$  is split. As the class of flat-cotorsion modules is closed under direct summands, the module  $T' = \text{Ker } \tau$  is flat-cotorsion. Let  $\varepsilon: G \rightarrow T$  be the composite  $G \xrightarrow{\tilde{\varepsilon}} \tilde{T} \hookrightarrow T$ ; it is an injective homomorphism with  $\text{Coker } \varepsilon = C \oplus T''$  Gorenstein flat-cotorsion, and  $\tau\varepsilon = \tilde{\tau}\tilde{\varepsilon} = \varepsilon''\alpha$ . Let  $\tau'$  be the inclusion of  $T'$  into  $T$  and  $\varepsilon': G' \rightarrow T'$  the injective homomorphism induced by  $\varepsilon$ . One now has a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & G' & \xrightarrow{\alpha'} & G & \xrightarrow{\alpha} & G'' \longrightarrow 0 \\ & & \varepsilon' \downarrow & & \varepsilon \downarrow & & \varepsilon'' \downarrow \\ 0 & \longrightarrow & T' & \xrightarrow{\tau'} & T & \xrightarrow{\tau} & T'' \longrightarrow 0 \end{array}$$

Consider the induced exact sequence  $0 \rightarrow \text{Coker } \varepsilon' \rightarrow \text{Coker } \varepsilon \rightarrow \text{Coker } \varepsilon'' \rightarrow 0$ . As  $G'$  and  $T'$  are cotorsion, so is the module  $\text{Coker } \varepsilon'$ . Since  $\text{Coker } \varepsilon$  and  $\text{Coker } \varepsilon''$  are Gorenstein flat-cotorsion, one has  $\text{Ext}_R^1(\text{Coker } \varepsilon', W) = 0$  for all flat-cotorsion  $R$ -modules  $W$  by Lemma 3.2. It follows that the sequence

$$0 \rightarrow G' \xrightarrow{\varepsilon'} T' \rightarrow \text{Coker } \varepsilon' \rightarrow 0$$

is  $\text{Hom}_R(-, W)$ -exact for all flat-cotorsion  $R$ -modules  $W$ . Repeating this process, one sees that the module  $G'$  satisfies condition (3) in Lemma 3.2. Moreover,  $G'$  satisfies condition (1) by assumption, and as  $G$  and  $G''$  satisfy (2), so does  $G'$ .

(b): The “only if” part follows from Lemma 3.2. For the converse, recall that there exists an exact sequence  $0 \rightarrow G' \rightarrow T' \rightarrow G''' \rightarrow 0$  with  $T'$  flat-cotorsion and  $G'''$  Gorenstein flat-cotorsion. Consider the push-out diagram

$$(1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & T' & \longrightarrow & X & \longrightarrow & G'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & G''' & = & G''' & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns. The second column and part (a) show that  $X$  is Gorenstein flat-cotorsion. As  $\text{Ext}_R^1(G'', T') = 0$  holds by assumption, the second row splits, whence  $G''$  is Gorenstein flat-cotorsion by Proposition 3.3.  $\square$

## 4. GORENSTEIN FLAT-COTORSION DIMENSION

We now turn to defining the advertised homological dimension, and subsequently prove that it behaves as one would expect for all associative rings.

**4.1 Definition.** Let  $M$  be an  $R$ -complex. The *Gorenstein flat-cotorsion dimension* of  $M$ , written  $\text{Gfcd}_R M$ , is defined as

$$\text{Gfcd}_R M = \inf \left\{ n \in \mathbb{Z} \mid \begin{array}{l} \text{There is a semi-flat-cotorsion replacement} \\ W \text{ of } M \text{ with } H_i(W) = 0 \text{ for all } i > n \text{ and} \\ \text{with } C_n(W) \text{ Gorenstein flat-cotorsion} \end{array} \right\}$$

with  $\inf \emptyset = \infty$  by convention. We say that  $\text{Gfcd}_R M$  is *finite* if  $\text{Gfcd}_R M < \infty$ .

**4.2.** Let  $M$  be an  $R$ -complex. For every semi-flat-cotorsion replacement  $W$  of  $M$  one has  $H(W) \cong H(M)$ ; the next (in)equalities are hence immediate from the definition,

$$\text{Gfcd}_R M \geq \sup M \quad \text{and} \quad \text{Gfcd}_R \Sigma^s M = \text{Gfcd}_R M + s \quad \text{for every } s \in \mathbb{Z}.$$

Moreover, one has  $\text{Gfcd}_R M = -\infty$  if and only if  $M$  is acyclic.

**4.3 Lemma.** *Let  $M$  be an  $R$ -complex. For every semi-flat-cotorsion replacement  $W$  of  $M$  and every  $n \geq \text{Gfcd}_R M$  the module  $C_n(W)$  is Gorenstein flat-cotorsion.*

**Proof.** Assume that  $\text{Gfcd}_R M = g$  holds for some integer  $g$ . By assumption there exists a semi-flat-cotorsion replacement  $W$  of  $M$  with  $C_g(W)$  Gorenstein flat-cotorsion and  $g \geq \sup W$ . By induction it follows from Proposition 3.4(a) that  $C_n(W)$  is Gorenstein flat-cotorsion for every  $n \geq g$ . Let  $W'$  be any semi-flat-cotorsion replacement of  $M$ . It follows from Lemma 2.7 and Proposition 3.3 that  $C_n(W')$  is Gorenstein flat-cotorsion for every  $n \geq g$ .  $\square$

**4.4 Proposition.** *Let  $M$  be an  $R$ -complex.*

(a) *For every bounded  $R$ -complex  $C$  of flat-cotorsion modules one has*

$$\inf \mathbf{R}\text{Hom}_R(M, C) \geq \inf \{i \in \mathbb{Z} \mid C_i \neq 0\} - \text{Gfcd}_R M.$$

(b) *For every complex  $F$  of finite flat dimension one has*

$$\sup \mathbf{R}\text{Hom}_R(F, M) \leq \text{Gfcd}_R M - \inf F.$$

**Proof.** Both assertions are trivial if  $M$  is acyclic so assume that it is not. The assertions are also trivial if  $\text{Gfcd}_R M$  is infinite, so assume that it is finite and set  $n = \text{Gfcd}_R M$ . Let  $W$  be a semi-flat-cotorsion replacement of  $M$ . There is a short exact sequence  $0 \rightarrow W' \rightarrow W \rightarrow W_{\leq n} \rightarrow 0$ ; by Lemma 4.3 the module  $C_n(W)$  is Gorenstein flat-cotorsion, so it follows from Proposition 3.4(a) that  $W'$  is a complex of Gorenstein flat-cotorsion modules, concentrated in degrees  $n$  and above. Further,  $W'$  is acyclic, as the map  $W \rightarrow W_{\leq n}$  is a quasi-isomorphism. By Lemma 4.3 the cycle modules in  $W'$  are Gorenstein flat-cotorsion, as one has  $Z_i(W') \cong C_{i+1}(W)$  for all  $i \geq n$ .

(a): For every flat-cotorsion module  $C'$  the complex  $\text{Hom}_R(W', C')$  is acyclic; see Lemma 3.2. As  $C$  is bounded above,  $\text{Hom}_R(W', C)$  is acyclic by [5, Lem. 2.5]. Thus Fact 1.5 yields an isomorphism  $\mathbf{R}\text{Hom}_R(M, C) \simeq \text{Hom}_R(W_{\leq n}, C)$  in the derived category. A standard argument—see for example the proof of [5, Thm. 3.1]—yields  $\text{Hom}_R(W_{\leq n}, C)_i = 0$  for  $i < \inf \{i \in \mathbb{Z} \mid C_i \neq 0\} - n$ .

(b) We can assume that the homology of  $F$  is bounded below, otherwise the inequality is trivial. By [2, Rmk. 1.7 and Thm. 2.4.F] we can further assume that  $F$  is a bounded complex of flat modules with  $F_i = 0$  for  $i < \inf F$ . For every  $i \in \mathbb{Z}$ , the complex  $\mathrm{Hom}_R(F_i, W')$  is acyclic, as the cycle modules in  $W'$  are cotorsion. As  $F$  is bounded, it follows from [5, Lem. 2.4] that  $\mathrm{Hom}_R(F, W')$  is acyclic. By Fact 1.5 there is thus an isomorphism  $\mathbf{R}\mathrm{Hom}_R(F, M) \simeq \mathrm{Hom}_R(F, W_{\subseteq n})$  in the derived category. A standard argument now yields  $\mathrm{Hom}_R(F, W_{\subseteq n})_i = 0$  for  $i > -\inf F + n$ .  $\square$

**4.5 Theorem.** *Let  $M$  be an  $R$ -complex and  $n$  be an integer. If  $\mathrm{Gfcd}_R M$  is finite and  $n \geq \sup M$ , then the following conditions are equivalent.*

- (i)  $\mathrm{Gfcd}_R M \leq n$ .
- (ii)  $-\inf \mathbf{R}\mathrm{Hom}_R(M, C) \leq n$  for every cotorsion  $R$ -module  $C$  of finite flat dimension.
- (iii)  $-\inf \mathbf{R}\mathrm{Hom}_R(M, C) \leq n$  for every flat-cotorsion  $R$ -module  $C$ .
- (iv)  $\mathrm{Ext}_R^{n+1}(M, C) = 0$  for every cotorsion  $R$ -module  $C$  of finite flat dimension.
- (v) For every semi-flat-cotorsion replacement  $W$  of  $M$  the module  $C_m(W)$  is Gorenstein flat-cotorsion for every  $m \geq n$ .

The proof relies on a technical lemma which follows at the end of the section.

**Proof.** The statement is trivial for an acyclic complex, so assume that  $M$  is not acyclic. The implications (ii)  $\implies$  (iii), (ii)  $\implies$  (iv), and (v)  $\implies$  (i) are clear. The implication (i)  $\implies$  (v) follows from Lemma 4.3.

(i)  $\implies$  (ii): Apply Proposition 4.4(a) to a finite flat resolution of  $C$  by flat-cotorsion modules; see Remark 2.6.

(iii)  $\implies$  (i): Let  $W$  be a semi-flat-cotorsion replacement of  $M$ . As  $g = \mathrm{Gfcd}_R M$  is finite, it follows from Lemma 4.3 that the module  $C_m(W)$  is Gorenstein flat-cotorsion for every integer  $m \geq g$ . Thus it is enough to show that  $n \geq g$  holds. Assume towards a contradiction that  $n < g$  holds. It now follows from the assumption  $n \geq \sup M$  that the module  $C_{g-1}(W)$  is not Gorenstein flat-cotorsion. By Lemma 2.8 the complex  $\Sigma^{1-g}W_{\geq g-1}$  is a semi-flat-cotorsion replacement of the module  $C_{g-1}(W)$ . Let  $C$  be a flat-cotorsion  $R$ -module; in view of Fact 1.5 one has

$$\begin{aligned} \mathrm{Ext}_R^1(C_{g-1}(W), C) &\cong \mathrm{H}_{-1}(\mathbf{R}\mathrm{Hom}_R(C_{g-1}(W), C)) \\ &= \mathrm{H}_{-1}(\mathrm{Hom}_R(\Sigma^{1-g}W_{\geq g-1}, C)) \\ &= \mathrm{H}_{-g}(\mathrm{Hom}_R(W_{\geq g-1}, C)) \\ &= \mathrm{H}_{-g}(\mathbf{R}\mathrm{Hom}_R(M, C)) \\ &= 0. \end{aligned}$$

It now follows by an application of Proposition 3.4(b) to the short exact sequence  $0 \rightarrow C_g(W) \rightarrow W_{g-1} \rightarrow C_{g-1}(W) \rightarrow 0$  that  $C_{g-1}(W)$  is Gorenstein flat-cotorsion; a contradiction.

(iv)  $\implies$  (i): Let  $W$  be a semi-flat-cotorsion replacement of  $M$ . As in the argument for (iii)  $\implies$  (i), set  $g = \mathrm{Gfcd}_R M$ , assume towards a contradiction that  $n < g$  holds, and notice that the module  $C_n(W)$  is not Gorenstein flat-cotorsion. There is an exact sequence  $0 \rightarrow C_g(W) \rightarrow W_{g-1} \rightarrow \cdots \rightarrow W_n \rightarrow C_n(W) \rightarrow 0$ , so by Lemma 4.7 there is an exact sequence of cotorsion  $R$ -modules

$$(1) \quad 0 \rightarrow K \rightarrow G \rightarrow C_n(W) \rightarrow 0$$

where  $G$  is Gorenstein flat-cotorsion. The complex  $\Sigma^{-n}W_{\geq n}$  is a semi-flat-cotorsion replacement of the module  $C_n(W)$ , see Lemma 2.8, so as above one has

$$\begin{aligned} \mathrm{Ext}_R^1(C_n(W), K) &\cong \mathrm{H}_{-1}(\mathbf{R}\mathrm{Hom}_R(C_n(W), K)) \\ &= \mathrm{H}_{-1}(\mathrm{Hom}_R(\Sigma^{-n}W_{\geq n}, K)) \\ &= \mathrm{H}_{-(n+1)}(\mathrm{Hom}_R(W_{\geq n}, K)) \\ &\cong \mathrm{Ext}_R^{n+1}(M, K) \\ &= 0. \end{aligned}$$

It follows that the sequence (1) splits, which by Proposition 3.3 implies that  $C_n(W)$  is Gorenstein flat-cotorsion; a contradiction.  $\square$

**4.6 Remark.** As one can surmise from the statement of the next lemma, not every  $R$ -module has a flat-cotorsion resolution—an example of such a module is provided in [18, Exa. 3.11]—but per Remark 2.6 cotorsion modules do. Thus, for a cotorsion  $R$ -module  $C$  one has  $\mathrm{Gfcd}_R C \leq n$  if and only if there is an exact sequence  $0 \rightarrow G \rightarrow W_{n-1} \rightarrow \cdots \rightarrow W_0 \rightarrow C \rightarrow 0$  where the modules  $W_i$  are flat-cotorsion and  $G$  is Gorenstein flat-cotorsion. In particular, a cotorsion  $R$ -module  $C$  is Gorenstein flat-cotorsion if and only if  $\mathrm{Gfcd}_R C = 0$  holds.

The proof of the next lemma is adapted from Holm [13].

**4.7 Lemma.** *Let  $M$  be an  $R$ -module. If there exists an exact sequence*

$$0 \longrightarrow G \longrightarrow W_{g-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow M \longrightarrow 0$$

*in which each module  $W_i$  is flat-cotorsion and  $G$  is Gorenstein flat-cotorsion, then there is an exact sequence of cotorsion  $R$ -modules  $0 \rightarrow K \rightarrow G' \rightarrow M \rightarrow 0$  and a quasi-isomorphism  $V \xrightarrow{\simeq} K$  where  $G'$  is Gorenstein flat-cotorsion and  $V$  is a semi-flat-cotorsion complex concentrated in degrees  $g-1, \dots, 0$ .*

**Proof.** As  $G$  is Gorenstein flat-cotorsion, the defining totally acyclic complex yields an exact sequence

$$0 \longrightarrow G \longrightarrow W'_{g-1} \longrightarrow \cdots \longrightarrow W'_0 \longrightarrow H \longrightarrow 0$$

where each module  $W'_i$  is flat-cotorsion and  $H$  is Gorenstein flat-cotorsion. Moreover, this sequence is  $\mathrm{Hom}_R(-, W'')$ -exact for every flat-cotorsion  $R$ -module  $W''$ . It follows that the identity on  $G$  lifts to a morphism of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & W'_{g-1} & \longrightarrow & \cdots & \longrightarrow & W'_0 & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W_{g-1} & \longrightarrow & \cdots & \longrightarrow & W_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

It is a quasi-isomorphism, so the mapping cone

$$0 \longrightarrow W'_{g-1} \longrightarrow W'_{g-2} \oplus W_{g-1} \longrightarrow \cdots \longrightarrow W'_0 \oplus W_1 \longrightarrow H \oplus W_0 \longrightarrow M \longrightarrow 0$$

is acyclic. The module  $G' = H \oplus W_0$  is Gorenstein flat-cotorsion by Proposition 3.3. The truncated complex  $V = 0 \rightarrow W'_{g-1} \rightarrow \cdots \rightarrow W'_0 \oplus W_1 \rightarrow 0$  is semi-flat-cotorsion, and with  $K = \mathrm{Ker}(G' \rightarrow M)$  the canonical map  $V \rightarrow K$  is a quasi-isomorphism. The modules  $K$  and  $M$  are both cotorsion as cotorsion modules are closed under cokernels of monomorphisms.  $\square$

**4.8 Remark.** As observed in [4, Prop. 4.2], a Gorenstein flat-cotorsion module is the same as a right  $\text{Flat}(R)$ -Gorenstein module in the sense of [4, Def. 2.1]. Given a sufficiently well-behaved cotorsion pair  $(\mathbf{U}, \mathbf{V})$  in the category  $\text{Mod}(R)$  of  $R$ -modules, one can adapt the definitions and arguments in Sections 1–4 of this paper to develop a right  $\mathbf{U}$ -Gorenstein dimension theory for  $R$ -complexes, and one can dualize them to develop a left  $\mathbf{V}$ -Gorenstein dimension theory.

Applied to the cotorsion pair  $(\text{Prj}(R), \text{Mod}(R))$  the right dimension is the classic Gorenstein projective dimension and the left dimension is trivial; it detects the homological infimum of a complex; see [4, Exmpl. 2.5]. Dually, the cotorsion pair  $(\text{Mod}(R), \text{Inj}(R))$  yields a right dimension that detects the homological supremum of a complex while the left dimension is the classic Gorenstein injective dimension.

For the cotorsion pair  $(\text{Flat}(R), \text{Cot}(R))$  the right  $\text{Flat}(R)$ -Gorenstein dimension is the one developed in this paper under a different name. A left  $\text{Cot}(R)$ -Gorenstein module is simply a flat-cotorsion module, see again [4, Prop. 4.2], so the left  $\text{Cot}(R)$ -Gorenstein dimension of a complex  $M$  is in this case the least  $n$  such that there is a semi-flat-cotorsion replacement  $F$  of  $M$  with  $F_i = 0$  for all  $i < -n$ .

## 5. COMPARISON TO THE GORENSTEIN FLAT DIMENSION

In this section, we put the newly minted Gorenstein flat-cotorsion dimension into context, showing that in most settings it is nothing but an avatar of the classic Gorenstein flat dimension. Recall that a complex of flat  $R$ -modules is called **F**-totally acyclic if it is acyclic and  $E \otimes_R F$  is acyclic for every injective  $R^\circ$ -module  $E$ . An  $R$ -module  $G$  is called *Gorenstein flat* if there exists an **F**-totally acyclic complex  $F$  with  $Z_0(F) = G$ .

A main result of [20] is that the class of Gorenstein flat modules is the left class in a hereditary cotorsion pair. From [20, Cor. 4.12] one can, in particular, extract the following statement.

**5.1 Fact.** The class of Gorenstein flat  $R$ -modules is closed under direct summands, extensions, and kernels of epimorphisms. Also, for every Gorenstein flat  $R$ -module  $G$  one has  $\text{Ext}_R^i(G, W) = 0$  for all  $i \geq 1$  and every flat-cotorsion  $R$ -module  $W$ .

The first step towards comparison of the dimensions is to compare Gorenstein flat modules to Gorenstein flat-cotorsion modules.

**5.2 Theorem.** *Let  $G$  be an  $R$ -module. If  $G$  is Gorenstein flat and cotorsion, then it is Gorenstein flat-cotorsion. The converse holds if  $R$  is right coherent.*

**Proof.** Assume that  $G$  is Gorenstein flat and cotorsion; it satisfies condition (1) in Lemma 3.2 and by Fact 5.1 also condition (2). Thus, it suffices to show that  $G$  satisfies condition (3) in Lemma 3.2. By definition there is a short exact sequence  $0 \rightarrow G \rightarrow F \rightarrow G' \rightarrow 0$  of  $R$ -modules with  $F$  flat and  $G'$  Gorenstein flat. There is also an exact sequence  $0 \rightarrow F \rightarrow W \rightarrow F' \rightarrow 0$  of  $R$ -modules with  $W$  flat-cotorsion

and  $F'$  flat. Consider the push-out diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & G & \longrightarrow & F & \longrightarrow & G' \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & G & \xrightarrow{\varepsilon_0} & W & \longrightarrow & X \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & F' & = & F' \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

with exact rows and columns. By Fact 5.1, exactness of the right-hand column implies that  $X$  is Gorenstein flat, and it follows that the middle row is  $\mathrm{Hom}_R(-, V)$ -exact for every flat-cotorsion  $R$ -module  $V$ . As  $G$  and  $W$  are cotorsion, so is  $X$ . Continuing this process, one gets an exact sequence

$$(1) \quad 0 \longrightarrow G \xrightarrow{\varepsilon_0} W_{-1} \longrightarrow W_{-2} \longrightarrow \cdots$$

that is  $\mathrm{Hom}_R(-, V)$ -exact for every flat-cotorsion  $R$ -module  $V$ . The modules  $W_n$  are flat-cotorsion, so  $W_{-1} \rightarrow W_{-2} \rightarrow \cdots$  is the desired complex, and  $\varepsilon_0$  is the non-zero component of the desired quasi-isomorphism.

If  $R$  is right coherent, then the converse holds by [4, Thm. 5.2].  $\square$

The definitions of Gorenstein flat dimension found in the literature all agree with the following definition; see also the discussion in [7, Rmk. 5.13].

**5.3 Definition.** Let  $M$  be an  $R$ -complex. The *Gorenstein flat dimension* of  $M$ , written  $\mathrm{Gfd}_R M$ , is defined as

$$\mathrm{Gfd}_R M = \inf \left\{ n \in \mathbb{Z} \mid \begin{array}{l} \text{There exists a semi-flat replacement } F \text{ of } M \text{ with} \\ \mathrm{H}_i(F) = 0 \text{ for all } i > n \text{ and } C_n(F) \text{ Gorenstein flat} \end{array} \right\}$$

with  $\inf \emptyset = \infty$  by convention. We say that  $\mathrm{Gfd}_R M$  is *finite* if  $\mathrm{Gfd}_R M < \infty$ .

A crucial consequence of Fact 5.1 is Lemma 5.4 below. We provide a direct proof but note that it can also be deduced by combining results from [16] and [20]. One can also obtain Lemma 5.4 from arguments by Sather-Wagstaff, Sharif, and White [19], still in view of [20].

**5.4 Lemma.** *Let  $M$  be an  $R$ -module. If  $M$  has finite Gorenstein flat dimension, then the following conditions are equivalent*

- (i)  $M$  is Gorenstein flat.
- (ii)  $\mathrm{Ext}_R^i(M, C) = 0$  holds for all  $i \geq 1$  and every flat-cotorsion  $R$ -module  $C$ .
- (iii)  $\mathrm{Ext}_R^i(M, C) = 0$  holds for all  $i \geq 1$  and every cotorsion  $R$ -module  $C$  of finite flat dimension.
- (iv)  $\mathrm{Ext}_R^1(M, C) = 0$  holds for every cotorsion  $R$ -module  $C$  of finite flat dimension.

**Proof.** The implication (iii)  $\implies$  (iv) is trivial and (i)  $\implies$  (ii) follows from Fact 5.1.

(ii)  $\implies$  (iii): A cotorsion module  $C$  of finite flat dimension has a bounded flat resolution by flat-cotorsion modules; see Remark 2.6. The claim now follows by dimension shifting.

(iv)  $\implies$  (i): We first reduce to the case where  $M$  is cotorsion. To this end, consider an exact sequence  $0 \rightarrow M \rightarrow C^M \rightarrow F^M \rightarrow 0$  of  $R$ -modules, where  $C^M$  is cotorsion and  $F^M$  is flat. By Fact 5.1 it suffices to prove that  $C^M$  is Gorenstein flat. From the Horseshoe Lemma for projective resolutions and Fact 5.1 it follows that  $C^M$  has finite Gorenstein flat dimension. We may now assume that  $M$  is cotorsion.

As  $M$  has finite Gorenstein flat dimension, there is an exact sequence

$$0 \longrightarrow G \longrightarrow W_{g-1} \longrightarrow \cdots \longrightarrow W_0 \longrightarrow M \longrightarrow 0$$

where the modules  $W_i$  are flat-cotorsion and  $G$  is Gorenstein flat. From the  $\mathbf{F}$ -totally acyclic complex defining  $G$  one gets an exact sequence

$$0 \longrightarrow G \longrightarrow F_{g-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow H \longrightarrow 0$$

where each module  $F_i$  is flat and  $H$  is Gorenstein flat. By Fact 5.1 this sequence is  $\text{Hom}_R(-, V)$ -exact for every flat-cotorsion  $R$ -module  $V$ . As in the proof of Lemma 4.7 one now gets an acyclic complex

$$0 \longrightarrow F_{g-1} \longrightarrow F_{g-2} \oplus W_{g-1} \longrightarrow \cdots \longrightarrow F_0 \oplus W_1 \longrightarrow H \oplus W_0 \longrightarrow M \longrightarrow 0.$$

The module  $G' = H \oplus W_0$  is Gorenstein flat, and the module  $K = \text{Ker}(G' \rightarrow M)$  has finite flat dimension. There is a short exact sequence  $0 \rightarrow K \rightarrow C^K \rightarrow F^K \rightarrow 0$  of  $R$ -modules with  $C^K$  cotorsion and  $F^K$  flat. It yields a pushout diagram

$$(1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & G' & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & C^K & \longrightarrow & X & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & F^K & = & F^K & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

with exact rows and columns. By Fact 5.1 the module  $X$  is Gorenstein flat. The module  $C^K$  is cotorsion of finite flat dimension, so by assumption the second row splits, which means that  $M$  is Gorenstein flat in view of Fact 5.1.  $\square$

**5.5 Proposition.** *Let  $M$  be an  $R$ -complex of finite Gorenstein flat dimension and  $n \geq \sup M$  an integer. One has  $\text{Gfd}_R M \leq n$  if and only if  $-\inf \mathbf{R}\text{Hom}_R(M, C) \leq n$  holds for every cotorsion  $R$ -module  $C$  of finite flat dimension.*

**Proof.** Assume that  $\text{Gfd}_R M \leq n$  holds and let  $F$  be a semi-flat replacement of  $M$  as in Definition 5.3. As in the proof of Theorem 4.5 there is a short exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F_{\leq n} \rightarrow 0$ . The complex  $F'$  is acyclic and concentrated in degrees  $n$  and above; by Fact 5.1 it is a complex of Gorenstein flat modules. For the same reason, the cycle modules in  $F'$  are Gorenstein flat. Let  $C$  be a cotorsion module of finite flat dimension. It follows from Lemma 5.4 that  $\text{Hom}_R(F', C)$  is acyclic, so there is an isomorphism  $\mathbf{R}\text{Hom}_R(M, C) \simeq \text{Hom}_R(F_{\leq n}, C)$  in the derived category. Evidently, one has  $\text{Hom}_R(F_{\leq n}, C)_i = 0$  for  $i < -n$ .

For the converse, set  $g = \text{Gfd}_R M$ . One has to show that  $n \geq g$  holds. Assume towards a contradiction that  $n < g$  holds. It now follows from the assumption  $n \geq$

$\sup M$  that the module  $C_{g-1}(F)$  is not Gorenstein flat. The complex  $\Sigma^{1-g}F_{\geq g-1}$  is a semi-flat replacement of the module  $C_{g-1}(F)$  which, therefore, is a module of finite Gorenstein flat dimension 1. Let  $C$  be a cotorsion  $R$ -module of finite flat dimension; in view of Fact 1.5 one has

$$\begin{aligned} \text{Ext}_R^1(C_{g-1}(F), C) &\cong H_{-1}(\mathbf{R}\text{Hom}_R(C_{g-1}(F), C)) \\ &= H_{-1}(\text{Hom}_R(\Sigma^{1-g}F_{\geq g-1}, C)) \\ &= H_{-g}(\text{Hom}_R(F_{\geq g-1}, C)) \\ &= H_{-g}(\text{Hom}_R(F, C)) \\ &= H_{-g}(\mathbf{R}\text{Hom}_R(M, C)) \\ &= 0. \end{aligned}$$

It now follows from Lemma 5.4 that  $C_{g-1}(F)$  is Gorenstein flat, a contradiction.  $\square$

**5.6 Lemma.** *Let  $C$  be a complex of cotorsion  $R$ -modules. For every  $n \geq \sup C$  the module  $C_n(C)$  is cotorsion.*

**Proof.** Let  $n \geq \sup C$ . Splicing a shifted injective resolution of  $C_n(C)$  with the acyclic complex  $\cdots \rightarrow C_{n+1} \rightarrow C_n \rightarrow C_n(C) \rightarrow 0$  one gets an acyclic complex  $X$  of cotorsion modules, and it follows from Fact 1.1 that the cycle module  $Z_{n-1}(X) \cong C_n(C)$  is cotorsion.  $\square$

The next theorem justifies the title of the paper.

**5.7 Theorem.** *Let  $M$  be an  $R$ -complex. There is an inequality*

$$\text{Gfcd}_R M \leq \text{Gfd}_R M,$$

*and equality holds if  $M$  has finite Gorenstein flat dimension.*

**Proof.** It is enough to prove that every complex of finite Gorenstein flat dimension has finite Gorenstein flat-cotorsion dimension, then Theorem 4.5 and Proposition 5.5 show that they agree. Assume that  $\text{Gfd}_R M = n$  holds for some integer  $n$ . That is, there exists a semi-flat replacement  $F$  of  $M$  with  $C_n(F)$  Gorenstein flat. Consider an approximation  $0 \rightarrow F \rightarrow C \rightarrow A \rightarrow 0$ , where  $A$  is acyclic and semi-flat and  $C$  is semi-cotorsion; see Fact 2.2. It follows that  $C$  is a semi-flat-cotorsion replacement of  $M$ ; see Fact 1.4. As  $A$  is acyclic there is an exact sequence  $0 \rightarrow C_n(F) \rightarrow C_n(C) \rightarrow C_n(A) \rightarrow 0$ . The module  $C_n(A)$  is flat and the class of Gorenstein flat modules is closed under extensions, see Fact 5.1, so  $C_n(C)$  is Gorenstein flat. By Lemma 5.6 it is also cotorsion, so by Theorem 5.2 it is Gorenstein flat-cotorsion.  $\square$

**5.8 Corollary.** *Let  $R$  be right coherent. For every  $R$ -complex  $M$  one has*

$$\text{Gfd}_R M = \text{Gfcd}_R M.$$

**Proof.** It suffices in view of Theorem 5.7 to prove that every complex of finite Gorenstein flat-cotorsion dimension has finite Gorenstein flat dimension, and that is immediate from the definitions and Theorem 5.2.  $\square$

**5.9 Remark (a caveat).** Every Gorenstein flat  $R$ -module  $G$  has Gorenstein flat dimension 0 and hence  $\text{Gfcd}_R G = 0$  as well. Thus, a module of Gorenstein flat-cotorsion dimension 0 need not be Gorenstein flat-cotorsion; any flat module that is not cotorsion exemplifies this; see also Remark 4.6.



So how far are Gorenstein flat modules from being Gorenstein flat-cotorsion? Fact 5.1 can be harnessed to provide an answer: Let  $G$  be a Gorenstein flat  $R$ -module. There are exact sequences of  $R$ -modules,

$$0 \longrightarrow G \longrightarrow H \longrightarrow F \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow H' \longrightarrow F' \longrightarrow G \longrightarrow 0,$$

where  $H$  and  $H'$  are Gorenstein flat-cotorsion and  $F$  and  $F'$  are flat. Indeed, the left-hand sequence exists with  $H$  cotorsion and  $F$  flat, and by Fact 5.1 the module  $H$  is Gorenstein flat and, hence, Gorenstein flat-cotorsion by Theorem 5.2. Similarly, the right-hand sequence exists with  $F'$  flat and  $H'$  cotorsion, and by Fact 5.1 the module  $H'$  is Gorenstein flat and, hence, Gorenstein flat-cotorsion by Theorem 5.2.

Recall that a noetherian (i.e. noetherian on both sides) ring is called Iwanaga–Gorenstein if both injective dimensions  $\text{id}_R R$  and  $\text{id}_{R^\circ} R$  are finite.

**5.10 Corollary.** *Let  $R$  be noetherian. The following conditions are equivalent.*

- (i)  $R$  is Iwanaga–Gorenstein.
- (ii) Every  $R$ - and every  $R^\circ$ -module has finite Gorenstein flat-cotorsion dimension.
- (iii) Every  $R$ - and every  $R^\circ$ -complex with bounded above homology has finite Gorenstein flat-cotorsion dimension.

**Proof.** As  $R$  is noetherian, it follows from Corollary 5.8 that the Gorenstein flat dimension coincides with the Gorenstein flat-cotorsion dimension over both  $R$  and  $R^\circ$ . The assertion now follows from results of Enochs and Jenda [9, Thm. 12.3.1] and Iacob [15, Thm. 3.2].  $\square$

**5.11 Lemma.** *Let  $M$  be an  $R$ -complex and  $n$  an integer. One has  $\text{fd}_R M \leq n$  if and only if  $M$  has a semi-flat-cotorsion replacement  $W$  with  $W_i = 0$  for all  $i > n$ .*

**Proof.** The “if” statement is evident. For the “only if,” let  $W$  be a semi-flat-cotorsion replacement of  $M$ . By assumption the module  $C_n(W)$  is flat, see [2, Thm. 2.4.F], so there is an exact sequence of complexes of flat  $R$ -modules

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W_{\leq n} \longrightarrow 0.$$

The sequence is degreewise pure exact, so it follows from [6, Prop. 6.2] that the complex  $W_{\leq n}$  is semi-flat. As one has  $n \geq \sup M$ , the module  $C_n(W)$  is cotorsion by Lemma 5.6. It follows that  $W_{\leq n}$  consists of cotorsion  $R$ -modules, whence it is semi-cotorsion by Fact 1.1. Thus  $W_{\leq n}$  is a semi-flat-cotorsion replacement of  $M$ .  $\square$

**5.12 Theorem.** *Let  $M$  be an  $R$ -complex. There is an inequality,*

$$\text{Gfcd}_R M \leq \text{fd}_R M,$$

*and equality holds if  $M$  has finite flat dimension.*

**Proof.** Without loss of generality assume that  $n := \text{fd}_R M$  is an integer. By Lemma 5.11 there is a semi-flat-cotorsion replacement  $W$  of  $M$  concentrated in degrees  $n$  and below. In particular, one has  $\text{Gfcd}_R M \leq n$ . Set  $m = \text{Gfcd}_R M$  and assume towards a contradiction that  $n > m$  holds. By Theorem 4.5 the module  $C_m(W)$  is Gorenstein flat-cotorsion, and as  $m \geq \sup M$  holds there is an exact sequence of  $R$ -modules,

$$0 \longrightarrow W_n \longrightarrow W_{n-1} \longrightarrow \cdots \longrightarrow W_m \longrightarrow C_m(W) \longrightarrow 0.$$

It follows that  $C_m(W)$  has finite flat dimension; in particular, it has finite Gorenstein flat dimension. By Theorem 5.7 one now has  $\text{Gfd}_R C_m(W) = 0$ , which means that  $C_m(W)$  is a Gorenstein flat module. Now it follows from Fact 5.1 that the module  $C_i(W)$  is Gorenstein flat for  $n > i \geq m$ . It follows from Lemma 5.4 that the following sequence splits

$$0 \longrightarrow W_n \longrightarrow W_{n-1} \longrightarrow C_{n-1}(W) \longrightarrow 0,$$

whence  $C_{n-1}(W)$  is flat-cotorsion. So one has  $\text{fd}_R M \leq n - 1$ , a contradiction.  $\square$

## 6. AN ILLUSTRATION: TATE COHOMOLOGY

In this final section, we demonstrate how one can utilize the Gorenstein flat-cotorsion dimension to generalize a result of Hu and Ding [14].

**6.1 Definition.** Let  $M$  be an  $R$ -complex. A *complete flat-cotorsion resolution* of  $M$  consists of the following data: a semi-flat-cotorsion replacement  $W$  of  $M$ , a totally acyclic complex of flat-cotorsion  $R$ -modules  $T$ , and a morphism  $\tau: T \rightarrow W$  of  $R$ -complexes such that  $\tau_i$  is an isomorphism for all  $i \gg 0$ .

**6.2 Proposition.** *Let  $M$  be an  $R$ -complex and  $n$  an integer. The following conditions are equivalent.*

- (i)  $\text{Gfcd}_R M \leq n$ .
- (ii) For every semi-flat-cotorsion replacement  $W$  of  $M$  there exists a complete flat-cotorsion resolution  $\tau: T \rightarrow W$  of  $M$  such that  $\tau_i$  is a split epimorphism for every  $i \in \mathbb{Z}$  and  $\tau_i = \text{id}_{W_i}$  holds for every  $i \geq n$ .
- (iii) There exists a complete flat-cotorsion resolution  $\tau: T \rightarrow W$  of  $M$  such that  $\tau_i$  is an isomorphism for every  $i \geq n$ .

**Proof.** The proof is cyclic; the implication (ii)  $\implies$  (iii) is trivial.

(i)  $\implies$  (ii): Let  $W$  be a semi-flat-cotorsion replacement of  $M$ . By assumption, the module  $C_n(W)$  is Gorenstein flat-cotorsion, so it follows from Lemma 3.2 that there is an acyclic complex concentrated in degrees  $n$  and below,

$$T' = 0 \longrightarrow C_n(W) \longrightarrow T_{n-1} \longrightarrow T_{n-2} \longrightarrow \cdots,$$

where the modules  $T_i$  are flat-cotorsion and such that  $\text{Hom}_R(T', V)$  is acyclic for every flat-cotorsion  $R$ -module  $V$ . Let  $T$  be the complex obtained by splicing together  $T'_{\leq n-1}$  and  $W_{\geq n}$  at  $C_n(W)$ . Now the identity on  $C_n(W)$  lifts to a morphism  $T_{\leq n-1} \rightarrow W_{\leq n-1}$ . The induced degreewise split surjective homomorphism  $T_{\leq n-1} \oplus \Sigma^{-1}\text{Cone } 1^{W_{\leq n-1}} \rightarrow W_{\leq n-1}$  together with the identity on  $T_{\geq n} = W_{\geq n}$  is the desired morphism  $\tau$ .

(iii)  $\implies$  (i): As  $\tau_i$  is an isomorphism for  $i \geq n$  one has  $C_n(W) \cong C_n(T)$ , and the latter module is Gorenstein flat-cotorsion.  $\square$

**6.3 Lemma.** *Let  $M$  be an  $R$ -complex of finite Gorenstein flat-cotorsion dimension. If  $\tau: T \rightarrow W$  and  $\tau': T' \rightarrow W'$  are complete flat-cotorsion resolutions of  $M$ , then there is a homotopy equivalence  $T \rightarrow T'$ .*

**Proof.** For an  $R$ -module  $X$  the disk complex  $0 \rightarrow X \xrightarrow{=} X \rightarrow 0$  concentrated in degrees  $i + 1$  and  $i$  is denoted  $D^i(X)$ . Fix  $i > \text{Gfcd}_R M$ ; as  $\text{Gfcd}_R M \geq \sup M$

holds, one has  $C_{i+1}(W) \cong Z_i(W)$  and  $C_{i+1}(W') \cong Z_i(W')$ . By Lemma 2.7 there exist flat-cotorsion modules  $V$  and  $V'$  with  $Z_i(W) \oplus V \cong Z_i(W') \oplus V'$ . Set

$$\widetilde{W} = W \oplus D^i(V), \quad \widetilde{W}' = W' \oplus D^i(V'), \quad \widetilde{T} = T \oplus D^i(V), \quad \text{and} \quad \widetilde{T}' = T' \oplus D^i(V').$$

The induced morphisms  $\tilde{\tau}: \widetilde{T} \rightarrow \widetilde{W}$  and  $\tilde{\tau}': \widetilde{T}' \rightarrow \widetilde{W}'$  are complete flat-cotorsion resolutions of  $M$ . The isomorphism  $Z_i(\widetilde{W}) \cong Z_i(\widetilde{W}')$  lifts by [4, Lem. 3.1 and Prop. 3.3] to a homotopy equivalence  $\widetilde{T} \rightarrow \widetilde{T}'$ . Evidently, there are homotopy equivalences  $T \rightarrow \widetilde{T}$  and  $\widetilde{T}' \rightarrow T'$ .  $\square$

In view of Lemma 6.3 and Proposition 2.5 one can now define a version of Tate cohomology.

**6.4 Definition.** Let  $M$  be an  $R$ -complex of finite Gorenstein flat-cotorsion dimension and  $\tau: T \rightarrow W$  a complete flat-cotorsion resolution of  $M$ . For an  $R$ -complex  $N$  and  $i \in \mathbb{Z}$  set

$$\widehat{\text{Ext}}_{\text{FC}}^i(M, N) = H_{-i}(\text{Hom}_R(T, N)).$$

For a complex of finite Gorenstein flat dimension, every complete flat-cotorsion resolution is per the next lemma a Tate flat resolution in the sense of [17, 7].

**6.5 Lemma.** *Let  $M$  be an  $R$ -complex of finite Gorenstein flat dimension. In every complete flat-cotorsion resolution  $T \rightarrow W$  of  $M$ , the complex  $T$  is  $\mathbf{F}$ -totally acyclic.*

**Proof.** Let  $T \rightarrow W$  be a complete flat-cotorsion resolution of  $M$ . As  $W$  is a semi-flat replacement of  $M$ , the modules  $C_i(T) \cong C_i(W)$  are Gorenstein flat for  $i \gg 0$ . Thus  $C_i(T)$  is a Gorenstein flat-cotorsion module of finite Gorenstein flat dimension for every  $i \in \mathbb{Z}$ . As the dimensions agree by Theorem 5.7, it follows that  $C_i(T)$  is Gorenstein flat for every  $i \in \mathbb{Z}$ , whence  $T$  is  $\mathbf{F}$ -totally acyclic, see [5, Lem. 2.3].  $\square$

**6.6 Remark.** Let  $M$  be an  $R$ -complex. By Fact 2.2 there is an exact sequence of  $R$ -complexes  $0 \rightarrow M \rightarrow C \rightarrow F' \rightarrow 0$  with  $C$  semi-cotorsion and  $F'$  acyclic and semi-flat and another exact sequence  $0 \rightarrow C' \rightarrow F \rightarrow C \rightarrow 0$  with  $F$  semi-flat and  $C'$  semi-cotorsion and acyclic. The diagram  $F \xrightarrow{\cong} C \xleftarrow{\cong} M$  is a semi-flat-cotorsion replacement of  $M$ . Assume that  $M$  has finite Gorenstein flat dimension. By Theorem 5.7 and Proposition 6.2 there is a complete flat-cotorsion resolution  $\tau: T \rightarrow W$  with  $\tau_i$  a split epimorphism for all  $i \in \mathbb{Z}$ . By Proposition 2.5 there is a homotopy equivalence  $W \rightarrow F$ , so by Lemma 6.5 the diagram

$$T \xrightarrow{\tau} W \xrightarrow{\cong} C \xleftarrow{\cong} M$$

is a complete flat resolution in the sense of [14]. It follows that for  $R$ ,  $M$ , and  $N$  as in [14, Thm. 5.5], our Definition 6.4 agrees with the definition in [14].

In view of Theorem 5.7 and Remark 6.6 the next result generalizes parts of [14, Thm. 1.5].

**6.7 Theorem.** *Let  $M$  be an  $R$ -complex. If  $M$  has finite Gorenstein flat-cotorsion dimension, then the following statements are equivalent.*

- (i)  $\text{fd}_R M = \text{Gfcd}_R M$  holds.
- (ii)  $\widehat{\text{Ext}}_{\text{FC}}^m(M, N) = 0$  holds for all  $m \in \mathbb{Z}$  and every  $R$ -complex  $N$ .
- (iii)  $\widehat{\text{Ext}}_{\text{FC}}^m(M, C) = 0$  holds for some  $m \in \mathbb{Z}$  and every cotorsion  $R$ -module  $C$ .

**Proof.** The proof is cyclic; the implication  $(ii) \implies (iii)$  is clear.

$(i) \implies (ii)$ : By Lemma 5.11 there is a semi-flat-cotorsion replacement  $W$  of  $M$  with  $W_i = 0$  for all  $i \gg 0$ . It follows that  $0 \rightarrow W$  is a complete flat-cotorsion resolution of  $M$ .

$(iii) \implies (i)$ : Since  $M$  has finite Gorenstein flat-cotorsion dimension, there exists by Proposition 6.2 a complete flat-cotorsion resolution  $\tau: T \rightarrow W$  of  $M$  with  $\tau_i = \text{id}_{F_i}$  for all  $i \geq g$ , where  $g$  is some integer. The module  $C_m(T)$  is cotorsion, so  $\widehat{\text{Ext}}_{\text{FC}}^m(M, C_m(T)) = H_{-m}(\text{Hom}_R(T, C_m(T))) = 0$  holds. This means that application of  $\text{Hom}_R(-, C_m(T))$  leaves the sequence

$$0 \rightarrow C_m(T) \rightarrow T_{m-1} \rightarrow C_{m-1}(T) \rightarrow 0$$

exact, and it follows that it splits. In particular,  $C_{m-1}(T)$  is a flat module and hence so is  $C_i(T)$  for every  $i \geq m-1$ . For  $i \gg 0$  the module  $C_i(W) = C_i(T)$  is flat, so  $M$  has finite flat dimension. Now invoke Theorem 5.12.  $\square$

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